

Classical Mechanics: A concise, mathematically oriented, summary

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Abstract

Classical mechanics is all about the analysis of motion: how and why the positions of objects in the three-dimensional space are changed over time. Motion is the essence of life — a world without motion is a dead world.

We choose to present classical mechanics by relying on three fundamental physical units: Length, Time, and Mass. The position of an object relies on the Length unit, and the movement of an object is described as a mathematical function, mapping time to position. The mass of an object is some measure of quantity — how much matter does the object have (indeed, the mass of two identical objects is twice the mass of the single object). The meaning of these fundamental physical units is taken for granted (similarly to "axioms" in mathematics). All the other quantities in classical mechanics (such as velocity, acceleration, momentum, force, energy, and more) are not fundamental: they can be defined, or derived mathematically, from the Length-Mass-Time (LMT) fundamental quantities.

The main source I have used to write this summary is Raz Kupferman's lecture notes ¹.

¹<http://math.huji.ac.il/~razk/Teaching/LectureNotes/LectureNotesMechanics.pdf>

Contents

1	Introduction	7
1.1	Units of Measurements	7
1.2	Dimensional Analysis of Physics Laws	7
1.2.1	Dimensions of physical quantities	7
1.2.2	The Fundamental Principle of the Laws of Physics	7
1.2.3	Dimension is always a power-law monomial	8
1.2.4	The Π theorem	9
2	Newton Mechanics	11
2.1	Kinematics	11
2.1.1	Radial and Tangential Acceleration	11
2.2	Dynamics: Newton's Laws	12
2.2.1	Force due to a vector field	12
2.3	Work and Energy	13
2.4	Angular momentum and Torque	13
2.5	Modeling interactions between several objects	14
2.5.1	Conservation of Momentum	14
2.5.2	Conservation of Angular Momentum	15
2.5.3	Conservation of Mechanical Energy	15
3	The Lagrangian and Euler-Lagrange Equations	17
3.1	Adding Constraints	17
3.2	Generalized Coordinates	17
3.3	d'Alembert's Principle of Virtual Work leads to Euler-Lagrange Equations	18
3.4	Deriving Euler-Lagrange Equations from Hamilton's Principle of Least Action via Variational Calculus	21
3.4.1	Hamilton's Principle of Least Action	21
3.4.2	Proof of Theorem 3.4.1 by Variational Calculus	22
4	Hamilton's Mechanics	25
4.1	Generalized Momentum and Phase Space	25
4.2	Fenchel-Legendre Transform	25
4.3	Hamiltonian and Hamilton's Equations	26
4.4	Conservation Quantities	27
A	Appendix	29
A.1	Derivatives and Their Notation in this Note	29
A.1.1	Derivative, partial derivative, gradient, and total derivative	29
A.1.2	Chain Rule	30
A.1.3	Derivative with Direct Dependencies	30

Chapter 1

Introduction

The most basic notion in mechanics is the position function, $r : t \rightarrow \mathbb{R}^3$. That is, $r(t)$ is the position of some object in the three-dimensional space at time t . Basic kinematics define "velocity" and "acceleration" as first and second derivatives of the position function. Newton's laws define what causes objects to change their velocity. E.g., Newton's first law states that the momentum (which is the product of the object's velocity with its mass) can only be changed by applying a force.

1.1 Units of Measurements

Units of measurements are divided to *fundamental units* and *derived units*. The choice of which measurements are fundamental is arbitrary. Once we picked them, then we should be able to express other measurements as combination of the fundamental units. For example, if we picked meters and seconds as fundamental units, then velocity is expressed by m/s and acceleration by m/s^2 .

A *system of units* is a set of fundamental units which is sufficient to express all the measurements we are interested in to describe some class of phenomena.

A *class of system of units* is determined by setting the types of fundamental units we want to have. E.g., LMT stands for Length-Mass-Time, and it contains the system of units meter-kg-second and cm-gram-second as two members of the class.

Throughout this book, we will mainly use the LMT class.

1.2 Dimensional Analysis of Physics Laws

1.2.1 Dimensions of physical quantities

Suppose we pick the class LMT, and then we change the units by dividing each by L, M, T respectively. E.g., moving from meter-kg-second to cm-gram-second means picking $L = 100, M = 1000, T = 1$. As a result, length measurements will be multiplied by L (e.g. $1m$ becomes $100cm$), mass will be multiplied by M , and time by T . Furthermore, every physical quantity is scaled by some $f(L, M, T)$. E.g., velocity will be changed by $f(L, M, T) = L/T$. Indeed, if the velocity is $1 m/s$ then, after the change, the velocity will be $100cm/s$, which satisfies $100 = 1 \cdot 100/1$. We call the function $f(L, M, T)$ the *dimension* of the physical quantity. If $f(L, M, T) = 1$ for some quantity (i.e., it has dimension 1), we call this physical quantity *dimensionless*. Note that the dimension depends on the choice of the class of system of units.

1.2.2 The Fundamental Principle of the Laws of Physics

The laws of physics are invariant under the choice of system of units.

Laws of physics are often expressed as an equation: expression 1 = expression 2. It follows that both expressions must have the same dimension, otherwise they won't be invariant to change of system of units.

1.2.3 Dimension is always a power-law monomial

Consider the LMT class as an example, and let a be some quantity. Its dimension is

$$[a] := f(L, M, T)$$

where $[a]$ is our notation for dimension, and we defined it as some function f . Consider change of units defined by constants L_1, M_1, T_1 , let $a \in \mathbb{R}$ be the value of the quantity before the change and let a_1 be its value after the change. Then, $a_1 = f(L_1, M_1, T_1) a$. Let's repeat the same with different constants L_2, M_2, T_2 , then $a_2 = f(L_2, M_2, T_2) a$. Dividing both equations, it follows that

$$\frac{a_2}{a_1} = \frac{f(L_2, M_2, T_2)}{f(L_1, M_1, T_1)}$$

But, we can equivalently look at the change of units from system 1 to system 2, which is defined by the constants $L_2/L_1, M_2/M_1, T_2/T_1$. So, we also have

$$a_2 = f(L_2/L_1, M_2/M_1, T_2/T_1) a_1$$

It follows that the following functional equation holds

$$\frac{f(L_2, M_2, T_2)}{f(L_1, M_1, T_1)} = f(L_2/L_1, M_2/M_1, T_2/T_1)$$

Differentiate both sides w.r.t. L_2 , and evaluate the expression for $L_1 = L_2 = L, M_2 = M_1 = M, T_2 = T_1 = T$, we get the equation

$$\frac{1}{f(L, M, T)} \frac{\partial f}{\partial L} f(L, M, T) = \frac{1}{L} \frac{\partial f}{\partial L} f(1, 1, 1) := \frac{1}{L} \alpha$$

where α is some constant that doesn't depend on L . For any fixed M, T , define $h(L) = f(L, M, T)$, and note that $h'(L) = \frac{\partial f}{\partial L} f(L, M, T)$. So, the above equation takes the form

$$\frac{h'(L)}{h(L)} = \frac{\alpha}{L} \Rightarrow h'(L) = h(L) \frac{\alpha}{L} \Rightarrow h(L) = L^\alpha c$$

for some constant c that depends on M, T but not on L . This is true because for such $h(L)$ we have $h'(L) = \alpha L^{\alpha-1} c = \alpha L^\alpha c/L = h(L) \alpha/c$. It follows that there's a function $g(M, T)$ such that

$$f(L, M, T) = L^\alpha g(M, T)$$

Substituting this in the functional equation where we set $L_1 = L_2 = L$, we get

$$\frac{g(M_2, T_2)}{g(M_1, T_1)} = 1^\alpha g(M_2/M_1, T_2/T_1)$$

If $\alpha \neq 0$, we can get rid of the term 1^α . Following similar rational one can show that $g(M, T) = M^\beta q(T)$ and that $q(T) = cT^\gamma$. So, we obtain

$$f(L, M, T) = c L^\alpha M^\beta T^\gamma$$

and since $f(1, 1, 1) = 1$ we get that $c = 1$. So, the bottom line is that we always have

$$f(L, M, T) = L^\alpha M^\beta T^\gamma$$

1.2.4 The Π theorem

A physical relationship between some dimensional parameter and several dimensional governing parameters can be rewritten as a relation between some dimensional parameter and several dimensionless products of the governing parameters.

