Introduction to Machine Learning (67577)

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Deep Learning

Outline

Gradient-Based Learning

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- SGD converges for convex problems. It may work for non-convex problems if we initialize "close enough" to a "good minimum"

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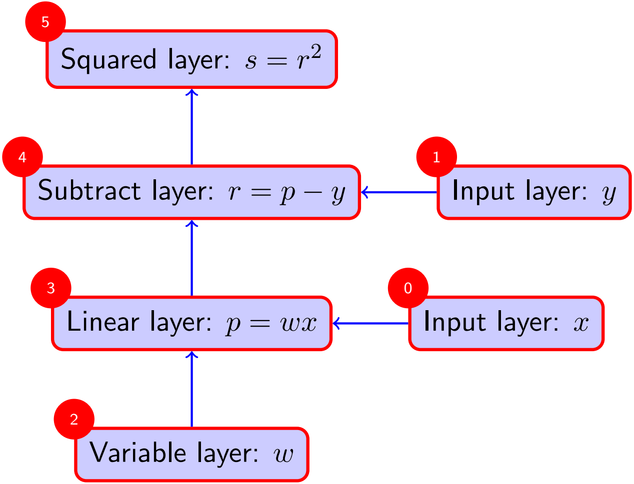
Computation Graph

A computation graph for a one dimensional Least Squares

(numbering of nodes corresponds to topological sort):



Gradient Calculation using the Chain Rule

- Fix x, y and write ℓ as a function of w by

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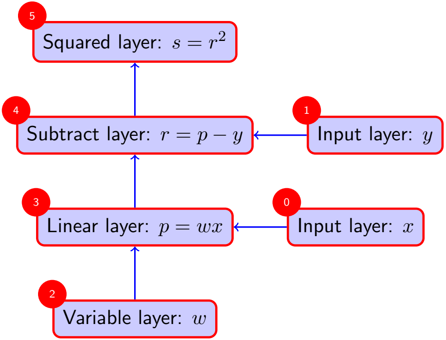
- Backpropagation: Calculate by a Forward-Backward pass over the graph

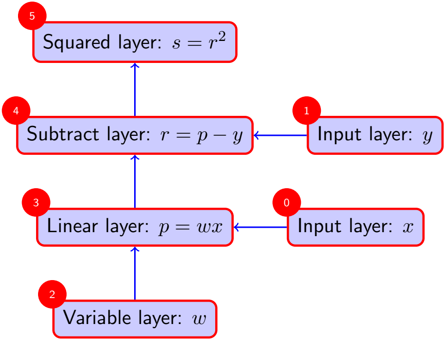
Computation Graph — Forward

- For $t = 0, 1, \ldots, T-1$
  - `Layer[t]->output = Layer[t]->function(Layer[t]->inputs)`



Computation Graph — Backward

- Recall: $\ell'(w) = s'(r_y(p_x(w))) \cdot r'_y(p_x(w)) \cdot p'_x(w)$
- `Layer[T-1]->delta = 1`
- For $t = T-1, T-2, \ldots, 0$
  - For `i in Layer[t]->inputs`:
    - `i->delta = Layer[t]->delta * Layer[t]->derivative(i,Layer[t]->inputs)`



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Main message

Computation graph enables us to construct very complicated functions from simple building blocks

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- The multiplication is matrix multiplication
- The correctness of the algorithm follows from the multivariate chain rule

$$J_{\mathbf{w}}(\mathbf{f} \circ \mathbf{g}) = J_{g(\mathbf{w})}(\mathbf{f}) J_{\mathbf{w}}(\mathbf{g})$$

Jacobian — Examples

- If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is element-wise application of $\sigma : \mathbb{R} \to \mathbb{R}$ then $J_{\mathbf{x}}(\mathbf{f}) = \operatorname{diag}((\sigma'(x_1), \ldots, \sigma'(x_n)))$.

