Introduction to Machine Learning (67577)
Reinforcement Learning

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Reinforcement Learning

Outline

1. Reinforcement Learning

2. Multi-Armed Bandit
   - ϵ-greedy exploration
   - EXP3
   - UCB

3. Markov Decision Process (MDP)
   - Value Iteration
   - Q-Learning
   - Deep-Q-Learning
   - Temporal Abstraction

Reinforcement Learning

Goal: Learn a policy, mapping from state space, S, to action space, A

Learning Process:

For $t = 1, 2, \ldots$

- Agent observes state $s_t \in S$
- Agent decides on action $a_t \in A$ based on the current policy
- Environment provides reward $r_t \in \mathbb{R}$
- Environment moves the agent to next state $s_{t+1} \in S$

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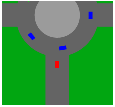
Many applications, e.g.: Robotics, Playing games, Finance, Inventory management, ...

Examples

Merge into traffic:

- Goal: Adjust the speed of the car according to traffic
- State is positions and velocities of the car and the preceding car
- Action is acceleration/braking command
- Reward is composed of avoiding accidents, smooth driving, and making progress



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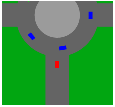

Playing Atari Game:

- https://www.youtube.com/watch?v=V1eYniJ0Rnk

Average Reward and Discounted Reward

Average Reward: Given time horizon T, the average reward of following a policy π is

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Discounted Reward: Given $\gamma \in (0,1)$, the discounted reward of following a policy π is

$$R_\gamma(\pi) \; = \; \mathbb{E} \sum_{t=1}^{\infty} \gamma^t \, r_t$$

Reinforcement Learning vs. Supervised Learning

SL is a special case of RL in which s_t is the "instance", a_t is the predicted label, $-r_t$ is the loss measuring the discrepancy between a_t and the "true" label, y_t, and s_{t+1} is chosen independent of s_t and a_t.

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Differences:

- In SL, actions do not effect the environment, therefore we can collect training examples in advance, and only then search for a policy
- In SL, the effect of actions is local, while in RL, actions have long-term effect
- In SL we are given the correct answer, while in RL we only observe a reward

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- Regret:

$$\mu^* - \mathbb{E}\, R_T(\pi)$$

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How to pick the next action?

- Exploitation: Choose the most promising action based on your current understanding
- Exploration: Maybe there is a better arm ?

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  - Regret:

$$\mu^* - \frac{m\bar{\mu} + (T-m)\mu_{\hat{i}}}{T} = (\mu^* - \mu_{\hat{i}}) + \tfrac{m}{T}(\mu_{\hat{i}} - \bar{\mu})$$

$$\le (\mu^* - \hat{\mu}_{i^*} + \hat{\mu}_{i^*} - \hat{\mu}_{\hat{i}} + \hat{\mu}_{\hat{i}} - \mu_{\hat{i}}) + \tfrac{m}{T} \le 2\epsilon + \frac{n \log(n)}{T\,\epsilon^2}$$

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  - For the best ϵ, the regret is order of $\left(\frac{n \log(n)}{T}\right)^{1/3}$

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- Remark: EXP3 works also in the adversarial setting

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- Regret can be shown to be bounded by $\frac{\log(T)}{T} \sum_{i:\Delta_i>0} \frac{1}{\Delta_i}$

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Markov Decision Process (MDP)

The Markovian Assumption:

- For every t, $s_{t+1} \sim \tau(s_t, a_t)$ where τ is a deterministic function over $S \times A$
- For every t, r_t is a random variable over $[0, 1]$ whose distribution depends deterministically only on (s_t, a_t) and we denote its expected value by $\rho(s_t, a_t)$,
- It follows that (s_{t+1}, r_t) is conditionally independent of $(s_{t-1}, a_{t-1}), (s_{t-2}, a_{t-2}), \ldots, (s_1, a_1)$ given (s_t, a_t)

MDP — algorithms

- Value Iteration: Find the optimal policy when τ and ρ are known
- Q-Learning: Find the optimal policy when τ and ρ are not known

The Value Function and the Q-Function

- The optimal value function is $V^* : S \to \mathbb{R}$ s.t. $V^*(s) = \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^t r_t \mid s_1 = s\right]$

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- In particular, the optimal a_t is a deterministic function of s_t

Value Iteration

- Iterative algorithm for finding V^*:
  Start with some arbitrary V_0 and update

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  - The proof follows from Banach's fixed point theorem

Naive Learner

- Step 1: Estimate τ and ρ by applying purely random policy
- Step 2: Apply Value Iteration to learn the optimal policy

Q-Learning

- Bellman's equation for the Q function:

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- Given (s_t, a_t, s_{t+1}, r_t), define

$$\delta_{s_t,a_t}(Q) = Q(s_t,a_t) - \left(r_t + \gamma \max_{a'} Q(s_{t+1},a')\right)$$

Q-Learning

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- The above update aims at converging to Bellman's equation

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- Speed of convergence can be improved if we balance the exploration-exploitation tradeoff (by one of the methods described previously)