Before proving the Π theorem, let us give an example. Consider an L system (only length). A right triangle is fully determined by the length of the hypotenuse, c , and its smallest angle, ϕ . Hence, its area can be written as $S = f(c, \phi)$. Indeed, if the legs of the triangle are a, b , then its area is $ab/2$ which can be rewritten as $c \sin(\phi) c \cos(\phi)/2$, so $S = c^2 \sin(\phi) \cos(\phi)/2$. The dimensions are $[S] = L^2, [c] = L, [\phi] = 1$. Consider a new variable $\alpha = S/c^2$. It is dimensionless. So, we can rephrase the formula as $\alpha = g(\phi)$, where in our case, $g(\phi) = \sin(\phi) \cos(\phi)/2$. In this new formula, all variables are dimensionless. The Π theorem is a systematic way to achieve a new formula in which all variables are dimensionless.

Let's turn to the formal statement and proof of the Π theorem. Consider a physical relation of the form

$$c = f(a_1, \dots, a_k, b_1, \dots, b_m)$$

Here, the distinction between the a 's and the b 's is through the assumption that there exists matrix $\alpha \in \mathbb{R}^{m,k}$ s.t. for each b_j , its dimension can be written as $[b_j] = \prod_i [a_i]^{\alpha_{j,i}}$. This is not true for the a_i , that is, their dimensions are independent. BTW, going to log scale, we can rewrite it as $\log([b_j]) = \sum_i \alpha_{j,i} \log([a_i])$, and the meaning of independence is the same as in linear algebra. We define the dimensionless quantities

$$\Pi_j = \frac{b_j}{\prod_i [a_i]^{\alpha_{j,i}}}$$

The first claim is that $[c] = \prod_i [a_i]^{\beta_i}$ for some vector β . This must be true, otherwise we could change units s.t. c is changed while all the a_i remains the same, contradicting the physical law. So, define the dimensionless quantity

$$\Pi = \frac{c}{\prod_i [a_i]^{\beta_i}}$$

Now, the Π theorem states that there is some function g which gives a rephrasing of the physical law as

$$\Pi = g(\Pi_1, \dots, \Pi_m)$$

That is, the law is rephrased using dimensionless quantities.

To prove the above, rewrite the law while replacing $b_j = \Pi_j \prod_i [a_i]^{\alpha_{j,i}}$ and $c = \Pi \prod_i [a_i]^{\beta_i}$ and rearrange terms to get

$$\Pi = \frac{1}{\prod_i [a_i]^{\beta_i}} f(a_1, \dots, a_k, \Pi_1 \prod_i [a_i]^{\alpha_{1,i}}, \dots, \Pi_m \prod_i [a_i]^{\alpha_{m,i}})$$

The right-hand side is some function $h(a_1, \dots, a_k, \Pi_1, \dots, \Pi_m)$. Since the left-hand side is dimensionless, h cannot actually use any of the a_i , as otherwise, we could change units so that only a_1 will change while the rest of the arguments in the equation are dimensionless, leading to contradiction. From this the Π theorem follows.

Why is it any good? It is sometimes convenient to switch to dimensionless formulas in order to ease the computation. Consider for example the following: a mass m is thrown vertically from the ground with initial velocity v . What will be its maximal height? We start with writing

$$h = f(m, v, g)$$

The dimensions are $[h] = L, [m] = M, [v] = L/T, [g] = L/T^2$. Note that all the arguments to f has independent dimensions. Hence, in the Π theorem, the right-hand side is a constant (because there are no b 's, only a 's). In addition, the definition of Π for the left-hand side transforms h into a dimensionless parameter $\Pi = \frac{h}{v^2/g}$. This yields

$$\Pi = \frac{h}{v^2/g} = \text{const}$$

The constant here happens to be $1/2$, but the fact that some quantity is a constant can help ease the calculations in many cases.

Chapter 2

Newton Mechanics

2.1 Kinematics

If $r(t) \in \mathbb{R}^3$ is the position of a particle in space at time t , then $v(t) = r'(t)$ and $a(t) = v'(t)$. Throughout this book, for every function of time, $f(t)$, where $f : \mathbb{R} \rightarrow \mathbb{R}^d$, we use the notation $f'(t)$ to denote the derivative of f with respect to time. That is, for $i \in [d]$, $f'_i(t) = \frac{d}{dt} f_i(t)$. For the position function, we often use $r(t) = (x(t), y(t), z(t))$.

2.1.1 Radial and Tangential Acceleration

From the definition of derivative, it is easy to see that the direction of $v(t)$ is tangent to the curve $r(t)$. Observe that if the speed size $\|v(t)\|$ is constant, then $a(t) \cdot v(t) = 0$, because

$$0 = (\|v(t)\|^2)' = 2v(t) \cdot v'(t) = 2v(t) \cdot a(t)$$

In the general case, we can always decompose $a(t) = \alpha(t)v(t)/\|v(t)\| + \beta(t)v^\dagger(t)$, where $v^\dagger(t)$ has a unit norm and is orthogonal to $v(t)$. The $\alpha(t)$ is often called tangential acceleration (because it points to the direction of the tangent of the curve on which the particle moves) and $\beta(t)$ is often called radial acceleration (because when $\|v(t)\|$ is constant, and the particle moves on a circle, then $v^\dagger(t)$ points to the center of the circle). We can find $\alpha(t) = a(t) \cdot v(t)/\|v(t)\|$ by observing that

$$(\|v(t)\|)' = (\sqrt{\|v(t)\|^2})' = \frac{2v(t) \cdot a(t)}{2\|v(t)\|} = \alpha(t)$$

That is, the tangential acceleration is the derivative of the size of the velocity vector.

When a particle moves on a plane (2 dimensions), we can write $v(t)$ in polar coordinates as

$$v(t) = (R(t) \cos(\theta(t)), R(t) \sin(\theta(t)))$$

Using the derivative of product, we obtain

$$v'(t) = R'(t) \underbrace{(\cos(\theta(t)), \sin(\theta(t)))}_{v(t)/\|v(t)\|} + R(t) \theta'(t) \underbrace{(-\sin(\theta(t)), \cos(\theta(t)))}_{v^\dagger(t)}$$

That is, we see again that the tangential acceleration is $R'(t) = (\|v(t)\|)'$. We also see that the radial acceleration is $R(t) \theta'(t)$. Observe that here, $\theta(t)$ is the angle of the vector $v(t)$, and not of the position of the particle.

When a particle moves at a constant speed $\|v(t)\| = \nu$ on a circle of radius ρ , its angular velocity is ν/ρ radians per second (since it completes a full circle in $2\pi\rho/\nu$ seconds). Note that this is the angular velocity of the position of the particle. However, at a circular motion, the angle $\theta(t)$ of the velocity vector is 90 degrees rotation of the angle of the position vector, so $\theta'(t) = \nu/\rho$ as well. It follows that the radial acceleration is $R(t) \theta'(t) = \|v(t)\| \nu/\rho = \nu^2/\rho$.

2.2 Dynamics: Newton's Laws

We will be working with the Length-Mass-Time (LMT) class as our fundamental quantities.

