A Measure of Symmetry Based on Shape Similarity

H. Zabrodsky[1] S. Peleg[1] D. Avnir[2]

[1]Institute of Computer Science
The Hebrew University of Jerusalem
91904 Jerusalem, Israel

[2]Department of Organic Chemistry
The Hebrew University of Jerusalem
91904 Jerusalem, Israel

Abstract

Symmetry is usually viewed as a discrete feature: an object is either symmetric or non-symmetric. We propose to view symmetry as a continuous feature and define a Continuous Symmetry Measure (CSM) to quantify the symmetry of objects. Some applications are also presented.

1 Introduction

The world is rich in symmetries. However, the exact mathematical definition of symmetry is not adequate to describe most symmetries, which are very seldom exact. Faces, and the human body, are a classic example for inexact symmetry. The projection of the world onto an image plane (the retina or a digital image) creates additional deviations from exact symmetry. To handle those inexact symmetries, we propose a "continuous symmetry measure" that can measure and quantify all types of symmetries of objects.

Definitions of Symmetry (see [8]):
A 2D object has a perfect **mirror-symmetry** if it is invariant under a reflection about a line (called the axis of mirror-symmetry) passing through the centroid of the object.

A 2D object has a perfect **rotational-symmetry** of order n (C_n) if it is invariant under rotations of $2\pi/n$ radians about its center of mass.

a. b.

Figure 1: a) C_8-symmetry. b) mirror-symmetry

2 Symmetry In Computer Vision

An intrinsic characteristic of objects and shapes, symmetry can be used to describe and discriminate objects [2]. Symmetrical features of images can be exploited for better image compression and symmetrical descriptions of shapes or symmetrical features of objects can be useful in guiding shape matching and model-based object matching [6]. Using symmetry as a constraint, reconstruction of 3D objects has also been implemented [7]. Detection of symmetries has been widely studied ([3, 5, 1] to name a few). Several studies even deal with quantification of mirror symmetry (chirality) in non symmetric shapes ([5, 4]). These symmetry detection and evaluation methods are each limited to a certain type of symmetry (mirror or circular symmetry). In this paper we present a general continuous symmetry measure for evaluating all types of symmetries.

3 A Continuous Symmetry Measure

Denote by Ω the space of all 2D shapes, where each shape P is represented by a sequence of n points $\{P_i\}_{i=0}^{n-1}$. We define a metric d on this space as follows:

$$d : \Omega \times \Omega \rightarrow R$$

$$d(P,Q) = d(\{P_i\},\{Q_i\}) = \frac{1}{n}\sum_{i=1}^{n} ||P_i - Q_i||^2$$

This metric defines a distance function between every two shapes in Ω.

We define the **Symmetry Transform** ST as the symmetric shape closest to P in terms of the metric d.

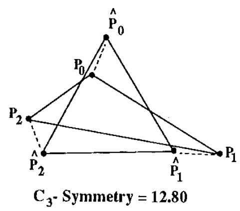
The **Continuous Symmetry Measure (CSM)** is now defined as the distance to the closest symmetric shape:

$$CSM(P) = d(P, ST(P))$$

The CSM of a shape $P = \{P_i\}_{i=0}^{n-1}$ is evaluated by finding the symmetry transform $\hat{P}$ of P (Fig.2) and computing: CSM(P) $= \frac{1}{n}\Sigma_{i=0}^{n-1}||P_i - \hat{P}_i||^2$.

0-8186-2855-3/92 $3.00 ©1992 IEEE

Figure 2: The symmetry transform of $\{P_0, P_1, P_2\}$ is $\{\hat{P}_0, \hat{P}_1, \hat{P}_2\}$. CSM $= \frac{1}{3}\Sigma_{i=0}^{2}||P_i - \hat{P}_i||^2$



An example of a 2D polygon and it's symmetry transforms and CSM values are shown in Fig. 3.

