Coverage and Vacuity in Network Formation Games

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Abstract

The frameworks of coverage and vacuity in formal verification analyze the effect of mutations applied to systems or their specifications. We adopt these notions to network formation games, analyzing the effect of a change in the cost of a resource. We consider two measures to be affected: the cost of the Social Optimum and extremums of costs of Nash Equilibria. Our results offer a formal framework to the effect of mutations in network formation games and include a complexity analysis of related decision problems. They also tighten the relation between algorithmic game theory and formal verification, suggesting refined definitions of coverage and vacuity for the latter.

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1 Introduction

Following the emergence of the Internet, there has been an explosion of studies employing game-theoretic analysis to explore applications such as network formation and routing in computer networks [21, 1, 20, 4]. In *network-formation games* (for a survey, see [38]), the network is modeled by a weighted graph. The weight of an edge indicates the cost of activating the transition it models, which is independent of the number of times the edge is used. Players have reachability objectives, each given by a source and a target vertex. Under the common Shapley cost-sharing mechanism, the cost of an edge is shared evenly by the players that use it. The players are selfish agents who attempt to minimize their own costs, rather than to optimize some global objective. In network-design settings, this would mean that the players selfishly select a path instead of being assigned one by a central authority. The study of networks from a game-theoretic point of view focuses on optimal strategies for the underlying players, stable outcomes of a given setting, namely equilibrium points, and outcomes that are optimal for the society as a whole.

A different type of reasoning about networks is the study of their on-going behaviors. In particular, in recent years we see growing use of formal-verification methods in the context of software-defined networks [34, 33]. The study of networks from a formal-verification point of view focuses on specification and verification of their behavior. The primary problem here is *model checking*: given a system (in particular, a network) and a specification for its desired behavior, decide whether the system satisfies the specification [18]. Typically, the system is given by means of a labeled graph and the specification is given by a temporal-logic formula. An important element in model-checking methodologies is an assessment of the

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quality of the modeling of the system and the specifications as well as the exhaustiveness of the model-checking process. Researchers have developed a number of *sanity checks*, aiming to detect errors in the modeling [27]. Two leading sanity checks are *vacuity* and *coverage*. In vacuity, the goal is to detect cases where the system satisfies the specification in some unintended trivial way [10, 31, 14]. In coverage, the goal is to increase the exhaustiveness of the specification by detecting components of the system that do not play a role in the verification process [24, 25, 16, 15]. Both vacuity and coverage checks are based on analyzing the effect of applying *local mutations* to the system or the specification. The intuition is that model checking of an exhaustive well-formed specification should be sensitive to such mutations.

Beyond the practical importance of sanity checks, their study highlights some general important theoretical properties regarding the sensitivity of systems and specifications to mutations. Examples to such properties include *duality* between mutations applied to the system and the specification [29], and trade-offs between desired and undesired insensitivity to mutations (for example, fault tolerance is associated with a desired insensitivity to mutations) [17]. A fundamental property of mutations in the context of formal verification is *monotonicity*: mutations to temporal-logic formulas are monotone, in the sense that if ψ is a formula and φ is a sub-formula of ψ that appears in a positive polarity (that is, nested in an even number of negations), then when we mutate ψ to ψ' by replacing φ by φ', then $\psi' \rightarrow \psi$ iff $\varphi' \rightarrow \varphi$. Monotonicity turns out to be a very helpful property in the context of vacuity checking. Indeed, the basic notion in vacuity is of a subfumula φ *not affecting* the satisfaction of a specification ψ. Formally, consider a system $\mathcal{S}$ satisfying a specification ψ. A subformula φ of ψ *does not affect* (the satisfaction of) ψ in $\mathcal{S}$ if $\mathcal{S}$ also satisfies all specifications obtained by mutating φ to some other subformula [10]. Thanks to monotonicity, we can check whether φ affects ψ by examining only the most challenging mutation, namely one that replaces φ by false and the most helpful mutation, namely one that replaces φ by true.

Our goal in this paper is to examine the sensitivity of network-formation games (NFGs, for short) to mutations applied to costs. While our study adopts from formal verification the notion of mutation-based analysis, we examine the effect of mutations on measures from game theory: the cost of stable and optimal outcomes. Recall that a strategy of a player in an NFG is a path from a source to a target vertex. A *profile* in the game is a vector of strategies, one for each player. A *Social Optimum* (SO) is a profile that minimizes the total cost to all players. A *Nash equilibrium* (NE) is a profile in which no player can decrease her cost by a unilateral deviation from her current strategy, that is, assuming that the strategies of the other players do not change.

Consider an NFG N. We say that the edge e of N *SO-affects* N if a change in the cost of e leads to a change in the cost of the SO. Formally, there exists $x \geq 0$ such that the cost of the SO profiles in N is different from the cost of the SO profiles in $N[e \leftarrow x]$, that is N with e being assigned cost x. We consider the function $cost^{e}_{SO}(N) : \mathbb{R} \rightarrow \mathbb{R}$, mapping a cost $x \geq 0$ to the cost of the SO profiles in $N[e \leftarrow x]$. That is, $cost^{e}_{SO}(N)$ describes the cost of the SO in N as a function of the cost of the edge e. We say that $cost_{SO}$ is monotonically increasing if for every NFG N and edge e of N, the function $cost^{e}_{SO}(N)$ is monotonically increasing. Likewise, $cost_{SO}$ is continuous if for every NFG N and edge e, the function $cost^{e}_{SO}(N)$ is continuous. For the best and worst NEs, we similarly define when an edge e *bNE-affects* and *wNE-affects* N, and define the functions $cost_{bNE}$ and $cost_{wNE}$, which describe the cost of the best and worst NEs as a function of the cost of an edge.

Our first set of results concerns the way edge costs affect the SO. Here, the results are quite expected: $cost_{SO}$ is monotonically increasing and continuous, which leads to simple solutions to related decision problems: as is the case with model checking and temporal-logic specifications, we can decide whether an edge e SO-affects N by checking the cost of the SO in $N[e \leftarrow 0]$ and $N[e \leftarrow \infty_N]$, for a sufficiently large cost ∞_N. This leads to Δ_2^P and Θ_2^P upper bounds (depending on whether costs are given in binary or unary, respectively), which we show to be tight. Also, we show that it is NP-complete and DP-complete to decide whether we can mutate a cost in a way that would cause the SO to be below or agree exactly with, respectively, a given threshold. The technically challenging results here are the Δ_2^P-lower bound (it is tempting to believe that thanks to monotonicity, we could decide whether e SO-affects N using only logarithmically many queries to an NP oracle that bounds the SO) and the DP upper bound (the upper and lower bounds on the SO that we can obtain by querying an NP and a co-NP oracle need not be associated with the same edge).

Things become unexpected when we turn to study effects on the costs of the best and worst NEs. Here an edge may affect the bNE without participating in profiles that are NEs, and may thus affect the bNE both positively and negatively. In model checking, this is related to coverage and vacuity in a setting with multiple occurrences of subformulas. For example, the atomic proposition p appears in the formula $\psi = (\varphi_1 \rightarrow p) \wedge (p \rightarrow \varphi_2)$ both positively and negatively. Consequently, we cannot decide whether p affects the satisfaction of ψ by examining its replacement by only true or false (in the context of vacuity), and we do not know the effect of mutating p in the system on the satisfaction of ψ (in the context of coverage). We show that $cost_{bNE}$ is neither monotone nor continuous, and in fact a change in the cost of an edge may incentivize players in surprising ways. In particular (see Figure 5), an edge e may not participate in any bNE in $N[e \leftarrow x]$, for all $x \geq 0$, and still the bNE may decrease as we increase the cost of e. We show that these challenges can be overcome by more restricted notions such as piecewise monotonicity and monotonicity on the participation of the mutated edge in bNE profiles. In particular, we show that these notions produce the same (tight) complexity bounds for the analogous decision problems we introduce for the SO. We note that while the general phenomenon of non-monotonicity is known (e.g., Braess' Paradox [12], the effectiveness of burning money [23, 36] or tax increase [19]), we are the first, to the best of our knowledge, to provide a comprehensive study of effects caused by cost mutation.

Our results on NFGs give rise to two research directions in coverage and vacuity in formal verification. The first arises from the segmentation of $\mathbb{R}^+$ induced by the non-monotonicity of the bNE, which suggests a similar segmentation in the context of multi-valued specification formalisms [2]. The second is a study of coverage and vacuity in formalisms for specifying strategic on-going behaviors [3, 13]. We discuss these research directions in Section 5.