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- Let $\mathbf{f}(\mathbf{x}, \mathbf{w}, b) = \mathbf{w}^\top \mathbf{x} + b$ for $\mathbf{w}, \mathbf{x} \in \mathbb{R}^n, b \in \mathbb{R}^1$. Then:

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- Let $\mathbf{f}(W, \mathbf{x}) = W\mathbf{x}$. Then:

$$J_{\mathbf{x}}(\mathbf{f}) = W \quad , \quad J_W(\mathbf{f}) = \begin{pmatrix} \mathbf{x}^\top & 0 & \cdots & 0 \\ 0 & \mathbf{x}^\top & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{x}^\top \end{pmatrix} \quad .$$

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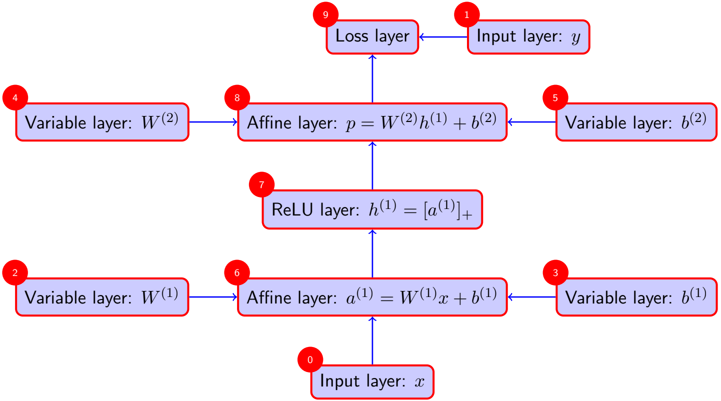
- Other ways to improve generalization is all sort of regularization

Expressiveness

- So far in the course we considered hypotheses of the form $x \mapsto w^\top x + b$
- Now, consider the following computation graph, known as "one hidden layer network":



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- Theorem: For every n, let $s(n)$ be the minimal integer such that there exists a one hidden layer network with $s(n)$ hidden neurons that implements all functions from $\{0,1\}^n$ to $\{0,1\}$. Then, $s(n)$ is exponential in n.

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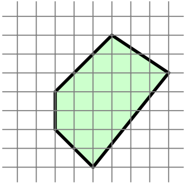
- Proof: Think on the VC dimension ...
- What type of functions can be implemented by small size networks?

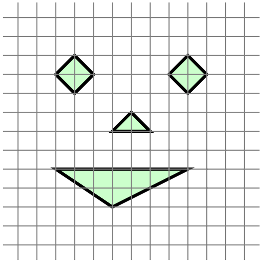
Geometric Intuition

- One hidden layer networks can express intersection of halfspaces



Geometric Intuition

- Two hidden layer networks can express unions of intersection of halfspaces



What can we express with T-depth networks ?

- Theorem: Let $T : \mathbb{N} \to \mathbb{N}$ and for every n, let $\mathcal{F}_n$ be the set of functions that can be implemented using a Turing machine using runtime of at most $T(n)$. Then, there exist constants $b, c \in \mathbb{R}_+$ such that for every n, there is a network of depth at most T and size at most $cT(n)^2 + b$ such that it implements all functions in $\mathcal{F}_n$.

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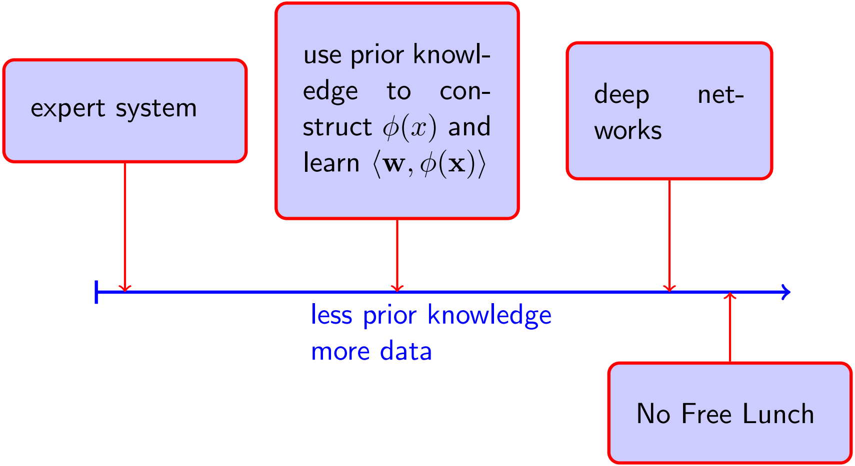
- Sample complexity is order of number of variables (in our case polynomial in T)
- Conclusion: A very weak notion of prior knowledge suffices — if we only care about functions that can be implemented in time $T(n)$, we can use neural networks of depth T and size $O(T(n)^2)$, and the sample complexity is also bounded by polynomial in $T(n)$!