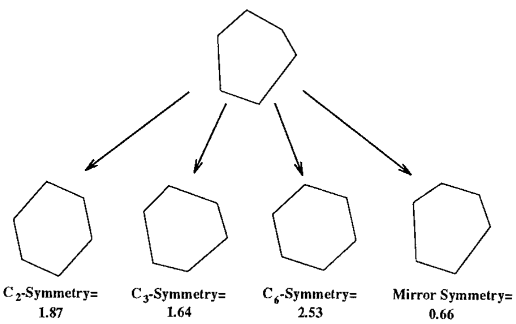


Figure 3: Symmetry Transforms of a 2D polygon and corresponding CSM values.

4 2D Symmetry Transform

Following is a geometrical algorithm for deriving the C_n symmetry transform of a shape P having n points. A mathematical derivation and proofs can be found in [9]. This transforms P into a regular n-gon, keeping the centroid in place.

1. *Fold* the points $\{P_i\}_{i=0}^{n-1}$ by rotating each point P_i counterclockwise about the centroid by $2\pi i/n$ radians (Fig. 4a).
2. Let $\hat{P}_0$ be the average of the points $\{\tilde{P}_i\}_{i=0}^{n-1}$.
3. *Unfold* the points, obtaining the C_n-symmetric points $\{\hat{P}_i\}_{i=0}^{n-1}$ by duplicating $\hat{P}_0$ and rotating clockwise about the centroid by $2\pi i/n$ radians (Fig. 4b).

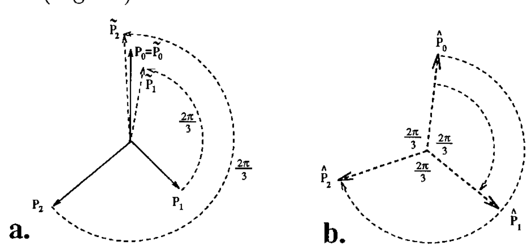


Figure 4: The C_3-symmetry Transform of 3 points: a) Folding $\{P_i\}_{i=0}^{2}$ into $\{\tilde{P}_i\}_{i=0}^{2}$. b) Unfolding the average point $\hat{P}_0 = \frac{1}{3}\sum_{i=0}^{2}\tilde{P}_i$ obtaining $\{\hat{P}_i\}_{i=0}^{2}$.

A 2D shape P having qn points is represented as q sets $\{S_r\}_{r=0}^{q-1}$ of n interlaced points $S_r = \{P_{rn+i}\}_{i=0}^{n-1}$. The C_n-symmetry transform of P is obtained as follows (Fig. 5):

1. Transform each set of n points into a regular n-gon by folding, averaging and unfolding.
2. Translate all regular n-gons so that their centroids coincide with the centroid of the original shape.

The procedure for evaluating the symmetry transform for mirror-symmetry is similar (see [9]).

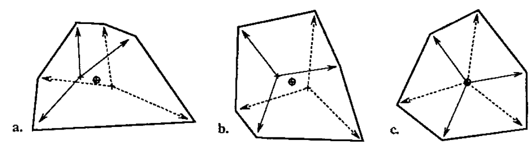


Figure 5: Geometric description of the C_3-symmetry transform for a 6-sided polygon. The centroid of the polygon is marked by $\oplus$. a) The original polygon shown as two sets of 3 points. b) Each set is transformed into a regular triangle leaving it's centroid (marked as $+$) invariant. c) Each regular triangle is translated so that its centroid coincides with the centroid of the entire shape. A C_3-symmetric shape is obtained.

5 Selecting the Points of the Shape

As symmetry has been defined on a sequence of points, representing a given shape by points must precede the application of the symmetry transform. If the shapes are polygons we can represent the shape by its vertices. In this case we can measure C_n-symmetries only for polygons whose number of vertices is a multiple of n.