Due to lack of space, some of the proofs are omitted, and can be found in the full version, as listed above.

2 Preliminaries

2.1 Network formation games

A *network formation game* (NFG) is $N = \langle k, V, E, c, \gamma \rangle$, where k is a number of players, V is a set of vertices, $E \subseteq V \times V$ is a set of directed edges, $c : E \rightarrow \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers including 0, is a cost function that maps each edge to the cost of forming it, and $\gamma = \{\langle s_1, t_1 \rangle, ..., \langle s_k, t_k \rangle\}$ is a set of objectives, each specifying a source and a target vertex per player. Thus, for all $1 \leq i \leq k$, the objective of player i is to form a path from s_i to t_i. A *strategy* for player i is a simple path $\pi_i \subseteq E$ from s_i to t_i. Note that since the path is simple, then π_i is indeed a subset of E. A *profile* $P = \langle \pi_1, ..., \pi_k \rangle$ is a vector of strategies, one for each player. For an edge $e \in E$, we denote by $used_P(e)$ the number of players that use e in their strategy in P, thus these with $e \in \pi_i$. We say that $e \in P$ if $used_P(e) > 0$.

Players pay the cost of forming edges they use. If players share an edge, they also share its cost. Thus, the cost of a strategy π_i in a profile P is $cost_{N,P}(\pi_i) = \sum_{e \in \pi_i} \frac{c(e)}{used_P(e)}$. Note that since c is positive, it is indeed sufficient to consider only simple paths as strategies. The *cost* of P in N is the sum of costs of its strategies, that is $cost(N,P) = \sum_{i=1}^{k} cost_{N,P}(\pi_i)$. Equivalently, $cost(N,P) = \sum_{e \in P} c(e)$.

A *Social Optimum (SO)* of N is a profile with minimal cost. That is, a profile P is an SO if for every other profile P' we have that $cost(N,P) \leq cost(N,P')$. Note that there may be several profiles that are a social optimum. We denote by $SO(N)$ and $cost_{SO}(N)$ the set of such profiles and their cost, respectively.

We say that the profile P is a *Nash Equilibrium (NE)* in N if no player can decrease her cost by deviating to another strategy assuming the other players stay in their strategies[1]. Formally, for all $1 \leq i \leq k$ and every $\pi_i' \neq \pi_i$, the cost of π_i' in $P' = \langle \pi_1, ..., \pi_{i-1}, \pi_i', \pi_{i+1}, ..., \pi_k \rangle$ is no lower than the cost of π_i in P, i.e. $cost_{N,P}(\pi_i) \leq cost_{N,P'}(\pi_i')$. A *best NE (bNE)* in N is an NE profile with minimal cost, i.e. a profile P is bNE iff P is an NE, and for every profile P' that is an NE, we have $cost(N,P) \leq cost(N,P')$. We denote by $bNE(N)$ and $cost_{bNE}(N)$ the set of profiles that are bNE, and their cost, respectively.

We dually define a *worst NE (wNE)* to be an NE profile with maximal cost, and denote by $wNE(N)$ and $cost_{wNE}(N)$ the set of such profiles and their cost, respectively. The *Price of Stability (PoS)* of N is the ratio between the cost of the bNE and the SO, that is, $PoS(N) = \frac{cost_{bNE}(N)}{cost_{SO}(N)}$.

▶ **Example 1.** Consider the NFG N appearing in Figure 1.

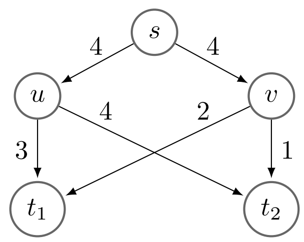

Figure 1 The NFG N.

Player 2 / Player 1	π_2^1 $s \to u \to t_2$	π_2^2 $s \to v \to t_2$
π_1^1 $s \to u \to t_1$	6 5	5 7
π_1^2 $s \to v \to t_1$	8 6	3 4

Table 1 Players' costs in N.

Assume that N is formed by two players. The first has objective $\langle s, t_1 \rangle$. The available strategies for her are $\pi_1^1 = \{(s,u),(u,t_1)\}$ and $\pi_1^2 = \{(s,v),(v,t_1)\}$. The second player has objective $\langle s, t_2 \rangle$. The available strategies for her are $\pi_2^1 = \{(s,u),(u,t_2)\}$ and $\pi_2^2 = \{(s,v),(v,t_2)\}$. If Player 1 choses the strategy π_1^1 and Player 2 uses the strategy π_2^1, then they share the cost of the edge (s,u), and their costs are $\frac{4}{2} + 3 = 5$ and $\frac{4}{2} + 4 = 6$ respectively. Table 1 describes the costs of the two players in the different profiles.

The profile with the lowest cost is $P = \langle \pi_1^2, \pi_2^2 \rangle$. Therefore, $SO(N) = \{P\}$, with cost $cost_{SO}(N) = 7$. Note that P is also the only NE in N. It is an NE since for the deviation $P' = \langle \pi_1^1, \pi_2^2 \rangle$, it holds that $4 = cost_{N,P}(\pi_1^2) < cost_{N,P'}(\pi_1^1) = 7$ and for the deviation $P'' = \langle \pi_1^2, \pi_2^1 \rangle$ it holds that $3 = cost_{N,P}(\pi_2^2) < cost_{N,P''}(\pi_2^1) = 8$. It is the only NE in N

[1] Throughout this paper, we consider pure strategies and pure deviations, as is the case for the vast literature on cost-sharing games.

since for every other profile there is a beneficial deviation. Therefore, P is both a bNE and a wNE. Since the bNE and the SO coincide, it follows that $PoS(N) = 1$. ◀

Consider an edge $e \in E$ and a value $x \in \mathbb{R}^+$. We denote by $c[e \leftarrow x]$ the cost function that agrees with c on every edge except e, which is assigned x. That is, $c[e \leftarrow x](e) = x$, and for all edge $e' \neq e$, we have $c[e \leftarrow x](e') = c(e')$. Let $N = \langle k, V, E, c, \gamma\rangle$, and let $e \in E$. We denote by $N[e \leftarrow x]$ the network obtained from N by changing the cost of e to x. Thus, $N[e \leftarrow x] = \langle k, V, E, c[e \leftarrow x], \gamma\rangle$.

Let c_1 and c_2 be cost functions. We say that c_2 *bounds* c_1 *from above*, denoted $c_1 \leq c_2$, if for all $e \in E$, we have $c_1(e) \leq c_2(e)$. We extend the notation to NFGs. Let $N_1 = \langle k, V, E, c_1, \gamma\rangle$ and $N_2 = \langle k, V, E, c_2, \gamma\rangle$ be two NFGs that differ only on their cost functions. If $c_1 \leq c_2$, we say that N_2 *bounds* N_1 *from above*, denoted $N_1 \leq N_2$.

▶ **Lemma 2.** *Let N_1 and N_2 be two NFGs that differ only on their cost functions. If $N_1 \leq N_2$, then for every profile P, we have $cost(N_1, P) \leq cost(N_2, P)$.*

2.2 Affecting edges in NFGs

Consider an NFG N and an edge e of N. We say that the edge e *SO-affects* N if there exists $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) \neq cost_{SO}(N)$. That is, when changing the cost of e to x, the cost of the SO profiles of N changes. We define *bNE-affects*, *wNE-affects*, and *PoS-affects* in a similar way, referring to the costs of the best and worst NEs, and the PoS.

▶ **Example 3.** Consider the NFG N from Example 1, and consider the edge $e = (s, v)$. The edge e SO-affects N, since, for example, for $N[e \leftarrow 2]$ we have that $\langle\pi_1^2, \pi_2^2\rangle$ is an SO with cost $5 < 7 = cost_{SO}(N)$. As another example, for $N[e \leftarrow 10]$ we have that $\langle\pi_1^1, \pi_2^1\rangle$ is an SO with cost $11 > 7 = cost_{SO}(N)$. Next, consider the edge $e = (u, t_1)$. For every $x \geq 0$, we have $cost(N[e \leftarrow x], \langle\pi_1^1, \pi_2^1\rangle) = x + 8$, $cost(N[e \leftarrow x], \langle\pi_1^1, \pi_2^2\rangle) = x + 9$, $cost(N[e \leftarrow x], \langle\pi_1^2, \pi_2^1\rangle) = 14$, and $cost(N[e \leftarrow x], \langle\pi_1^2, \pi_2^2\rangle) = 7$. Therefore, $cost_{SO}(N[e \leftarrow x]) = \min\{x + 8, x + 9, 14, 7\} = 7 = cost_{SO}(N)$, and so e does not SO-affect N.