Momentum and Force: The momentum of a particle is defined as

$$p(t) := m(t) v(t)$$

We sometimes omit the dependence on t and simply write $p = mv$, but the precise meaning always involve a function of time. Momentum is a vector, and each of its components has dimension ML/T . The Force on a particle at time t is defined as

$$f(t) := p'(t)$$

By the derivative of product, we can write

$$f(t) = m'(t) v(t) + m(t) a(t)$$

If the mass is constant we get the popular form of Newton's second law

$$f(t) = m a(t)$$

The dimension of the force is ML/T^2 . The unit of force 1 Newton is the amount of force needed to change the momentum of a $1kg$ particle by $1m/s$.

Newton's laws are in fact definitions: Historically, Newton's first law states that the momentum of a particle will remain constant unless a force is act upon it, and Newton's second law states that force is defined as the change of momentum. Even Newton realized that he could only give a circular definition to momentum and force: Force is what causes momentum to change, and momentum is fixed unless a force is applied to it. In fact, momentum and force are "definitions" and not "rules", in the same manner that velocity and acceleration are definitions and not rules.

2.2.1 Force due to a vector field

Often, the force on a point particle is a function of its position, meaning we can write $f(t) = g(r(t))$, for some $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. We call g a *vector field*. In this case, Newton's second law becomes a second-order differential equation

$$m r''(t) = g(r(t))$$

Example 2.2.1 Consider a mass m particle on a vertical spring. The force is given by

$$f(t) = (-mg - kz(t))e_3$$

where $r(t) = (x(t), y(t), z(t))$, $g = 9.8$ is the gravity on earth, and k is a constant associated with the spring. Therefore, $r(t)$ satisfies the equation

$$mz''(t) = -mg - kz(t)$$

This can be rewritten as

$$z''(t) = -\frac{k}{m} \left(z(t) + \frac{mg}{k} \right)$$

Define $w(t) = z(t) + mg/k$, then the above yields

$$w''(t) = -\frac{k}{m} w(t)$$

The solution is

$$w(t) = w(0) \cos(\omega t) + \frac{w'(0)}{\omega} \sin(\omega t)$$

where $\omega = \sqrt{k/m}$. That is, a mass suspended by a string will perform a harmonic motion around the point $z = -mg/k$.

2.3 Work and Energy

The *work* exerted on a body at a time interval $[t_1, t_2]$ is defined as

$$W := \int_{t_1}^{t_2} f(t) \cdot v(t) dt$$

When the mass is constant, $f(t) = m a(t)$, so it follows that

$$W = \int_{t_1}^{t_2} m a(t) \cdot v(t) dt$$

Define the scalar function $g(t) = m a(t) \cdot v(t)$ and observe that for $E(t) = 0.5 m \|v(t)\|^2$ we have

$$E'(t) = m v(t) \cdot v'(t) = m v(t) \cdot a(t) = g(t)$$

Define $T(v) = \frac{1}{2} m \|v\|^2$, we obtain that

$$W = \int_{t_1}^{t_2} g(t) dt = E(t_2) - E(t_1) = T(v(t_2)) - T(v(t_1))$$

The function $T(v)$ is called the *kinetic energy of the particle*, so we obtained that the work is the difference of kinetic energy.

Next, suppose that the force is due to a force field $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, namely, for every time t , $f(t) = \phi(r(t))$. Suppose also that there exists a function Φ such that $\phi = -\nabla\Phi$. Such function Φ is called the *potential field* or the *potential energy*. Observe that

$$(-\Phi(r(t)))' = -\phi(r(t)) \cdot r'(t) = \phi(r(t)) \cdot v(t) = f(t) \cdot v(t)$$

Hence, in such case the work can also be rewritten as a difference in potential energy:

$$W = -\Phi(r(t_2)) + \Phi(r(t_1))$$

Comparing the two expressions for the work we conclude that

$$-\Phi(r(t_2)) + \Phi(r(t_1)) = T(v(t_2)) - T(v(t_1)) \Rightarrow T(v(t_1)) + \Phi(r(t_1)) = T(v(t_2)) + \Phi(r(t_2))$$

This is an "energy preservation" result: the total mechanical energy, $\Phi(r) + T(v)$, remains constant.

Example 2.3.1 *A body is dropped (at rest) from a height h to the ground. At what speed will it hit the ground? We could solve it using equations of motion. But, we can also solve it by energies. The force applied on the body is $-mge_3$, so we'll take $\Phi(z) = mgz$. Take $t_1 = 0$ and t_2 the time the body hits ground, then from the conservation of energy we get*

$$T(0) + \Phi(h e_3) = T(v(t_2)) + \Phi(0) \Rightarrow \frac{1}{2} m \|v(t_2)\|^2 = mgh \Rightarrow \|v(t_2)\| = \sqrt{2gh}$$

2.4 Angular momentum and Torque

The *angular momentum* is

$$L(t) = r(t) \times p(t)$$

where $p(t) = m(t)v(t)$ is the momentum, and for vectors a, b , the vector $a \times b$ is their cross product.

Recall that the cross product of a, b is the vector whose direction is orthogonal to a and b (and the sign is by the "right-hand-rule"), and whose size is $\|a\| \|b\| \sin(\theta)$, where θ is the angle between a and b . The product rule applies to the derivative of cross product.

The *torque* is the derivative of the angular momentum, and equals to:

$$\tau(t) := L'(t) = r'(t) \times p(t) + r(t) \times p'(t) = v(t) \times (mv(t)) + r(t) \times f(t) = r(t) \times f(t)$$

where the last equality is because the cross product of two vectors of the same direction is 0.

The relation between torque and angular momentum is the same as the relation between force and momentum: the former is a derivative of the latter.

To get some intuition, consider a fixed mass particle that moves on a circle at a constant speed. Its regular momentum vector has fixed length but its angle changes, hence the momentum is not constant, and a force is applied on the particle (since the force is the derivative of momentum). However, its angular momentum is fixed (because $\|r(t)\|$, $\|p(t)\|$, and $r(t) \cdot p(t)$ are all constants, and the plane $r(t), p(t)$ defines is also constant). Hence, there is zero torque.

2.5 Modeling interactions between several objects

So far, we've taken an egocentric view with respect to a single object. For example, when a body is dropped from height h to the ground, we analyzed the forces acting upon it as some external vector field (e.g. the gravity due to earth), and we didn't care about the forces the body applies on other objects.

We can choose to add additional objects to our model, and directly analyze the interactions between the objects we model. The choice of which objects to add to the model and which forces are external is somewhat an arbitrary modeling choice.

So, let's consider a system comprising n particles of mass m_1, \dots, m_n . We denote the trajectory of the i -th particle by $r_i(t) = (x_i(t), y_i(t), z_i(t))$. For other vector quantities, such as velocity, we will use the notation $v_i(t) = (v_{1,i}(t), v_{2,i}(t), v_{3,i}(t))$. We denote by $f_i^e(t)$ the "external" force acting on object i and by $f_{j,i}(t)$ the "internal force" that object j applies on object i . The total force acting on object i is therefore

$$f_i(t) = f_i^e(t) + \sum_{j \neq i} f_{j,i}(t)$$

Using this notation, Newton's third law (action and reaction) is

$$f_{j,i}(t) = -f_{i,j}(t)$$

We make the following assumptions:

Assumption 2.5.1 $f_i^e(t)$ only depends on $r_i(t), v_i(t)$, and $f_{j,i}(t)$ only depends on $r_j(t), r_i(t), v_j(t), v_i(t)$.

We therefore slightly overload notation and use f_i^e to denote both the function from t to \mathbb{R}^3 representing the external force on particle i at time t , and the function from $\mathbb{R}^3 \times \mathbb{R}^3$ to \mathbb{R}^3 representing the dependence of $f_i^e(t)$ on $r_i(t)$ and $v_i(t)$. That is, $f_i^e(t) = f_i^e(r_i(t), v_i(t))$. Similarly, $f_{j,i}(t) = f_{j,i}(r_i(t), v_i(t), r_j(t), v_j(t))$.