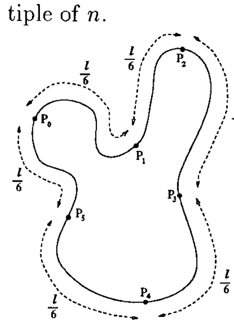


Figure 6: Point selection by length: points are selected along the contour such that each point is equidistant to the next in terms of curve length. In this example six points are distributed along the contour spaced by $\frac{1}{6}$ of the full contour length.

There are several ways to select a sequence of points to represent general 2D shapes. Using **selection-by-length**, the shape is represented by points along its contour, equidistant in terms of curve length of the contour (Fig. 6). When a simple contour length is

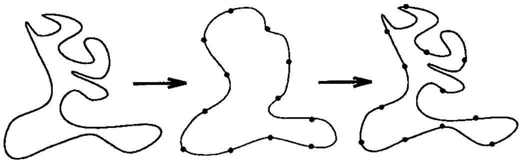


Figure 7: Selection by smoothing: contour is smoothed, points are selected by length along the smoothed contour and then projected on to the original contour.

not meaningful (as in noisy shapes in Fig. 8), we select points on a smoothed version of the contour. The selected points are then projected onto the original contour (Fig. 7). The level of smoothing can vary and for maximal smoothing, when the contour is reduced to a circle, we get the special case of **selection-by-angle** (Fig. 8): points are selected on the contour at equal angles around the centroid.

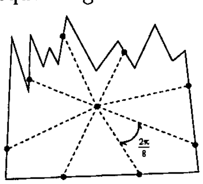


Figure 8: Selection-by-angle: points are distributed along the contour at regular angular intervals around the centroid.

6 Center of Symmetry

We use the CSM to locate the **Center of Symmetry** of occluded shapes. In general, the center of symmetry of symmetric and almost symmetric shapes aligns with the centroid of the shape. However, the center of symmetry of truncated or occluded objects does not. Using selection-by-angle, the CSM value is smaller for selection about the center of symmetry than about the centroid. To locate the center of symmetry we use an iterative procedure of gradient descent that converges from the centroid to the center of symmetry by minimizing the CSM values (Fig. 9).

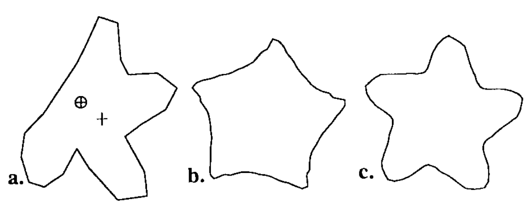


Figure 9: a) Original occluded shape, its centroid (+) and its center of symmetry($\oplus$). b,c) The C_5-symmetry transform of the shape using selection-by-angle about the centroid (b) and about the center of symmetry (c).

7 Applications

7.1 2D Method

A simple generalization of the 2D symmetry transform to grey level images is to view the image as a "topographic" map of its grey levels (Fig. 10). The symmetry transform is applied separately to each topographic contour giving a symmetric image.

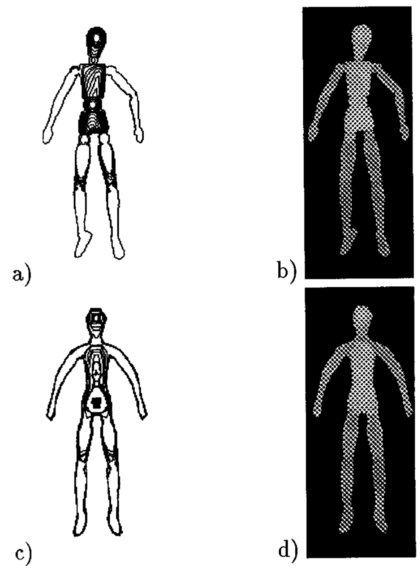


Figure 10: Finding the symmetry transform for mirror-symmetry.

a) Image as a "topographic" map.
b) The grey level image corresponding to (a).
c) Performing the mirror symmetry transform separately for each contour.
d) The grey level image corresponding to (c).