We proceed to bNE and wNE. Here, the change may affect the stability of profiles, and not just their cost. Consider the edge $e = (s, u)$. Table 2 describes the costs of the different profiles of $N[e \leftarrow (1 - \varepsilon)]$, for some $0 < \varepsilon < 1$.

Player 2 / Player 1	π_2^1: $s \to u \to t_2$	π_2^2: $s \to v \to t_2$
π_1^1: $s \to u \to t_1$	Player 2: $4\frac{1}{2} - \frac{\varepsilon}{2}$; Player 1: $3\frac{1}{2} - \frac{\varepsilon}{2}$	Player 2: 5; Player 1: $4 - \varepsilon$
π_1^2: $s \to v \to t_1$	Player 2: $5 - \varepsilon$; Player 1: 6	Player 2: 3; Player 1: 4

Table 2 Costs in $N[\langle s, u\rangle \leftarrow (1 - \varepsilon)]$.

Player 2 / Player 1	π_2^1: $s \to u \to t_2$	π_2^2: $s \to v \to t_2$
π_1^1: $s \to u \to t_1$	Player 2: 6; Player 1: $2 + x$	Player 2: 5; Player 1: $4 + x$
π_1^2: $s \to v \to t_1$	Player 2: 8; Player 1: 6	Player 2: 3; Player 1: 4

Table 3 Costs in $N[\langle u, t_1\rangle \leftarrow x]$.

We previously saw that the only NE profile in N is $P = \langle\pi_1^2, \pi_2^2\rangle$, with cost 7, and therefore it is both the bNE and the wNE. We can see that the cost of P is minimal for $N[e \leftarrow (1 - \varepsilon)]$. However, P is no longer an NE. Indeed, for the profile $P' = \langle\pi_1^1, \pi_2^2\rangle$, obtained by a deviation of Player 1, we have that $4 - \varepsilon = cost_{N[e \leftarrow 1-\varepsilon], P'}(\pi_1^1) < cost_{N[e \leftarrow 1-\varepsilon], P}(\pi_1^2) = 4$. For $N[e \leftarrow (1 - \varepsilon)]$, the only NE profile is $\langle\pi_1^1, \pi_2^1\rangle$, with cost $8 - \varepsilon$. For $0 < \varepsilon < 1$ it therefore holds that $7 = cost_{bNE}(N) < cost_{bNE}(N[e \leftarrow 1 - \varepsilon]) = 8 - \varepsilon$, and the same for wNE. Therefore, the edge e both bNE-affects and wNE-affects N. Furthermore, e PoS-affects N, as $PoS(N) = 1$ and $PoS(N[e \leftarrow 1 - \varepsilon]) = \frac{8-\varepsilon}{7} > 1$.

Next, consider the edge $e = (u, t_1)$. We show that e does not bNE-affect nor does it wNE-affect N. To see this, consider the costs of the different profiles of $N[e \leftarrow x]$ for $x \geq 0$, described in Table 3. It can be easily verified that, for all $x \geq 0$, the only NE in $N[e \leftarrow x]$ is $\langle \pi_1^2, \pi_2^2 \rangle$. Therefore, $cost_{bNE}(N[e \leftarrow x]) = cost_{wNE}(N[e \leftarrow x]) = 7$. As e neither SO-affect nor bNE-affect N, it follows that e does not PoS-affect N.

It is also worth noting that it is not always the case that an edge either both bNE-affects and wNE-affects or both does not bNE-affect and wNE-affect N. As an example, consider the edge $e = (u, t_2)$. The cost table of $N[e \leftarrow x]$ appears in Table 4.

Player 2 Player 1	π_2^1 $s \to u \to t_2$	π_2^2 $s \to v \to t_2$
π_1^1 $s \to u \to t_1$	$2 + x$ 5	5 7
π_1^2 $s \to v \to t_1$	$4 + x$ 6	3 4

Table 4 Costs in $N[\langle u, t_2 \rangle \leftarrow x]$.

It is not hard to see that for $0 \leq x \leq 3$, it holds that $P_1 = \langle \pi_1^1, \pi_2^1 \rangle$ and $P_2 = \langle \pi_1^2, \pi_2^2 \rangle$ are NEs in $N[e \leftarrow x]$. However, $cost(N[e \leftarrow x], P_1) = 7 + x$ and $cost(N[e \leftarrow x], P_2) = 7$. Therefore, $cost_{bNE}(N[e \leftarrow x]) = \min\{7 + x, 7\} = 7$, and $cost_{wNE}(N[e \leftarrow x]) = \max\{7 + x, 7\} = 7 + x$. Since for all $x > 3$, the profile P_2 is the only NE in $N[e \leftarrow x]$, it follows that e does not bNE-affect N, and e wNE-affects N. ◀

2.3 Monotonicity and continuity

Consider a function $f : \mathbb{R} \to \mathbb{R}$. We say that f is *monotonically increasing* if for all $x_1, x_2 \in \mathbb{R}$, we have that $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$. For $x_0 \in \mathbb{R}$, we say that f is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. Then, we say that f is *continuous* if f is continuous at x_0 for all $x_0 \in \mathbb{R}$.

For an edge $e \in E$, we define the function $cost_{SO}^e(N) : \mathbb{R} \to \mathbb{R}$ by $cost_{SO}^e(N)(x) = cost_{SO}(N[e \leftarrow x])$ if $x \geq 0$, and $cost_{SO}^e(N)(x) = cost_{SO}(N[e \leftarrow 0])$ otherwise. That is, $cost_{SO}^e(N)$ is the cost of the SO in N as a function of the cost of the edge e. We say that $cost_{SO}$ is monotonically increasing, if for every NFG N and edge e of N, the function $cost_{SO}^e(N)$ is monotonically increasing. That is, $cost_{SO}$ is monotonically increasing if an increase in the cost of any edge, for any NFG, can only cause an increase in the cost of the SO. Likewise, $cost_{SO}$ is continuous, if for every NFG N and edge e, the function $cost_{SO}^e(N)$ is continuous. We define the monotonicity and the continuity of $cost_{bNE}$, $cost_{wNE}$ and PoS in a similar way.

3 Affecting the Social Optimum

In this section we study the sensitivity of the SO to cost mutations. We first study the monotonicity and continuity of $cost_{SO}$, and then the complexity of relevant decision problems.

3.1 Monotonicity and continuity of the SO

▶ **Theorem 4.** [$cost_{SO}$ **is monotone**] *For every NFG N and edge e of N, the function $cost_{SO}^e(N)$ is monotone.*

Proof. Let N_1 and N_2 be NFGs that differ only in their cost functions. We prove that if $N_1 \leq N_2$, then $cost_{SO}(N_1) \leq cost_{SO}(N_2)$. In particular, this holds for N_1 and N_2 being N with cost functions that differ only in the cost of e. Let $P_1 \in SO(N_1)$ and let $P_2 \in SO(N_2)$. By the minimality of the SO for N_1, we get that $cost(N_1, P_1) \leq cost(N_1, P_2)$. By Lemma 2, as $N_1 \leq N_2$, we have that $cost(N_1, P_2) \leq cost(N_2, P_2)$. Therefore, $cost(N_1, P_1) \leq cost(N_2, P_2)$, and hence $cost_{SO}(N_1) \leq cost_{SO}(N_2)$. ◀

Since $cost_{SO}$ is monotonically increasing, a sufficient condition for an edge not to SO-affect the network is based on comparing the cost of the SO in the two extreme costs for the edge. The lowest cost is 0. For the highest cost, let ∞_N be a sufficiently large value for a cost of an edge to be considered extreme in N, in the sense that if an edge e with cost ∞_N is in some strategy, then the cost of that strategy is guaranteed to be larger than the cost of all strategies that do not contain e. For example, we can define ∞_N to be $1 + \sum_{e \in E} c(e)$.