Under the above assumptions, if our system consists of all bodies in the universe then the present state fully determines the future and past by solving the second order differential equations of how positions of all bodies evolve. This observation was made by Laplace.

2.5.1 Conservation of Momentum

Consider a system of n particles. The *center of mass* of the system is the weighted average of the position, where the weighting is by the mass:

$$R(t) := \frac{\sum_{i=1}^n m_i r_i(t)}{\sum_{i=1}^n m_i}$$

The velocity of the system is the derivative

$$V(t) := R'(t) = \frac{\sum_{i=1}^n m_i v_i(t)}{\sum_{i=1}^n m_i}$$

Denote the total mass by

$$M = \sum_{i=1}^n m_i$$

The total momentum is

$$P(t) := MV(t) = \sum_{i=1}^n m_i v_i(t) = \sum_{i=1}^n p_i(t)$$

That is, the total momentum equals to the sum of individual momentum. The rate of change of the total momentum is

$$P'(t) = \sum_{i=1}^n p_i'(t) = \sum_{i=1}^n \left(f_i^e(t) + \sum_{j \neq i} f_{j,i}(t) \right) = \sum_{i=1}^n f_i^e(t)$$

where the last equality follows from the law of action and reaction. We conclude the *law of Preservation of Momentum*: in the absence of external force, the total momentum of the system is conserved.

2.5.2 Conservation of Angular Momentum

The total angular momentum of the system is

$$\sum_i r_i(t) \times p_i(t)$$

The derivative is

$$\sum_i r_i(t) \times \left(f_i^e(t) + \sum_{j \neq i} f_{j,i}(t) \right)$$

Using $f_{j,i} = -f_{i,j}$ we have (where we omitted the dependence on t for simplicity)

$$\sum_i r_i \times \sum_{j \neq i} f_{j,i} = \frac{1}{2} \sum_{i \neq j} (r_i - r_j) \times f_{j,i}$$

The **Strong law of action and reaction** states that the direction of $f_{j,i}$ is parallel to $r_i - r_j$, hence the above sum is zero, and we obtain that the derivative of angular momentum becomes

$$\sum_i r_i(t) \times f_i^e(t)$$

We conclude the **Law of preservation of Angular Momentum**: in the absence of external torque, the total angular momentum of the system is conserved.

2.5.3 Conservation of Mechanical Energy

The total work done by all forces at time interval $[t_1, t_2]$ is

$$W = \sum_{i=1}^n \int_{t_1}^{t_2} f_i(t) \cdot v_i(t) dt$$

As in the case of a single particle, it can be shown that this equals to diff of *total kinetic energy*, defined as

$$T(v(t)) = \sum_{i=1}^n T(v_i(t)) = \sum_{i=1}^n \frac{1}{2} m \|v_i(t)\|^2$$

We usually assume that all the forces are *conservative*, i.e.

$$f_i^e(t) = f_i^e(r_i(t)) = -\nabla\Phi_i^e(r_i(t))$$

and

$$f_{j,i}(t) = -\nabla_1\Phi_{i,j}(r_i(t), r_j(t)) \quad , \quad f_{i,j}(t) = -\nabla_2\Phi_{i,j}(r_i(t), r_j(t))$$

where ∇_1 is gradient w.r.t. the first 3-dim vector and ∇_2 is gradient w.r.t. the second 3-dim vector. By the law of action and reaction, we get that

$$\nabla_1\Phi_{i,j}(r_i(t), r_j(t)) = -\nabla_2\Phi_{i,j}(r_i(t), r_j(t))$$

By the chain rule, this implies that the potential depends on the difference between r_i and r_j , that is,

$$\Phi_{i,j}(r_i(t), r_j(t)) = \Phi_{i,j}(r_i(t) - r_j(t))$$

In fact, by the strong law of action and reaction it is not hard to see that $\Phi_{i,j}$ only depends on the distance between r_i and r_j , that is,

$$\Phi_{i,j}(r_i(t), r_j(t)) = \Phi_{i,j}(\|r_i(t) - r_j(t)\|)$$

Anyway, back to energy calculations, it is not too hard to show that

$$W = -\Phi(r(t_2)) + \Phi(r(t_1))$$

where

$$\Phi(r(t)) = \sum_{i=1}^n \Phi_i^e(r_i(t)) + \sum_{i \neq j} \Phi_{i,j}(\|r_i(t) - r_j(t)\|)$$

is called the *total potential energy*. Comparing the two expressions for the Work, we conclude the law of conservation of total mechanical energy, defined as,

$$E(r, v) = \Phi(r) + T(v)$$

Chapter 3

The Lagrangian and Euler-Lagrange Equations

3.1 Adding Constraints

The presentation we've layout so far is seemingly sufficient to describe systems of particles. There are two missing things:

- Friction: Friction is a force that converts mechanical energy into thermal energy. Thermal energy is a term describing the microscopic motion of the molecules that form a physical body. Conceptually, if we included in our model every single molecule (ignoring the laws of quantum mechanics), there would be no such thing as thermal energy. Friction is outside the scope of this write-up.
- Constraints: Often, external forces constrain the movement of particles in some manner. For example, consider a train moving on some tracks. The structure of the tracks constrain the train's movement to a particular curve in the surface. Another example is a simple pendulum. Yet another example is the movement of a rigid body, where the distance between different particles must remain constant. While we could work with the original Euclidean coordinates and explicitly model the external forces that impose the constraints, it is highly non-convenient. This section shows how to impose constraints in a more convenient manner.

Holonomic Constraints: A holonomic constraint is any constraint that can be written as

$$h(r, t) = 0$$

where $h : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$. The function h constrain the position $r(t)$ at time t to be on the set $\{r : h(r, t) = 0\}$. Some examples:

- A simple pendulum: A bead is attached to a rod. At any time t , the position must be on a circle whose center is at the pivot and whose radius is the length of the rod. So, $h(r, t) = \|r - c\|^2 - \rho$ for some fixed $c \in \mathbb{R}^3$ and $\rho \in \mathbb{R}$.
- A rigid body: for any two particles we have $h_{i,j}(r, t) = \|r_i - r_j\| - d_{i,j}$

3.2 Generalized Coordinates

A system of n particles in \mathbb{R}^3 is described by $3n$ functions of the time (the $x(t), y(t), z(t)$ of the n particles). With k holonomic constraints of the form $h_i(r, t) = 0$, the position at time t is restricted to a $(3n - k)$ -dimensional Manifold, $\mathcal{M}(t)$. We can therefore re-parameterize $r(t)$ by a vector $q(t) \in U(t) \subseteq \mathbb{R}^{3n-k}$ such that at time t , there exists a

one-to-one mapping $\varphi(t) : U(t) \rightarrow \mathcal{M}(t)$. Mathematicians call the pair (U, φ) *local coordinates* for the manifold \mathcal{M} . Physicists call the coordinates q *generalized coordinates* (because, unlike regular coordinates which has units of length, generalized coordinates may have other units).

As an example, consider two particles that are connected by a rod of length ℓ . The regular representation is by $(r_1, r_2) \in \mathbb{R}^6$, but if we take into account the constraint, then we can represent the positions as $r_1 \in \mathbb{R}^3$ and $(\alpha, \beta) \in [-\pi, \pi]^2$ representing the polar angles of r_2 w.r.t. a center of circle whose center is r_1 . The generalized coordinates in this case are the vector $q \in \mathbb{R}^3 \times [-\pi, \pi]^2$.

The goal is to come up with a new formulation of mechanics that will not require the knowledge of forces whose sole goal is to keep the constraints, but instead, we will inject the constraint in another way and focus on the other forces.