The ultimate hypothesis class



Outline

Runtime of learning neural networks

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- Theorem: Under some average case complexity assumption, it is hard to learn one hidden layer networks with $\omega(\log(d))$ hidden neurons even improperly

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  - Easier than optimizing over Python programs ...
  - Need to apply some tricks (initialization, learning rate, mini-batching, architecture), and need some luck

Outline

Deep Learning Golden age in Vision

- 2012-2014 Imagenet results:

CNN
non-CNN

2012 Teams	%error
Supervision (Toronto)	15.3
ISI (Tokyo)	26.1
VGG (Oxford)	26.9
XRCE/INRIA	27.0
UvA (Amsterdam)	29.6
INRIA/LEAR	33.4

2013 Teams	%error
Clarifai (NYU spinoff)	11.7
NUS (singapore)	12.9
Zeiler-Fergus (NYU)	13.5
A. Howard	13.5
OverFeat (NYU)	14.1
UvA (Amsterdam)	14.2
Adobe	15.2
VGG (Oxford)	15.2
VGG (Oxford)	23.0

2014 Teams	%error
GoogLeNet	6.6
VGG (Oxford)	7.3
MSRA	8.0
A. Howard	8.1
DeeperVision	9.5
NUS-BST	9.7
TTIC-ECP	10.2
XYZ	11.2
UvA	12.1

- 2015 results: MSRA under **3.5%** error. (using a CNN with 150 layers!)

figures from Yann LeCun's CVPR'15 plenary

Convolution Layer

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- Observe: can be implemented as a combination of Im2Col layer and Affine layer

Im2Col Layer

- Im2Col for 3×3 convolution

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

					0	0	1																	
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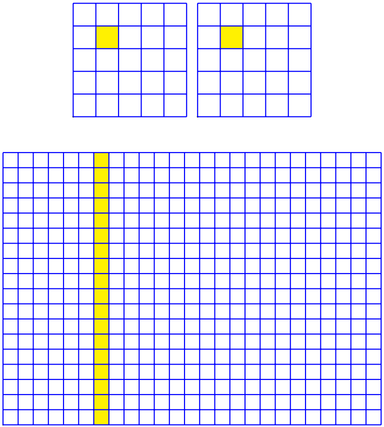
Im2Col Layer

- Im2Col for 3×3 convolution with 2 input channels



Parameters of Convolutions layer

- Kernel height and kernel width
- Stride height and stride width
- zero padding (True or False)
- Number of output channels

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- Discuss: how to calculate derivative ?

Outline

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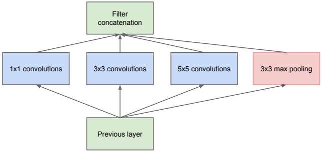
  - LogLoss: If the correct label is y then the loss is $-\log(p_y) = \log\left(\sum_j \exp(h_j(x) - h_i(x))\right)$

Outline

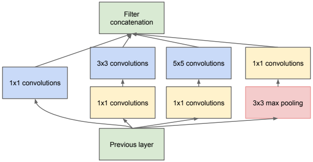
Reduction Layers

- The complexity of convolutional layers is $C_{\text{in}} \times C_{\text{out}} \times H \times W$
- A "reduction layer" is a 1×1 convolution aiming at reducing C_{in}
- It can greatly reduce the computational complexity (less time) and sample complexity (fewer parameters)

Inception modules



(a) Inception module, naïve version



(b) Inception module with dimensionality reduction

Figure 2: Inception module

- Szegedy et al (Google)
- Won the ImageNet 2014 challenge (6.67% error)

Residual Networks

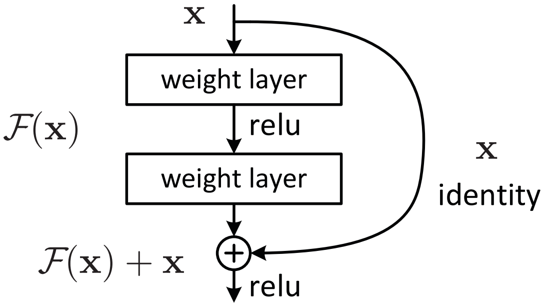


Figure 2. Residual learning: a building block.

- He, Zhang, Ren, Sun (Microsoft)
- Won the ImageNet 2015 challenge with a 152 layers network (3.57% error)

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- Learning rate: Choice of learning rate is important. One way is to start with some fixed η and decrease it by $1/2$ whenever the training stops making progress.
- Variants of SGD: There are plenty of variants that work better than vanilla SGD.

Failures of Deep Learning

- Parity of more than 30 bits
- Multiplication of large numbers
- Matrix inversion
- ...

Summary

- Deep Learning can be used to construct the ultimate hypothesis class
- Worst-case complexity is exponential
- ... but, empirically, it works reasonably well and leads to state-of-the-art on many real world problems