▶ **Lemma 5.** *For every NFG N and edge e of N, the edge e does not SO-affect N iff $cost_{SO}(N[e \leftarrow 0]) = cost_{SO}(N[e \leftarrow \infty_N])$.*

Proof. Since $N[e \leftarrow 0] \leq N[e \leftarrow \infty_N]$ and the function $cost_{SO}(N)$ is monotonically increasing, then $cost_{SO}(N[e \leftarrow 0]) = cost_{SO}(N[e \leftarrow \infty_N])$ implies that for all $x \geq 0$, we have $cost_{SO}(N[e \leftarrow 0]) = cost_{SO}(N[e \leftarrow x]) = cost_{SO}(N[e \leftarrow \infty_N])$. Thus, for all $x \geq 0$, we have $cost_{SO}(N) = cost_{SO}(N[e \leftarrow x])$, so the cost of e does not SO-affect N. For the other direction, if the cost of e does not SO-affect N, then, by definition, for all $x \geq 0$, we have that $cost_{SO}(N) = cost_{SO}(N[e \leftarrow x])$. In particular, $cost_{SO}(N[e \leftarrow 0]) = cost_{SO}(N[e \leftarrow \infty_N])$, and we are done. ◀

Note that it follows that for an NFG N and edge e in it, if there is a profile $P \in SO(N)$ such that $e \in P$ and $c(e) > 0$, then e SO-affects N, as reducing its cost to 0 reduces also the cost of the SO.

In case e SO-affects N, we can characterize the behavior of $cost_{SO}(N[e \leftarrow x])$ as follows.

▶ **Lemma 6.** *Consider an NFG N and an edge e of N. If e SO-affects N, then there is a value $x \in \mathbb{R}$ such that the following hold.*

1. *For all values y with $y > x$, the edge e does not participate in any profile in $SO(N[e \leftarrow y])$ and $cost_{SO}(N[e \leftarrow y]) = x + cost_{SO}(N[e \leftarrow 0])$.*
2. *For all values y with $y < x$, the edge e participates in at least one profile in $SO(N[e \leftarrow y])$ and $cost_{SO}(N[e \leftarrow y]) = y + cost_{SO}(N[e \leftarrow 0])$.*
3. *The edge e participates in at least one profile in $SO(N[e \leftarrow x])$ and $cost_{SO}(N[e \leftarrow x]) = x + cost_{SO}(N[e \leftarrow 0])$.*

Proof. Since e SO-affects N, then, by Lemma 5, we have that $cost_{SO}(N[e \leftarrow 0]) < cost_{SO}(N[e \leftarrow \infty_N])$. It is not hard to see that taking x to be $\min\{y : cost_{SO}(N[e \leftarrow y]) = cost_{SO}(N[e \leftarrow \infty_N])\}$ satisfies the conditions in the lemma. In particular, when e participates in all profiles in the SO, then $x = \min \emptyset = \infty$. ◀

▶ **Theorem 7.** *For every NFG N and edge e of N, the function $cost^e_{SO}(N)$ is continuous.*

Proof. Consider an NFG N and edge e of N. First, if the edge e does not SO-affect N, then $cost^e_{SO}(N)$ is constant and therefore continuous. Otherwise, by Lemma 6, there is a value $x \in \mathbb{R}$ such that for all values y with $y \geq x$, we have that $cost_{SO}(N[e \leftarrow y]) = x + cost_{SO}(N[e \leftarrow 0])$, and for all values y with $y < x$, we have that $cost_{SO}(N[e \leftarrow y]) = y + cost_{SO}(N[e \leftarrow 0])$. Thus, continuity in all points except x follows immediately from continuity of linear functions. For the point x, Lemma 6 implies that for all $\varepsilon > 0$, we have that $f(x + \epsilon) - f(x) = 0$, and $f(x) - f(x - \epsilon) = \epsilon$, so $cost^e_{SO}(N)$ is continuous also at x. ◀

3.2 Decision problems

The SO-cost decision problem is the problem of deciding, given an NFG N and a threshold $\kappa \geq 0$, whether $cost_{SO}(N) \leq \kappa$. The SO-cost problem is NP-complete [38]. In this section we study the following related decision problems.

1. Edge-SO-affects: Given an NFG N and an edge e of N, does e SO-affect N? Thus, Edge-SO-affects $= \{\langle N, e\rangle \mid e \text{ SO-affects } N\}$.
2. Edge-SO-optimization: Given an NFG N, an edge e of N, and a threshold $\kappa \geq 0$, is there a value $x \geq 0$, such that $cost_{SO}(N[e \leftarrow x]) \leq \kappa$? Thus, Edge-SO-optimization $= \{\langle N, e, \kappa\rangle \mid$ there exists $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) \leq \kappa\}$.
3. SO-optimization: Given an NFG N and a threshold $\kappa \geq 0$, is there an edge e of N and a value $x \geq 0$, such that $cost_{SO}(N[e \leftarrow x]) \leq \kappa$? Thus, SO-optimization$= \{\langle N, \kappa\rangle \mid$ there exist e and $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) \leq \kappa\}$.
4. SO-control: Given an NFG N and a threshold $\kappa \geq 0$, is there an edge e of N and a value $x \geq 0$, such that $cost_{SO}(N[e \leftarrow x]) = \kappa$? Thus, SO-control$= \{\langle N, \kappa\rangle \mid$ there exist e and $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) = \kappa\}$.

Analyzing the complexity of the problems, we assume that the costs of an NFG are given in binary. As we shall note below, this affects the complexity of the problems. In addition to the classes NP and co-NP, we are going to refer to the class $\Delta_2^P = P^{NP}$ (Θ_2^P), of decision problems that can be decided by a polynomial-time deterministic Turing machine that has access to polynomially many (logarithmically many, respectively) queries to an oracle to an NP-complete problem, and the class DP, of decision problems that are the intersection of an NP and a co-NP problem. That is, a decision problem $\mathcal{L}$ is in DP if there are decision problems L_1, L_2 such that $L_1 \in$ NP, $L_2 \in$ co-NP and $\mathcal{L} = L_1 \cap L_2$.

▶ **Theorem 8.** *The* Edge-SO-affects *problem is Δ_2^P-complete, and is Θ_2^P complete when costs are given in unary.*

Proof. We start with membership in Δ_2^P. Given an NFG N and an edge e in N, a deterministic Turing machine can use an oracle to SO-cost, calculate $cost_{SO}(N[e \leftarrow 0])$ and $cost_{SO}(N[e \leftarrow \infty_N])$ and compare them. Since the maximal cost of a profile is $\sum_{e\in E} c(e)$, and $cost_{SO}$ is the sum of costs of a subset of edges, rather than an arbitrary number in $\mathbb{R}$, the Turing machine can proceed by a binary search and thus the number of oracle calls is logarithmic in $\sum_{e\in E} c(e)$. When costs are given in binary, $\sum_{e\in E} c(e)$ is exponential in input, hence there are polynomially-many oracle calls. Thus, Edge-SO-affects$\in \Delta_2^P$. However, when costs are given in unary, $\sum_{e\in E} c(e)$ is polynomial in input, hence there are logarithmically-many oracle calls. Thus, Edge-SO-affects$\in \Theta_2^P$.

In the full version, we prove that the problem is Δ_2^P-hard by a reduction from maximum-satisfying-assignment, namely the problem of deciding, given a 3CNF formula φ if the lexicographically maximal assignment that satisfies φ has LSB that equals 1. It was shown by [26] that maximum-satisfying-assignment is Δ_2^P-complete. Essentially, given φ, we construct an NFG N such that profiles corresponds to assignments, and the cost of a profile decreases with lexicographically greater satisfying assignments. The edge e participates in profiles that correspond to assignments in which the LSB is 1, and is minimal only when the maximal lexicographic assignment has LSB 1. Consequently, $\langle N, e\rangle \in$ Edge-SO-affects iff $\varphi \in$ maximum-satisfying-assignment.

In the full version, we prove that when costs are given in unary, the problem is Θ_2^P-hard. The proof is by a reduction from VC-compare, namely the problem of deciding, given two undirected graphs $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, whether the size of a minimal vertex cover of G_1 is less than or equal to the size of a minimal vertex cover of G_2. Essentially, given G_1 and G_2, we construct an NFG N that subsumes both graphs and the objectives of the players are defined so that profiles correspond to choosing a vertex cover in one of the graphs. The edge e participates in profiles in which the players choose to proceed with a cover in G_1, which happens only when the size of a minimal vertex cover of G_1 is less than or equal to the size of a minimal vertex cover of G_2. Consequently, $\langle N, e\rangle \in$ Edge-SO-affects iff $\langle G_1, G_2\rangle \in$ VC-compare. ◀

We continue to the optimization problems. The proof is easy and can be found in the full version. In particular, the lower bounds are by a reduction from the SO-cost problem.