3.3 d'Alembert's Principle of Virtual Work leads to Euler-Lagrange Equations

The positions at time t of the n particles is denoted by $r(t) \in \mathbb{R}^{3n}$ and in generalized coordinates we define it by $q(t) \in \mathbb{R}^d$ for $d < 3n$. The relation between the regular coordinates and generalized coordinates are by

$$r(t) = \varphi(q(t), t) \quad (3.1)$$

It is convenient to define $\hat{q}(t) = (q(t), t) \in \mathbb{R}^{d+1}$. Hence,

$$r(t) = \varphi(\hat{q}(t))$$

We denote by $J^\varphi(\hat{q}(t)) \in \mathbb{R}^{3n, d+1}$ the Jacobian matrix of φ at $\hat{q}(t)$, that is, the i, j element of this matrix is the partial derivative of φ_i w.r.t. the j 'th element of \hat{q} , evaluated at $\hat{q}(t)$. We denote by $J_j^\varphi(\hat{q}(t))$ the j 'th column of the matrix and use $J_{:-1}^\varphi(\hat{q}(t))$ to denote the matrix obtained by removing the last column of $J^\varphi(\hat{q}(t))$.

d'Alembert's Principle of Virtual Work states that at any time t , the velocity vector is constrained to be in a d -dimensional hyper-plane. The "forces of constrains", according to d'Alembert's principle, are normal to this hyper-plane. From mathematician's perspective, the first part of d'Alembert's principle is a physicist's interpretation of the geometric statement that the first d columns of $J^\varphi(\hat{q}(t))$ are *tangent* to the constrained manifold, $\mathcal{M}(t)$.

The second part of d'Alembert's principle is that the force, $f(r(t), t)$, can be decomposed into two orthogonal forces, one is on the d -dimensional subspace spanned by the columns of $J_{:-1}^\varphi(\hat{q}(t))$, and the other one is the orthogonal complement. The first vector is denoted $f^a(r(t), t)$, and summarizes the "applied" forces, and the second vector is denoted $f^c(r(t), t)$, and summarizes the "forces of constraints". So, $f(r(t), t) = f^a(r(t), t) + f^c(r(t), t)$.

We can write Newton's second law as

$$M r''(t) = f^a(r(t), t) + f^c(r(t), t), \quad \text{where } M = \text{diag}(m_1, m_1, m_1, m_2, m_2, m_2, \dots, m_n, m_n, m_n) \quad (3.2)$$

Multiply both sides from left by $J_{:-1}^\varphi(\hat{q}(t))^\top$, and Using d'Alembert's principle, we obtain

$$J_{:-1}^\varphi(\hat{q}(t))^\top M r''(t) = J_{:-1}^\varphi(\hat{q}(t))^\top f^a(r(t), t) \quad (3.3)$$

where we have omitted $f^c(r(t), t)$ because it is orthogonal to every column of $J_{:-1}^\varphi(\hat{q}(t))$. This gives a set of d equations.

Deriving Euler-Lagrange Equations Our next step is to take the d equations given by (3.3) and rewrite them using q alone (without r). This will give us the analog of Newton laws but with the generalized coordinates.

Comment about notation: In the following, when a function's input is divided into several variable names, e.g. $g(a, b, c)$ for vectors a, b, c of arbitrary dimension, when we write $\nabla_b g(\alpha, \beta, \gamma)$ we refer to the vector containing the partial derivatives of $g(a, b, c)$ w.r.t. the elements of the vector b , evaluated at (α, β, γ) .

We assume that f^a takes the form $f^a(r, t) = -\nabla_r F^a(r, t)$ for some $F^a : \mathbb{R}^{3n} \times \mathbb{R} \rightarrow \mathbb{R}$. Denote,

$$\Phi(q, t) = F^a(\varphi(q, t), t) \quad (3.4)$$

Then, using the chain rule,

$$-\nabla_q \Phi(q, t) = -J_{:-1}^\varphi(q, t)^\top \nabla_r F^a(\varphi(q, t), t) = J_{:-1}^\varphi(q, t)^\top f^a(\varphi(q, t), t)$$

So, the right-hand side of the Newton equation becomes

$$-\nabla_q \Phi(q(t), t) \quad (3.5)$$

Next, introduce the functions $\nu : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^{3n}$ and $T : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\nu(q, \xi, t) = J_{:-1}^\varphi(q, t)^\top \xi + J_{:,d+1}^\varphi(q, t) \quad (3.6)$$

and

$$T(q, \xi, t) = \frac{1}{2} \nu(q, \xi, t)^\top M \nu(q, \xi, t) \quad (3.7)$$

Observe that

$$v(t) = \frac{d}{dt} \varphi(q(t), t) = J_{:-1}^\varphi(q(t), t)^\top q'(t) + J_{:,d+1}^\varphi(q(t), t) = \nu(q(t), q'(t), t) \quad (3.8)$$

and from this we also get that $T(q(t), q'(t), t)$ is the kinetic energy of the system.

Lemma 3.3.1 *The left-hand side of (3.3) equals to*

$$\frac{d}{dt} \nabla_\xi T(q(t), q'(t), t) - \nabla_q T(q(t), q'(t), t)$$

Proof To see this, first observe that using the chain rule,

$$\nabla_q T(q, \xi, t) = \nu(q, \xi, t)^\top M \nabla_q \nu(q, \xi, t)$$

Second, observe that

$$\nabla_\xi \nu(q, \xi, t) = J_{:-1}^\varphi(q, t)^\top \in \mathbb{R}^{3n, d}$$

and so

$$\nabla_\xi T(q, \xi, t) = \nu(q, \xi, t)^\top M \nabla_\xi \nu(q, \xi, t) = \nu(q, \xi, t)^\top M J_{:-1}^\varphi(q, t)^\top$$

Third,

$$\begin{aligned} \frac{d}{dt} \nabla_\xi T(q(t), q'(t), t) &= \left(\frac{d}{dt} \nu(q(t), q'(t), t)^\top \right) M J_{:-1}^\varphi(q, t)^\top + \nu(q(t), q'(t), t)^\top M \left(\frac{d}{dt} J_{:-1}^\varphi(q(t), t) \right)^\top \\ &= r''(t)^\top M J_{:-1}^\varphi(\hat{q}(t))^\top + \nu(q(t), q'(t), t)^\top M \left(\frac{d}{dt} J_{:-1}^\varphi(q(t), t) \right)^\top \end{aligned}$$

Observing that the (i, j) element of $J_{:-1}^\varphi(q(t), t)$ is $\frac{\partial \varphi_i(q(t), t)}{\partial q_j}$, we get that the (i, j) element of $\frac{d}{dt} J_{:-1}^\varphi(q(t), t)$ is

$$\frac{d}{dt} \frac{\partial}{\partial q_j} \varphi_i(q(t), t) = \frac{\partial}{\partial q_j} \frac{d}{dt} \varphi_i(q(t), t) = \frac{\partial}{\partial q_j} \frac{d}{dt} r_i(t) = \frac{\partial}{\partial q_j} v_i(t) = \nabla_{q_j} \nu_i(q(t), q'(t), t)$$

so we get that

$$\frac{d}{dt} \nabla_\xi T(q(t), q'(t), t) = r''(t)^\top M J_{:-1}^\varphi(\hat{q}(t))^\top + \nabla_q T(q(t), q'(t), t)$$

Rearranging the above yields

$$r''(t)^\top M J_{:-1}^\varphi(\hat{q}(t))^\top = \frac{d}{dt} \nabla_\xi T(q(t), q'(t), t) - \nabla_q T(q(t), q'(t), t)$$

■

Based on (3.5) and the above lemma, (3.3) can be rewritten as

$$\frac{d}{dt} \nabla_{\xi} T(q(t), q'(t), t) - \nabla_q T(q(t), q'(t), t) = -\nabla_q \Phi(\varphi(\hat{q}(t)), t) \quad (3.9)$$

Finally, define the function

$$L(q, \xi, t) = T(q, \xi, t) - \Phi(q, t) \quad (3.10)$$

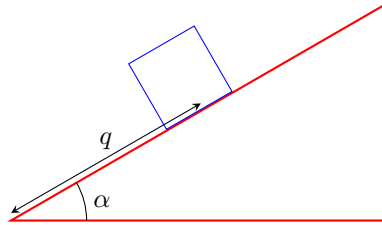
we can rewrite (3.9) as

$$\frac{d}{dt} (\nabla_{\xi} L(q, q'(t), t)) - \nabla_q L(q, q'(t), t) = 0 \quad (3.11)$$

The function L is called the *Lagrangian*, and (3.11) is called the Lagrange-Euler equations.