The Curse of Dimensionality

- The Q function is a table of size $|S| \times |A|$
- This size grows exponentially with the dimensions of S and A
- The convergence of the "tabular" Q-learning (namely, maintaing Q is a table of size $|S| \times |A|$) becomes very slow
- We describe two approaches to overcome this problem:
  - Function Approximation
  - Temporal Abstractions

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- Since we want to minimize $\frac{1}{2}\delta_{s_t,a_t}(\theta)^2$ we take a gradient step:

$$\theta_{t+1} = \theta_t - \eta_t \delta_{s_t,a_t}(\theta_t)\, \nabla Q_\theta(s_t, a_t)$$

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- Memory replay: After executing a_t and observing r_t, s_{t+1} we store the example (s_t, a_t, r_t, s_{t+1}) in a database. Instead of updating just based on the last example, update based on a mini-batch of random examples from the database

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- Freezing Q: Every C step, freeze the value of Q_θ and denote it by $\hat{Q}$. Then, redefine δ to be

$$\delta_{s_t,a_t}(\theta) = Q_\theta(s_t, a_t) - \left(r_t + \gamma \max_{a'} \hat{Q}(s_{t+1}, a')\right)$$

  This has some stabilization effect on the algorithm

Intuition: Structuring a State Space

- Consider some state space $S \subset \mathbb{R}^d$
- Suppose we partition it to $S = S_1 \mathbin{\dot\cup} S_2 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} S_k$
- Assuming homogenous actions within each S_i, we can apply Q learning while using $[k]$ as a new state space
- One can think of Deep-Q-Learning as automatically finding the partition (the first layers of the network)

Temporal Abstraction

- Decisions are often structured into sub-tasks with a broad range of time scale. E.g.:
  - Task: Call a taxi
  - Step 1: finding my phone
  - Step 2: finding the number
  - Step 3: dialing the first digit
  - ...
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- Options: (Sutton, Precup, Singh)
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    - $\pi : S \to A$ is the policy to apply while within the "option"
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  - That is, we should learn a policy over options, $\mu : S \to O$
  - We can learn μ similarly to how we learn a vanilla policy, and the advantage is that mt may be easier to pick O than picking A

Limitations of MDPs

- The Markovian assumption is mathematically convenient but rarely holds in practice
- POMDP = Partially Observed MDP: There is a hidden Markovian state, but we only observe a view that depends on it
- Another approach is "direct policy search", that do not necessarily rely on the Markovian assumption.

Summary

- Reinforcement Learning is a powerful and useful learning setting, but is much harder than Supervised Learning
- The Exploration-Exploitation Tradeoff
- MDP: Connecting the future rewards to current actions using a Markovian assumption

Appendix

Stationary Distribution of an MDP

- A MDP and a deterministic policy function π induces a Markov chain over S, because $\mathbb{P}[s_{t+1}|s_t, a_t, \ldots, s_1, a_1] = \mathbb{P}[s_{t+1}|s_t]$
- The stationary distribution over S is the probability vector q such that $q_s = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} 1[s_t = s]$
- We have that $q_s = \sum_{s'} q_{s'}\, \mathbb{P}[s|s']$
- We have $R_T(\pi) \to \sum_s q_s \rho_s$ where $\rho_s = (s, \pi(s))$
- Using P to denote the matrix s.t. $P_{s,s'} = \mathbb{P}[s|s']$, we obtain that the average reward is the solution of the following Linear Program (LP):

$$\min_q \langle q, -\rho\rangle \text{ s.t. } q \geq 0, \langle q, 1\rangle = 1, (P - I)q = 0$$

The Dual Problem and the Value Function

- Primal

$$\min_{q \in \mathbb{R}^{|S|}} \langle q, -\rho \rangle \text{ s.t. } q \geq 0, \langle q, 1 \rangle = 1, (P - I)q = 0$$

- Dual: define $A = [(P^\top - I), \mathbf{1}]$

$$\max_{v \in \mathbb{R}^{|S|+1}} \langle v, [0, \ldots, 0, 1] \rangle \text{ s.t. } Av \leq -\rho$$

- Equivalently:

$$\max_{v \in \mathbb{R}^{|S|}, \beta \in \mathbb{R}} \beta \text{ s.t. } \beta \leq -\rho + (I - P^\top)v = v - [\rho + P^\top v]$$

- Equivalently (since at the optimum, $\beta = \min_s [v_s - (\rho_s + (P^\top v)_s)]$)

$$\max_{v \in \mathbb{R}^{|S|}} \min_s [v_s - (\rho_s + (P^\top v)_s)]$$

Solution

- Assumption: rewards are ≥ 0
- Claim: If there's a solution to $(I - P^\top)v = \rho$, then it is an optimal solution for which $\beta = 0$
- Proof: For any v, choose s s.t. v_s is minimal, then $(P^\top v)_s \geq v_s$, because the rows of $P^\top$ are probabilities vector. Since $\rho_s \geq 0$, we have that for this s, $v_s - (\rho_s + (P^\top v)_s) \leq 0$, so $\beta \leq 0$, which concludes our proof.