7.2 3D Symmetry

When analyzing depth maps, where pixel values denote elevation, we measure the 3D symmetry by extending the symmetry transform and CSM to 3D symmetries (see [9]). We applied the CSM to find the direction of gaze of artificially and real face images by finding their mirror-symmetry transform. The 3D shape is represented by a set of 3D points. (Fig. 11). The mirror-symmetry transform of the image is obtained by minimizing the CSM over all possible reflection planes. Results are shown in Fig. 12.

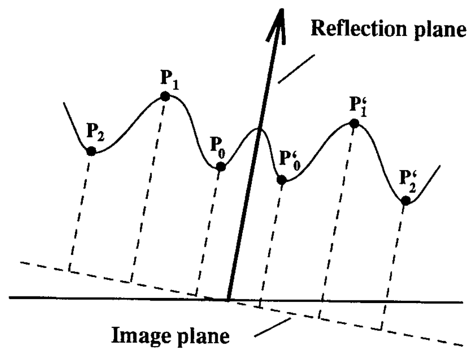


Figure 11: Finding 3D mirror-symmetry. The reflection plane minimizing the CSM indicates the closest mirror-symmetry.

8 Conclusion

In this paper we have viewed symmetry as a continuous feature, and defined a Continuous Symmetry Measure (CSM) of shapes. The general definition of symmetry measure enables a comparison of the "amount" of symmetry of different shapes and the "amount" of different symmetries of a single shape. Furthermore, the CSM is associated with the symmetric shape which is 'closest' to the given one, enabling visual evaluation of the CSM.

References

[1] J. Bigün. Recognition of local symmetries in gray value images by harmonic functions. In *9-ICPR*, pages 345–347, 1988.

[2] M. Brady and H. Asada. Smoothed local symmetries and their implementation. *Int. J. Robotics Research*, 3(3):36–61, 1984.

[3] P. Eades. Symmetry finding algorithms. In G.T. Toussaint, editor, *Computational Morphology*, pages 41–51. Elsevier, 1988.

[4] Y. Hel-Or, S. Peleg, and D. Avnir. Characterization of right handed and left handed shapes. *CVGIP:Image Understanding*, 53(2), 1991.

[5] G. Marola. On the detection of the axes of symmetry of symmetric and almost symmetric planar images. *IEEET-PAMI*, 11(1):104–108, 1989.

[6] W.G. Oh, M. Asada, and S. Tsuji. Model based matching using skewed symmetry information. In *9-ICPR*, pages 1043–1045, 1988.

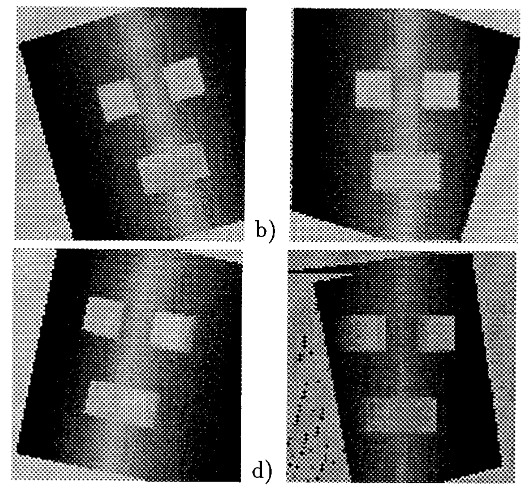
a) b) c) d)


Figure 12: Applying the 3D mirror-symmetry.
a,c) original depth maps.
b,d) The symmetry reflection plane has been found and the image rotated to a frontal vertical view.

[7] D. Terzopoulos, A. Witkin, and M. Kass. Symmetry seeking models and object reconstruction. *Int. J. Computer Vision*, 1:211–221, 1987.

[8] H. Weyl. *Symmetry.* Princeton Univ. Press, 1952.

[9] H. Zabrodsky, S. Peleg, and D. Avnir. A continuous symmetry measure. Technical Report TR-91-9, Hebrew University, August 1991.