Corollary 3.3.1 Consider a system with n objects where the relation between general coordinates and real coordinates is given by $r(t) = \varphi(q(t), t)$, where $\varphi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^{3n}$. Assume that all the applied forces in the system only depend on the position vector, and let $\Phi(q, t)$ be defined as in (3.4). Let the functions v, T be defined as in (3.6), (3.7), respectively. The Lagrangian function is defined in (3.10), and the d Euler-Lagrange equations given in (3.11) must be satisfied.

Example 3.3.1 Cart on a Slope: Consider a mass m cart on a friction-less slope of a constant angle α .



The position of the car is fully determined by the distance of the cart from the bottom of the slope, denoted $q(t)$. That is, $\varphi(q) = q (\cos(\alpha), 0, \sin(\alpha))$. The force acting on the cart is $(0, 0, -mg)$, which equals to $-\nabla_r F(r)$ for $F(r) = mgz$. Hence, the potential energy is

$$\Phi(q(t), t) = mg \sin(\alpha) q(t)$$

So the Lagrangian becomes

$$L(q, \xi, t) = \frac{m}{2} (\xi)^2 - mg \sin(\alpha) q$$

The Euler-Lagrange equation is therefore

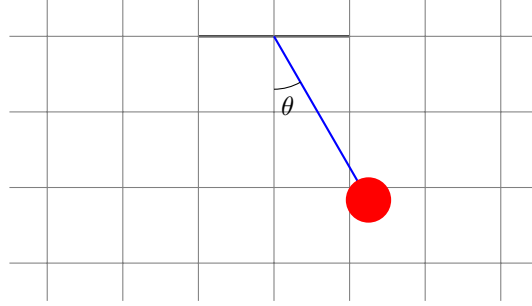
$$\frac{d}{dt} (mq'(t)) + mg \sin(\alpha) = 0$$

or

$$q''(t) = -g \sin(\alpha)$$

That is, the acceleration is multiplied by the sine of the slope. We could arrive to the same equation by decomposing the gravitation force to a component perpendicular to the slope (which is cancelled out due to the law of action and a reaction) and a component tangent to the slope (which is $-mg \sin(\alpha)$).

Example 3.3.2 Simple Pendulum: A mass m is attached to a rigid rod of length ℓ , whose upper side is stick to a fixed position. The position of the mass is therefore fully described by a single generalized coordinate — the angle the rod makes with the vertical axis.



The φ function is:

$$\varphi(\theta) = (\ell \sin(\theta), 0, -\ell \cos(\theta))$$

The velocity vector is

$$v(t) = v(\theta(t)) = (\ell \cos(\theta(t)) \theta'(t), 0, \ell \sin(\theta(t)) \theta'(t))$$

and the kinetic energy becomes

$$T = \frac{m \ell^2}{2} (\theta'(t))^2$$

The force acting on the mass is $(0, 0, -mg)$, which equals to $-\nabla_r F(r)$ for $F(r) = mgz$. Hence, the potential energy is

$$\Phi(\theta(t), t) = mg \cdot (-\ell \cos(\theta(t))) = -mg \ell \cos(\theta(t))$$

So the Lagrangian becomes

$$L(q, \xi, t) = \frac{m \ell^2}{2} \xi^2 + mg \ell \cos(q)$$

The Euler-Lagrange equation is therefore

$$\frac{d}{dt}(m \ell^2 \theta'(t)) - mg \ell \sin(\theta(t)) = 0$$

or

$$\theta''(t) = \frac{g}{\ell} \sin(\theta(t))$$

3.4 Deriving Euler-Lagrange Equations from Hamilton's Principle of Least Action via Variational Calculus

3.4.1 Hamilton's Principle of Least Action

Let $q : \mathbb{R} \rightarrow \mathbb{R}^d$ be some function that describes the value of the generalized coordinates as a function of time. Let $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be some arbitrary function. The *action* of q w.r.t. f is defined as,

$$I[q] = \int_{t=t_1}^{t_2} f(t, q(t), q'(t)) dt$$

Note that I is a functional: it is a function over a space of functions.

Theorem 3.4.1 *The function q that minimizes the action functional $I[q]$ satisfies the Euler-Lagrange equations:*

$$\frac{d}{dt} \nabla_{(d+1):2d} f(q(t), q'(t), t) - \nabla_{1:d} f(q(t), q'(t), t) = 0$$

where $\nabla_{i:j} f$ is the vector of partial derivatives w.r.t. elements $i, (i+1), \dots, j$ of the input to f .

Note that picking f to be the Lagrangian function, the above yields the Euler-Lagrange equations. That is, among all possible q functions, the "real one" is the one that minimizes the action w.r.t. the Lagrangian function.

3.4.2 Proof of Theorem 3.4.1 by Variational Calculus

Variational Calculus

Variational Calculus deals with necessary conditions on a function to be a minimizer of a functional. In a sense, we are going to generalize the notion of "derivative" from functions to functionals, as well as to show that this derivative vanishes at a minimal point.

Definition 3.4.1 Let \mathcal{H} be some set of functions. The Gâteaux Differential of a functional $I : \mathcal{H} \rightarrow \mathbb{R}$ at a function $q \in \mathcal{H}$ is the (continuous and linear) functional $\delta I(q)$ s.t.

$$\forall g \in \mathcal{H}, \quad \delta I(q)[g] := \lim_{\epsilon \rightarrow 0} \frac{I[q + \epsilon g] - I[q]}{\epsilon}$$

To get some intuition on the definition, suppose \mathcal{H} is a set of functions from \mathbb{R}^k to \mathbb{R}^d , and let us approximate this set by the set of functions $\hat{\mathcal{H}} = \{\sum_{i=1}^n \alpha_i g_i : \alpha \in \mathbb{R}^n\}$, where g_1, \dots, g_n are n functions, each of which from \mathbb{R}^k to \mathbb{R}^d . Since every $g \in \hat{\mathcal{H}}$ emerged from $\alpha \in \mathbb{R}^n$, we can define $\hat{I} : \mathbb{R}^n \rightarrow \mathbb{R}$ s.t. $\hat{I}(\alpha) = I[g(\alpha)]$, where $g(\alpha) = \sum_{i=1}^n \alpha_i g_i$. The gradient of \hat{I} at $q \in \hat{\mathcal{H}}$ is an n -dimensional vector whose i 'th element is

$$\lim_{\epsilon \rightarrow 0} \frac{I[q + \epsilon g_i] - I[q]}{\epsilon}$$

In other words,

$$\forall i \in [n], \quad \nabla \hat{I}(q)[i] = \lim_{\epsilon \rightarrow 0} \frac{I[q + \epsilon g_i] - I[q]}{\epsilon}$$

We immediately see the similarity to the Gâteaux differential.