▶ **Theorem 9.** *The* Edge-SO-optimization *and* SO-optimization *problems are NP-complete.*

For the upper-bound of the SO-control problem, we first need the following lemma.

▶ **Lemma 10.** *Let N be an NFG and let $\kappa \geq 0$ be a threshold. If there are (not necessarily distinct) edges e_1 and e_2 of N such that $cost_{SO}(N[e_1 \leftarrow 0]) \geq \kappa$ and $cost_{SO}(N[e_2 \leftarrow \infty]) \leq \kappa$, then there is an edge e of N and a value $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) = \kappa$.*

Proof. Assume towards contradiction that for all edges e of N and value $x \geq 0$, it holds that $cost_{SO}(N[e \leftarrow x]) \neq \kappa$. In particular, this means that $cost_{SO}(N[e_1 \leftarrow 0]) > \kappa$ and $cost_{SO}(N[e_2 \leftarrow \infty]) < \kappa$. Hence, by monotonicity of $cost^{e}_{SO}(N)$, we get that $cost_{SO}(N) = cost_{SO}(N[e_2 \leftarrow c(e_2)]) \leq cost_{SO}(N[e_2 \leftarrow \infty]) < \kappa < cost_{SO}(N[e_1 \leftarrow 0]) \leq cost_{SO}(N[e_1 \leftarrow c(e_1)]) = cost_{SO}(N)$. ◀

▶ **Theorem 11.** *The* SO-control *problem is DP-complete.*

Proof. We start with membership. Let $L_1 = \{\langle N, \kappa\rangle \mid$ there exist an edge e and $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) \leq \kappa\}$ and $L_2 = \{\langle N, \kappa\rangle \mid$ there exist an edge e and $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) \geq \kappa\}$. Note that L_1 is SO-optimization and is therefore in NP. We show that L_2 is in co-NP. The complement of L_2 is $L_2^c = \{\langle N, \kappa\rangle \mid$ for all edges e and $x \geq 0$ we have $cost_{SO}(N[e \leftarrow x] < \kappa)\}$. A witness for membership in L_2^c is a set S of $|E| = m$ profiles, one for each edge, satisfying $cost(N[e \leftarrow \infty], P_e) < \kappa$ for each $P_e \in S$. The witness is polynomial since we only require m profiles. By monotonicity, it holds that if such a profile P_e exists for an edge e, then for every $x \geq 0$, we have that $cost_{SO}(N[e \leftarrow x]) \leq cost(N[e \leftarrow x], P_e) \leq cost(N[e \leftarrow \infty], P_e) < \kappa$. If this holds for every edge, then $\langle N, \kappa\rangle \in L_2^c$. In the other direction, if there is an edge e such that for every profile P it holds that $cost(N[e \leftarrow \infty], P) \geq \kappa$, then $cost_{SO}(N[e \leftarrow \infty]) \geq \kappa$, and therefore $\langle N, \kappa\rangle \notin L_2^c$. Therefore, L_2^c is in NP, hence L_2 is in co-NP. We show that $L_1 \cap L_2 =$SO-control.

For the first direction, let $\langle N, \kappa\rangle \in$ SO-control. Therefore, there is an edge $e \in E$ and a value $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) = \kappa$. In particular, we have that $cost_{SO}(N[e \leftarrow x]) \leq \kappa$, therefore $\langle N\kappa\rangle \in L_1$. Furtheremore, $cost_{SO}(N[e \leftarrow x]) \geq \kappa$, therefore $\langle N, \kappa\rangle \in L_2$. Hence, $\langle N, \kappa\rangle \in L_1 \cap L_2$.

For the other direction, let $\langle N, \kappa\rangle \in L_1 \cap L_2$. Since $\langle N, \kappa\rangle \in L_1$, there is $e_1 \in E$ and $x_1 \geq 0$ such that $cost_{SO}(N[e_1 \leftarrow x_1]) \leq \kappa$. If $cost_{SO}(N[e_1 \leftarrow \infty]) \geq \kappa$, then by continuity and the intermediate value theorem, there is $x \geq 0$ such that $cost_{SO}(N[e_1 \leftarrow x]) = \kappa$, hence $\langle N, \kappa\rangle \in$ SO-control. If $cost_{SO}(N[e_1 \leftarrow \infty]) < \kappa$, we use the fact that $\langle N, \kappa\rangle \in L_2$. Hence, there is $e_2 \in E$ and $x_2 \geq 0$ such that $cost_{SO}(N[e_2 \leftarrow x_2]) \geq \kappa$. If $cost_{SO}(N[e_2 \leftarrow 0]) \leq \kappa$, then again by continuity and the intermediate value theorem, there is $x \geq 0$ such that $cost_{SO}(N[e_2 \leftarrow x]) = \kappa$. If $cost_{SO}(N[e_2 \leftarrow 0]) > \kappa$, then since $cost_{SO}(N[e_1 \leftarrow \infty]) < \kappa$ by Lemma 10, there is an edge $e \in E$ and a value $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) = \kappa$, and therefore $\langle N, \kappa\rangle \in$ SO-control.

We turn to prove that the problem is DP-hard. We reduce SAT-UNSAT to SO-control. SAT-UNSAT is the problem of deciding, given two 3CNF formulas φ_1 and φ_2, whether φ_1 is satisfiable and φ_2 is not satisfiable. That is, $\langle \varphi_1, \varphi_2 \rangle \in$ SAT-UNSAT iff there exists an assignment f_1 to the variables of φ_1 such that f_1 satisfies φ_1, and for all assignments f_2 to the variables of φ_2, it holds that f_2 does not satisfy φ_2. It was shown in [35] that SAT-UNSAT is DP-complete.

We propose the following reduction. For each formula φ_i, with $i \in \{1, 2\}$, we add a fresh variable z_i. We first construct a new formula φ'_i in the following way. For each clause, we disjunct the clause with z_i. We also conjunct the entire formula with $\neg z_i$. Note that if φ_i is satisfied by an assignment f_i, then φ'_i is satisfied by the assignment that agrees with f_i on all the variables in φ_i, and has $z_i = \textbf{false}$. Furthermore, if φ_i is unsatisfiable, then φ'_i is unsatisfiable. Indeed, an assignment that satisfies φ'_i must have $z_i = \textbf{false}$, implying that all other clauses are satisfied by an assignment that satisfies φ_i as well. Next, we construct an NFG $N_i = \langle k_i, V_i, E_i, c_i, \gamma_i \rangle$, for $i \in \{1, 2\}$, as follows (see Figure 2).

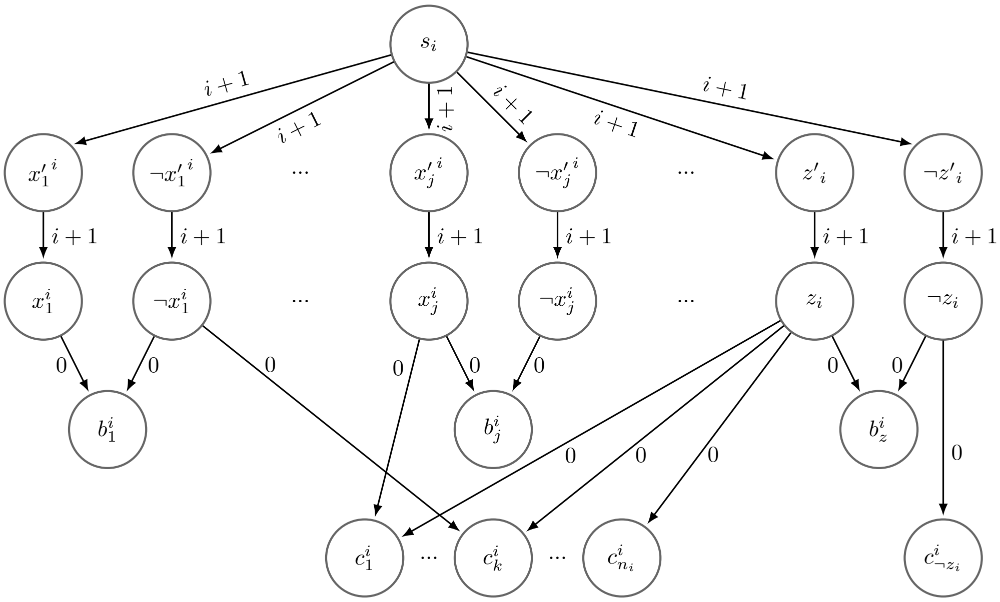

Figure 2 The NFG N_i; each edge denotes a set of two parallel edges with the same cost.