Lemma 3.4.1 Let $I : \mathcal{H} \rightarrow \mathbb{R}$ be a functional and assume that it is Gâteaux differentiable at some q . Assume that q is a local minimum of I . Then, for every $h \in \mathcal{H}$ we have that $\delta I(q)[h] = 0$.

Proof Assume the contrary, that is, assume that q is a local minimum, but there is some $h \in \mathcal{H}$ s.t. $\delta I(q)[h] \neq 0$. That is, there is some $\beta \neq 0$ s.t.

$$\lim_{\epsilon \rightarrow 0} \frac{I[q + \epsilon g] - I[q]}{\epsilon} = \beta$$

Suppose first that $\beta < 0$. Then, there should be $\epsilon < 0$ such that

$$\left| \frac{I[q + \epsilon g] - I[q]}{\epsilon} - \beta \right| \leq -\beta \Rightarrow \beta - \frac{I[q + \epsilon g] - I[q]}{\epsilon} \leq -\beta \Rightarrow I[q + \epsilon g] - I[q] \leq 2\epsilon\beta < 0$$

We got a contradiction to $I[q]$ being a minimum. A similar argument holds for the case $\beta > 0$. ■

Proof of Theorem 3.4.1

We are now ready to prove Theorem 3.4.1. The Gâteaux differential of the action functional is

$$\begin{aligned} \delta I(q)[g] &= \lim_{\epsilon \rightarrow 0} \frac{I[q + \epsilon g] - I[q]}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\int_{t=t_1}^{t_2} f(t, q(t) + \epsilon g(t), q'(t) + \epsilon g'(t)) dt - \int_{t=t_1}^{t_2} f(t, q(t), q'(t)) dt}{\epsilon} \\ &= \int_{t=t_1}^{t_2} \lim_{\epsilon \rightarrow 0} \frac{f(t, q(t) + \epsilon g(t), q'(t) + \epsilon g'(t)) - f(t, q(t), q'(t))}{\epsilon} dt \\ &= \int_{t=t_1}^{t_2} [\nabla_{1:d} f(q(t), q'(t), t) \cdot g(t) + \nabla_{d+2:d} f(q(t), q'(t), t) \cdot g'(t)] dt \end{aligned}$$

Since Theorem 3.4.1 assumes that q minimizes the action functional we must have that $\delta I(q)[g] = 0$ for every g . In particular, it equals 0 for every g for which $g(t_1) = g(t_2) = \mathbf{0} \in \mathbb{R}^d$. Therefore, using integration by parts on the second term above, we have

$$\begin{aligned} & \int_{t=t_1}^{t_2} \nabla_{d:2d} f(q(t), q'(t), t) \cdot g'(t) dt \\ &= [\nabla_{d:2d} f(q(t), q'(t), t) \cdot g(t)]_{t_1}^{t_2} - \int_{t=t_1}^{t_2} \left(\frac{d}{dt} \nabla_{d:2d} f(q(t), q'(t), t) \right) \cdot g(t) dt \\ &= - \int_{t=t_1}^{t_2} \left(\frac{d}{dt} \nabla_{d:2d} f(q(t), q'(t), t) \right) \cdot g(t) dt \end{aligned}$$

Combining all the above we obtain that

$$\begin{aligned} 0 &= \delta I(q)[g] \\ &= \int_{t=t_1}^{t_2} \left[\nabla_{1:d} f(q(t), q'(t), t) \cdot g(t) - \left(\frac{d}{dt} \nabla_{d:2d} f(q(t), q'(t), t) \right) \cdot g(t) \right] dt \\ &= \int_{t=t_1}^{t_2} \left[\nabla_{1:d} f(q(t), q'(t), t) - \frac{d}{dt} \nabla_{d:2d} f(q(t), q'(t), t) \right] \cdot g(t) dt \end{aligned}$$

Since this should hold for every g that satisfies the boundary condition, we need that for every t the vector inside the parentheses must equal to the zero vector. This yields the Euler-Lagrange equations and concludes the proof of Theorem 3.4.1.

Chapter 4

Hamilton's Mechanics

4.1 Generalized Momentum and Phase Space

Recall that the generalized coordinates of a system at time t is the vector $q(t) \in \mathbb{R}^d$ such that $r(t) = \varphi(q(t), t)$. We have defined the Lagrangian function as a function $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$, where we use the notation $L(q, \xi, t)$ to denote the inputs of L .

Definition 4.1.1 *The generalized momentum is the vector function $p : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ defined as $p(q, z, t) = \nabla_{\xi} L(q, \xi, t)$.*

The following lemma shows that the generalized momentum is indeed a generalization of the vanilla definition of momentum.

Lemma 4.1.1 *Suppose that $\varphi(q, t) = q$, that is $q(t) = r(t)$ for every t . Then, $p(q(t), q'(t), t) = M v(t)$.*

Proof Since $L(q, \xi, t) = T(q, \xi, t) - \Phi(q, t)$, we have that $\nabla_{\xi} L(q, \xi, t) = \nabla_{\xi} T(q, \xi, t)$. In our particular case, $T(q, \xi, t) = \frac{1}{2} \xi^{\top} M \xi$ hence $\nabla_{\xi} T(q, \xi, t) = M \xi$. The claim follows by noting that $q'(t) = v(t)$. ■

In the Hamiltonian formulation of the laws of mechanics, we describe the state of the system with the pair of variables (q, p) . This pair is called the *phase space* of the system. Our goal is therefore to rewrite the equations of motion as equations of the functions $q(t), p(t)$. To do so, we will rely on the Fenchel-Legendre Transform, which is the topic of the next section.

4.2 Fenchel-Legendre Transform

Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, its Fenchel-Legendre conjugate is the function $f^* : \mathbb{R}^d \rightarrow \mathbb{R}$ defined as

$$f^*(m) = \max_x x^{\top} m - f(x)$$

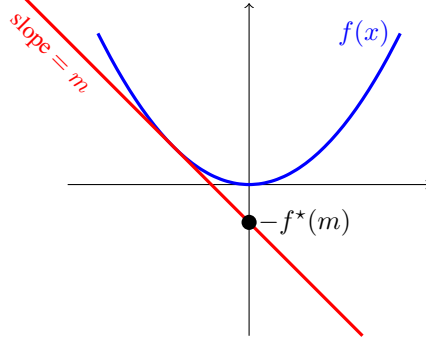
Obviously, for every x , $f^*(m) \geq x^{\top} m - f(x)$, which yields Fenchel-Young inequality,

$$x^{\top} m \leq f(x) + f^*(m)$$

The function f^* is a convex function (since it is a maximum of affine functions).

If f is differentiable, then the maximizer x in the definition of f^* must satisfy $m - \nabla f(x) = 0$ or $m = \nabla f(x)$. If the gradient is invertible, this can be rewritten as $x = (\nabla f)^{-1}(m)$. Plugging this value in the objective in the definition of $f^*(m)$ yields $f^*(m) = (\nabla f)^{-1}(m)^{\top} m - f((\nabla f)^{-1}(m))$.

If f is one dimensional, there is a simple geometrical interpretation to the conjugate function: it maps a slope of a tangent of f to the point it intersects the y -axis. An illustration is given below.



4.3 Hamiltonian and Hamilton's Equations

The *Hamiltonian* is the Fenchel-Legendre conjugate of the Lagrangian with respect to the variable ξ , that is,

$$H(q, p, t) = \max_{\xi} \xi^{\top} p - L(q, \xi, t) \quad (4.1)$$

Denote $\xi(q, p, t) = \operatorname{argmax}_{\xi} \xi^{\top} p - L(q, \xi, t)$ and note that

$$p = \nabla_{\xi} L(q, \xi(q, p, t), t) \quad (4.2)$$

or in other words, $\xi(q, p, t)$ is the inverse relation to $p = \nabla_{\xi} L(q, \xi, t)$. We can therefore rewrite the Hamiltonian as

$$H(q, p, t) = p^{\top} \xi(q, p, t) - L(q, \xi(q, p, t), t) \quad (4.3)$$

Observe that the Hamiltonian is a function of the phase space of the system. Next, the following theorem shows an analogue of the Euler-Lagrange equations.

Theorem 4.3.1 (Hamilton's Equations) *The dynamics of the system should satisfy the following equations:*

$$\begin{aligned} q'(t) &= \nabla_p H(q(t), p(t), t) \\ p'(t) &= -\nabla_q H(q(t), p(t), t) \end{aligned}$$

Proof Using (4.3) we have,

$$\nabla_p H(q, p, t) = \xi(q, p, t) + p^{\top} \nabla_p \xi(q, p, t) - \nabla_{\xi} L(q, \xi(q, p, t), t)^{\top} \nabla_p \xi(q, p, t)$$

Using (4.2), the second and third terms cancel, which gives $\nabla_p H(q, p, t) = \xi(q, p, t)$. But since $\xi(q, p, t)$ is the inverse relation to $p = \nabla_{\xi} L(q, \xi, t)$, by plugging the arguments $q = q(t), p = p(t) = \nabla_{\xi} L(q(t), \xi(q(t), p(t), t), t)$, we get that $\xi(q(t), p(t), t) = q'(t)$, which proves the first equation in the theorem. For the second equation, observe that

$$\nabla_q H(q, p, t) = p^{\top} \nabla_q \xi(q, p, t) - \nabla_q L(q, \xi(q, p, t), t) - \nabla_{\xi} L(q, \xi(q, p, t), t)^{\top} \nabla_q \xi(q, p, t)$$

Again, using (4.2), the first and third terms cancel, and we get $\nabla_q H(q, p, t) = -\nabla_q L(q, \xi(q, p, t), t)$, which again gives $\nabla_q H(q(t), p(t), t) = -\nabla_q L(q(t), q'(t), t)$. Using Euler-Lagrange equations and the definition of the generalized momentum, we have

$$\nabla_q L(q(t), q'(t), t) = \frac{d}{dt} (\nabla_{\xi} L(q(t), q'(t), t)) = \frac{d}{dt} p(t) = p'(t)$$

and therefore

$$-\nabla_q H(q(t), p(t), t) = p'(t)$$

which concludes our proof. ■

4.4 Conservation Quantities

A quantity is called *conserved* if it does not depend on the time t . We have seen the laws of conservation of momentum and energy. We now show a more general way to derive conservation theorems. We start with a law of preservation of generalized momentum.

Theorem 4.4.1 (Preservation of Generalized Momentum) *Assume that $L(q(t), q'(t), t)$ does not depend on its j 'th input (that is, $L(q(t), q'(t), t) = L(q(t) + \alpha e_j, q'(t), t)$ for every α). Then, the j 'th element of the generalized momentum, $p_j(t) = \nabla_{d+j} L(q(t), q'(t), t)$, is conserved.*

Proof By the j 'th Euler-Lagrange equation

$$\frac{d}{dt} \nabla_{d+j} L(q(t), q'(t), t) = \nabla_j L(q(t), q'(t), t)$$

The left-hand side of the above is $p_j'(t)$ and the right-hand side of the above is obviously 0. ■

The above theorem generalizes the law of preservation of momentum. To see this, recall that the momentum is $p(t) = M v(t)$, where M is the mass matrix as in (3.2). Consider the Lagrangian w.r.t. the original coordinates, that is, $q(t) = r(t)$. Suppose also that there are no applied forces, hence, $L(q(t), q'(t), t) = T(q(t), q'(t), t)$, and in our formulation, $T(q(t), q'(t), t) = \frac{1}{2} q'(t)^\top M q'(t) = \frac{1}{2} v(t)^\top M v(t)$. In this case, the Lagrangian does not depend on $q(t)$ and t , but only on $q'(t)$. Hence, the momentum is preserved.

A generalized coordinate q_j that satisfies the conditions of Theorem 4.4.1 is called a *cyclic* coordinate. From the definition of the Hamiltonian, it is easy to see that if q_j is a cyclic coordinate then H does not depend on q_j . In this case, the number of Hamilton's equations is reduced by 1.

Noether's theorem states that any differentiable symmetry of the action of a physical system has a corresponding conservation law.

Appendix A

Appendix

A.1 Derivatives and Their Notation in this Note

A.1.1 Derivative, partial derivative, gradient, and total derivative

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the derivative at point x is denoted

$$\frac{df}{dx}(x) := \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon) - f(x)}{\epsilon}$$

For a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, the partial derivative w.r.t. the i 'th input at vector x is denoted

$$\frac{\partial f}{\partial x_i}(x) := \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon e_i) - f(x)}{\epsilon}$$

where e_i is the all zeros vector except 1 at the i 'th coordinate. We also use the *gradient* notation

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right)$$

and

$$\nabla_i f(x) := \frac{\partial f}{\partial x_i}(x)$$

It is sometimes convenient to write functions as $f(x, y)$ where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$. That is, $f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}$. In such cases, we also write

$$\nabla_y f(x, y) := \left(\frac{\partial f}{\partial y_1}(x, y), \dots, \frac{\partial f}{\partial y_n}(x, y) \right)$$

and similarly for $\nabla_x f(x, y)$.

For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we can refer to it as m functions, f_1, \dots, f_m , each one from \mathbb{R}^n to \mathbb{R} , and calculate the gradient for each one, which yields a *Jacobian* matrix $\nabla f(x)$ whose i 'th row is $\nabla f_i(x)$.

The *total derivative* of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ at $x \in \mathbb{R}^n$ is defined as the function $df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that satisfies

$$\lim_{\epsilon \rightarrow 0} \frac{\|f(x + \epsilon) - (f(x) + df_x(\epsilon))\|}{\|\epsilon\|} = 0$$

Note that in the above, ϵ is an n -dimensional vector. If a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has partial derivatives, then it is possible to show that we can take $df_x(\epsilon) = \nabla f(x) \cdot \epsilon$. That is, the total derivative function is a linear function defined by the gradient of f .

A.1.2 Chain Rule

The chain rule has a particularly elegant statement for the total derivative:

$$d(g \circ f)_a = dg_{f(a)} \cdot df_a$$

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \rightarrow \mathbb{R}$ then

$$\nabla(g \circ f)(a) = \nabla g(f(a)) \cdot \nabla f(a)$$

A.1.3 Derivative with Direct Dependencies

Suppose that $h(x) = f(g(x), y(x))$. Then, we can write it as $h(x) = f(\gamma(x))$ where $\gamma(x) = (g(x), y(x))$. Therefore,

$$\begin{aligned} d(f \circ \gamma)_{x_0} &= df_{\gamma(x_0)} \cdot d\gamma_{x_0} \\ &= df_{g(x_0), y(x_0)} \cdot d\gamma_{x_0} \\ &= \nabla f(g(x_0), y(x_0)) \cdot \nabla \gamma(x_0) \\ &= \nabla f_{y(x_0)}(g(x_0)) \cdot \nabla g(x) + \nabla f_{g(x_0)}(y(x_0)) \cdot \nabla y(x) \end{aligned}$$

where $f_b(a) = f(a, b)$ and $f_a(b) = f(a, b)$. In particular, in the special case in which $g(x) = x$ and $x \in \mathbb{R}$, we obtain that the derivative is

$$\frac{dh}{dx}(x_0) = \nabla_a f(x_0, y(x_0)) + \nabla_b f(x_0, y(x_0)) \cdot \nabla y(x)$$

Here, $\nabla_a f$ is the partial derivative of f w.r.t. its first argument and $\nabla_b f$ is the vector of partial derivatives of f w.r.t. its second argument.