Let n_i be the number of variables in φ_i, and let m_i be the number of clauses in φ_i. Thus, the number of variables in φ'_i is $n_i + 1$, and the number of clauses in φ'_i is $m_i + 1$. We define $V_i = \bigcup_{1 \le j \le n_i+1} \{x^i_j, \neg x^i_j, {x'}^i_j, \neg {x'}^i_j, b^i_j\} \bigcup_{1 \le k \le m_i+1} \{c^i_k\} \cup \{s_i\}$. That is, for each variable x^i_j of φ'_i, we have in V_i two vertices for the variable x^i_j, denoted $x^i_j, {x'}^i_j$, two vertices for its negation $\neg x^i_j$, denoted $\neg x^i_j, \neg {x'}^i_j$, and another vertex, denoted b^i_j. We also have a vertex for each clause, and a source vertex. The edges and costs are as follows. There are two parallel edges, each with cost $i + 1$, from s_i to both ${x'}^i_j, \neg {x'}^i_j$ for every variable x^i_j of φ'_i. There are two parallel edges, each with cost $i+1$, from ${x'}^i_j$ to x^i_j and from $\neg {x'}^i_j$ to $\neg x^i_j$ for every variable x^i_j of φ'_i. There are two parallel edges, each with cost 0 from both $x^i_j, \neg x^i_j$ to b^i_j. Finally, for every clause c^i_k, there are two parallel edges, each with cost 0, from every literal appearing in c^i_k to the vertex c^i_k. Note that, in particular, this means that there are two parallel edges with cost 0 from z_i to all clauses except the clause $\neg z_i$. Finally, we have $k_i = n_i + 1 + m_i + 1$ players. The first $n_i + 1$ players are clause players, and the objective of Player $1 \le k \le n_i + 1$ is $\langle s_i, c^i_k \rangle$. The rest are variable players, and the objective of Player $n_i + 2 \le j \le n_i + m_i + 2$

is $\langle s_i, b^i_j \rangle$. To complete the construction, we fix $N = \langle k_1 + k_2, V_1 \cup V_2, E_1 \cup E_2, c_1 \cup c_2, \gamma_1 \cup \gamma_2 \rangle$ and $\kappa = 4n_1 + 6n_2 + 16$.

Note that since N_1 and N_2 are disjoint, it holds that $cost_{SO}(N) = cost_{SO}(N_1) + cost_{SO}(N_2)$. We argue that if φ_i, for $i \in [1, 2]$, is satisfiable, then $cost_{SO}(N_i) = 2(i+1)\cdot(n_i+1)$, and otherwise $cost_{SO}(N_i) = 2(i + 1) \cdot (n_i + 2)$. Thus, N has a distinct SO-cost to every combination of {SAT, UNSAT} × {SAT, UNSAT}, which enables us to point to a threshold κ such that $\langle \varphi_1, \varphi_2 \rangle \in$ SAT-UNSAT iff $\langle N, \kappa \rangle \in$ SO-control. Details can be found in the full version. ◀

4 Affecting the Best Nash Equilibrium

In this section we study the sensitivity of the best NE to cost mutations. As we shall see, while the setting is less clean than in the SO case, we are able to obtain the same complexity bounds for analogous decision problems.

4.1 Monotonicity and continuity of the bNE

▶ **Theorem 12.** [$cost_{bNE}$ **is not monotone**] *There is an NFG N and an edge e of N, such that the function $cost^e_{bNE}(N)$ is not monotone.*

Proof. Consider the NFG N appearing in Figure 3. The game is played between two players, with objectives $\langle s, t_1 \rangle$ and $\langle s, t_2 \rangle$. Let $e = \langle s, t_2 \rangle$. Table 5 describes the costs of the players in the possible four profiles of $N[e \leftarrow x]$. When $x \in [0, 1)$, the only NE is $\langle \pi^2_1, \pi^1_2 \rangle$, with cost $x + 2$. When $x > 1$, the only NE is $\langle \pi^2_1, \pi^2_2 \rangle$, with cost 2. So, for all $x \in (0, 1)$, we have that $cost_{bNE}(N[e \leftarrow x]) = 2 + x > 2 = cost_{bNE}(N[e \leftarrow 1])$, and thus $cost^e_{bNE}(N)$ is not monotone. ◀

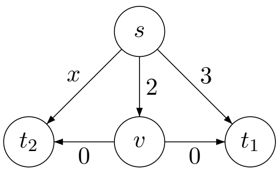

Figure 3 The NFG N.

Player 2 / Player 1	π^1_2 $s \to t_2$	π^2_2 $s \to v \to t_2$
π^1_1 $s \to t_1$	3, x	3, 2
π^2_1 $s \to v \to t_1$	2, x	1, 1

Table 5 Players' costs in N.

▶ **Theorem 13.** [$cost_{bNE}$ **is not continuous**] *There is an NFG N and an edge e of N, such that the function $cost^e_{bNE}(N)$ is not continuous.*

Proof. We use the same NFG N and edge e as in the proof of Theorem 12. It is easy to see that $cost^e_{bNE}(N)$ is not continuous at 1. ◀

While $cost_{bNE}$ is neither monotonous nor continuous, we now show that it is composed of finitely many linear segments. We say that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *composed of linear segments* if there is a segmentation $0 = x_0 < x_1 < ... < x_n < x_{n+1} = \infty$ of $\mathbb{R}^+$, for some $n \geq 0$, such that for every $0 \leq i \leq n$ there is a linear function $f_i : \mathbb{R} \to \mathbb{R}$ such that for all $x \in [x_i, x_{i+1}]$ it holds that $f(x) = f_i(x)$. We call $x_0, x_1, ..., x_{n+1}$ the *edge points* of f. Given an NFG N, a profile P, and an edge e, the cost of P is a linear function with respect to the cost of e. Indeed, $cost(N, P) = \sum_{e' \in P \setminus \{e\}} c(e') + \mathbb{1}_{P,e} c(e)$, where $\mathbb{1}_{P,e} \in \{0, 1\}$ is an indicator of e being used in P. In particular, when $\mathbb{1}_{P,e} = 0$, then $cost(N, P)$ is a constant function.

▶ **Lemma 14.** *Given an NFG N, an edge e, and a profile P, the range of values x such that P is an NE in $N[e \leftarrow x]$ is a single (possibly empty) segment.*

Proof. By definition, a profile P is an NE if for every i and for every profile P' obtained from P by a deviation π_i' of Player i that $cost_{N,P}(\pi_i) \leq cost_{N,P'}(\pi_i')$. Hence, P is an NE in $N[e \leftarrow x]$ in values x for which the set of constraints of the form $cost_{N,P}(\pi_i) \leq cost_{N,P'}(\pi_i')$ holds. As each constraint is a linear inequality in a single variable (that is, x), the solution set is a single (perhaps empty) segment. ◀

We denote by $bumps(P)$ the set of edge points of the segment along which P is an NE in $N[e \leftarrow x]$. That is, $bumps(P) = \{a, b\}$ if P is an NE in $N[e \leftarrow x]$ for exactly all $a \leq x \leq b$. By Lemma 14, $bumps(P)$ contains at most two points. We further denote by $Bumps(N, e) = \bigcup_P bumps(P)$. Since the number of strategies per player and the number of players are finite, the number of profiles is finite as well. Hence, since $|bumps(P)| \leq 2$ for every profile P, we get that $Bumps(N, e)$ is finite.

Consider two profiles $P_1 \neq P_2$ in N. For an edge e, we say that a value $x \geq 0$ is an *intersection point* for e, P_1, and P_2, if $cost(N[e \leftarrow x], P_1) = cost(N[e \leftarrow x], P_2)$. Note that since $cost(N[e \leftarrow x], P)$ is linear for every profile P, there is at most one intersection point for every edge and two profiles. Let $Ints(N, e)$ be the set of all intersection points for e and pairs of profiles in N. Since the number of different profiles is finite, so is $Ints(N, e)$.

▶ **Theorem 15.** *Consider an NFG N and an edge e in N. Then, $cost_{bNE}(N[e \leftarrow x])$ is composed of finitely many linear segments, and is monotonically increasing within each segment.*

Proof. Recall that $cost^e_{bNE}(N)(x) = cost_{bNE}(N[e \leftarrow x]) = \min_{P \in bNE(N[e \leftarrow x])} cost(N[e \leftarrow x], P) = \min_{P \in bNE(N[e \leftarrow x])} \sum_{e' \in P \setminus \{e\}} c(e') + \mathbb{1}_{P,e} x$. Hence, $cost_{bNE}(N[e \leftarrow x])$ is composed of linear segments. The set of edge points refines $bumps(N, e) \cup Ints(N, e)$, and since it is finite, so are the number of segments. Furthermore, as $cost(N[e \leftarrow x], P)$ is monotonically increasing for every P, we get that $cost_{bNE}(N[e \leftarrow x])$ is monotonically increasing within each segment. ◀

Figure 4 below contains plots[2] of the function $cost_{bNE}(N[e \leftarrow x])$. The left plot describes $cost_{bNE}(N[e \leftarrow x])$ where N is the NFG from Example 1 and $e = \langle s, u \rangle$. To its right, we describe a three-player NFG N and the plot of $cost_{bNE}(N[e \leftarrow x])$ with $e = \langle s, v_2 \rangle$.

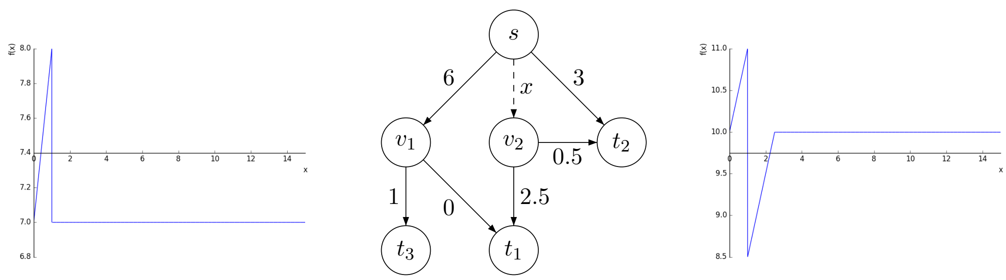

Figure 4 Plots for $cost_{bNE}(N[e \leftarrow x])$.

[2] The plots were generated by a simple Python program that gets as input an NFG by means of a NetworkX weighted directed graph, and naively follows the segmentation from Theorem 15.

4.2 Decision problems

The bNE-cost decision problem is the problem of deciding, given an NFG N and a threshold $\kappa \geq 0$, whether $cost_{bNE}(N) \leq \kappa$. The bNE-cost problem is NP-complete [4]. In this section we study the following related decision problems.

1. Edge-bNE-affects: Given an NFG N and an edge e of N, does e bNE-affect N? Thus, Edge-bNE-affects $= \{\langle N, e\rangle \mid e \text{ bNE-affects } N\}$.
2. Edge-bNE-optimization: Given an NFG N, an edge e of N, and a threshold $\kappa \geq 0$, is there a value $x \geq 0$, such that $cost_{bNE}(N[e \leftarrow x]) \leq \kappa$? Thus, Edge-bNE-optimization $= \{\langle N, e, \kappa\rangle \mid \text{there exists } x \geq 0 \text{ such that } cost_{bNE}(N[e \leftarrow x]) \leq \kappa\}$.
3. bNE-optimization: Given an NFG N and a threshold $\kappa \geq 0$, is there an edge e of N and a value $x \geq 0$, such that $cost_{bNE}(N[e \leftarrow x]) \leq \kappa$? Thus, bNE-optimization$= \{\langle N, \kappa\rangle \mid$ there exist e and $x \geq 0$ such that $cost_{bNE}(N[e \leftarrow x]) \leq \kappa\}$.

Before we turn to analyze the complexity of the problems, let us illustrate the non-intuitive behavior of $cost_{bNE}$. Consider the NFG N appearing in Figure 5, and let $e = \langle s, v_2\rangle$. As can be seen in Table 6, the profile $\langle \pi_1^3, \pi_2^3\rangle$ is an NE with cost 10 independent of the value of x. Then, when $0 \leq x \leq \frac{1}{2}$, the profile $\langle \pi_1^2, \pi_2^1\rangle$ is an NE with cost $10.5 + x$, and when $x \geq \frac{1}{2}$, the profile $\langle \pi_1^1, \pi_2^1\rangle$ is an NE with cost 9. Accordingly, $cost_{bNE}(N[e \leftarrow x])$ is 10 when $0 \leq x < \frac{1}{2}$, and is 9 when $x \geq \frac{1}{2}$. Though observations of the non-intuitive behavior of network exists in literature (e.g., Braess' Paradox [12]), it is common that added/removed edges participate in equilibria profiles either before or after changing the network. In this example, however, the edge e, which bNE-affects N, does not participate in any bNE profile! Thus, $cost_{bNE}$ is fixed in the two segments $[0, \frac{1}{2})$ and $[\frac{1}{2}, \infty]$, yet still e bNE affects N.

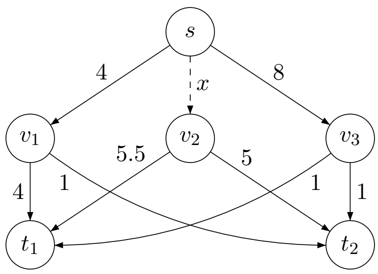

Figure 5 The NFG N.

Player 2 / Player 1	π_2^1 s, v_1, t_2	π_2^2 s, v_2, t_2	π_2^3 s, v_3, t_2
π_1^1 s, v_1, t_1	3 6	$5 + x$ 8	9 8
π_1^2 s, v_2, t_1	5 $5.5 + x$	$5 + \frac{x}{2}$ $5.5 + \frac{x}{2}$	9 $5.5 + x$
π_1^3 s, v_3, t_1	5 9	$5 + x$ 9	5 5

Table 6 Players' costs in N.

▶ **Lemma 16.** *Let N be an NFG, and let e be an edge in N. If there is an NE profile P such that $e \notin P$, then for all $x \geq c(e)$, we have that P is an NE in $N[e \leftarrow x]$.*

Proof. Assume towards contradiction that there is $x > c(e)$ such that P is not an NE. Then, there is a player i with strategy π_i in P that has an incentive to unilaterally deviate to another strategy π_i'. Denote by P' the deviation profile resulting from i's deviation. Since P is an NE in N, we have that $cost_{N,P}(\pi_i) \leq cost_{N,P'}(\pi_i')$. Since $e \notin P$, we have that $cost_{N[e\leftarrow x],P}(\pi_i) = cost_{N,P}(\pi_i)$. Since $x > c(e)$ we have that $cost_{N,P'}(\pi_i') \leq cost_{N[e\leftarrow x],P'}(\pi_i')$. Therefore $cost_{N[e\leftarrow x],P}(\pi_i) \leq cost_{N[e\leftarrow x],P'}(\pi_i')$, in contradiction to the fact that Player i has an incentive to deviate. ◀

Lemma 16, together with the segmentation of $bNE(N[e \leftarrow x])$, is used for proving the following characterization of an edge that does not bNE-affect N. The proof is based on a careful consideration of all cases and can be found in the full version.

▶ **Theorem 17.** *Let N be an NFG. An edge e in N does not bNE-affect N iff there is a profile $P \in bNE(N[e \leftarrow 0])$ such that $e \notin P$ and for all $x \geq 0$ it holds that $cost_{bNE}(N[e \leftarrow x]) \geq cost_{bNE}(N[e \leftarrow 0])$.*

▶ **Theorem 18.** *The* Edge-bNE-affects *problem is Δ_2^P-complete, and is Θ_2^P-complete when costs are given in unary.*

Proof. We start with membership. First, note that given an NFG N, and edge e of N, and a value $\kappa \geq 0$, we can decide in NP whether there is a profile P such that $e \notin P$ and $cost(N, P) = \kappa$.

Let $OPT_0 = cost_{bNE}(N[e \leftarrow 0])$. As argued in the membership claim for Theorem 8, we can find OPT_0 using polynomially-many queries to an NP oracle when costs are given in binary, and using logarithmically-many queries when costs are given in unary. Then, using a single query to Edge-bNE-optimization (with modification to strictly smaller) with input N, e, and OPT_0, we can decide if there is a value $x \geq 0$ such that $cost_{bNE}(N[e \leftarrow x]) < OPT_0$. If so, then e affects N. Otherwise, use a single query to ask if there is a profile P such that $e \notin P$ and $cost(N[e \leftarrow 0], P) = OPT_0$. By Theorem 17, we have that e bNE-affects N iff the answer is no.

The hardness results for Δ_2^P and Θ_2^P can be found in the full version. In both cases we use the same reduction as in the hardness results for Theorem 8. In the case of Δ_2^P we make a slight variation. Then we show that the profiles described for the SO is a superset of the bNE profiles. ◀

Finally, for the optimization problems, the analysis is similar to the one in Theorem 9, except that we also have to argue that the witness value x is polynomial in input. The details can be found in the full version.

▶ **Theorem 19.** *The* edge-bNE-optimization *and* bNE-optimization *problems are NP-complete.*

▶ Remark 20. **[On the PoS and the worst NE]** Recall that $PoS(N) = \frac{cost_{bNE}(N)}{cost_{SO}(N)}$. If an edge e bNE-affects N, it does not necessarily imply that e PoS-affects N. Indeed, e may participate also in the SO. Nevertheless, the NFG N used in the proofs of Theorems 12 and 13 demonstrates that PoS is neither monotone nor continuous. To see this, note that for all $x \geq 0$, we have that $cost_{SO}(N[e \leftarrow x]) = 2$, we get that for $x \in [0, 1)$, we have that $PoS(N[e \leftarrow x]) = 1 + \frac{x}{2}$, and for $x \geq 1$, we have that $PoS(N[e \leftarrow x]) = 1$.

As for the worst NE, since the NFG N used in the proofs of Theorems 12 and 13 is such that $N[e \leftarrow x]$ has a single NE for all values of x, the considerations about the best and worst NE coincide, and thus N demonstrate that $cost_{wNE}$ is neither monotone nor continuous.

5 Discussion and Future Work

We studied the effect of mutations applied to the cost of edges in network formation games. Our results about monotonicity and continuity of the SO and NE are aligned with similar folk results in similar settings in game theory. We are, however, the first to introduce a formal framework to study these phenomena, and to provide a complexity analysis of the decision problems they induce. We also point to new surprising effects of the mutations.

The mutations we study for NFGs are of a restricted type: an unbounded change in the cost of a single resource in the game. As has been the case in coverage and vacuity in formal verification, richer types of mutations reflect practical bounds on the possible mutations. For example, it would be interesting to study how one can control the bNE by a budget-restricted mutation of several edges. Also, while our definition of affect is Boolean, namely an edge SO-, bNE-, or wNE-affects a network or it does not, it is interesting to examine a quantitative approach, where we care how much an edge affects these measures. Finally, while our optimization problems care about an upper bound to the costs of the SO and bNE, in some applications it is interesting to control these values by both an upper and lower bound. We leave the richer setting and variants for future research.

Both game theory and formal verification aim at reasoning about behaviors of interacting entities, yet consider different aspects of the interaction. We view this work as another chain in an exciting transfer of concepts and ideas between the two areas [28]. In the context of game theory, this includes an extension of NFGs to objectives that are richer than reachability [9], to a timed setting [6], and to a setting where the strategies of the players are dynamic [7]. Beyond richer settings, it is shown in [30, 5] how ideas used in formal verification for abstraction and symbolic presentation of huge systems can be used for reasoning about NFGs. In the other direction, concepts from game theory are used in the formalization of strategic behaviors in formal verification (e.g., rational verification and synthesis [22, 39]). In the more economic view, cost-sharing mechanisms from NFGs are used in [8] in order to augment the problem of synthesis from component libraries by cost considerations.

Our contribution here started with the transfer of concepts from formal verification to game theory, yet our results suggest new research directions in coverage and vacuity in formal verification, and logic in general. Studies of coverage and vacuity so far concern Boolean specification formalisms [27]. In contrast, the objectives of the players in typical game-theoretic settings, in particular NFGs, are quantitative. Recently, there is growing interest in *multi-valued* specification formalisms, which specify the *quality* of systems, and not only their correctness [2]. Moreover, the systems we reason about may be multi-valued too. For the multi-valued setting, we need to develop a theory of quantified multi-valued propositions. In particular, the segmentation of values in $\mathbb{R}^+$ we perform for bNE, is analogous to a segmentation of $[0, 1]$ – the domain of values of atomic propositions and sub-formulas in typical multi-valued formalisms. Indeed, while mutations of sub-formulas that appear in a positive or negative polarity behave monotonically, sub-formulas with a mixed polarity may induce a non-trivial segmentation. Moreover, as has been the case with $bumps(P)$ in the bNE segmentation, the edge points of the segments may not be constants that appear in the formula. For example, when sub-formulas and atomic propositions take values in $[0, 1]$, then the maximal satisfaction value of the formula $p \wedge (\neg p)$ is when the satisfaction value of p is $\frac{1}{2}$.

Furthermore, the need to reason formally about multi-agent systems has led to a development of specification formalisms that enable reasoning about on-going strategic behaviors [3, 13, 32, 11]. Essentially, these formalisms, most notably ATL, ATL*, and Strategy Logic (SL), include quantification of strategies of the different agents and of the computations they may force the system into, making it possible to specify concepts like SO and NE. While coverage and vacuity are traditionally viewed as sanity checks in model checking, in the context of SL specifications, they can also be used for revealing properties of games and strategic behaviors. Out work demonstrates how SL formulas that specify concepts like SO and NE explain properties like monotonicity. Indeed, non-monotonicity of the bNE corresponds to the mixed polarity of the objectives in the SL formula that describes an NE: a negative occurrence (left-hand side of an implication) when we refer to a deviation and a positive one (right-hand side of that implication) in for the current strategy. In contrast, in the formula for the SO, all occurrences of the objectives are positive, implying monotonicity. Moreover, for a specific given game, reasoning about the effect of mutations can be reduced to checking the coverage of SL formulas that specify properties of the game. Thus, a framework for coverage and vacuity in SL is interesting for both formal verification and game theory.

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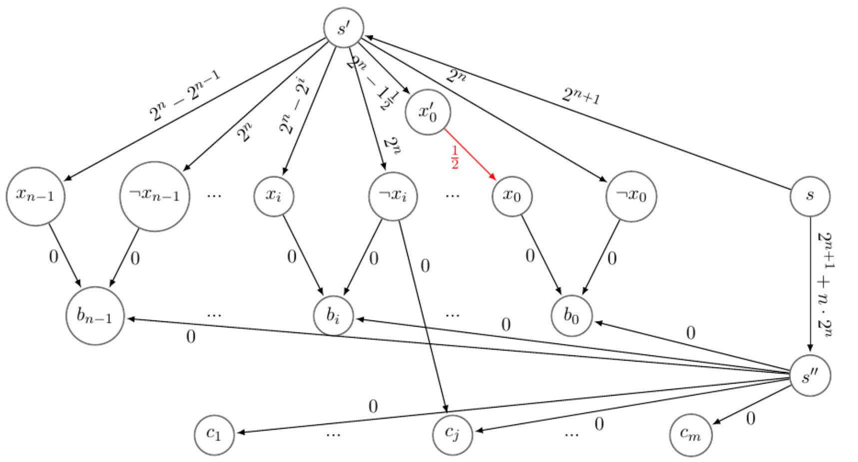
A Proofs

A.1 The Δ_2^P lower bound in Theorem 8

For a 3CNF formula φ, denote by n and m the number of variables and clauses in φ, respectively. We assume that $n > 2$. We assume that some order $x_{n-1}, \ldots, x_1, x_0$ over the variables, with x_0 being minimal in the order, and denote the clauses by $c_1, \ldots, c_m$. Given φ, we define the NFG $N = \langle n+m, V, E, c, \gamma \rangle$ as follows. The set of vertices is $V = \{x_i, \neg x_i, b_i\}_{i=0}^{n-1} \cup \{c_j\}_{j=1}^{m} \cup \{x_0', s, s', s''\}$. The edges and their costs are as follows. There is an edge with cost 2^{n+1} from s to s', and an edge with cost $2^n + n \cdot 2^n$ from s to s''. Next, there is an edge with cost 0 from s'' to b_i for all $0 \leq i \leq n-1$ and to c_j for all $1 \leq j \leq m$. For all $1 \leq i \leq n-1$, there is an edge with cost $2^n - 2^i$ from s' to x_i, an edge with cost 2^n from s' to $\neg x_i$, and an edge with cost 0 from both $x_i, \neg x_i$ to b_i. There is an edge with cost 2^n from s' to $\neg x_0$, an edge with cost $2^n - 2^0 - \frac{1}{2} = 2^n - 1\frac{1}{2}$ from s' to x_0', and an edge with cost $\frac{1}{2}$ from x_0' to x_0. As for all other variables, there is an edge with cost 0 from both $x_0, \neg x_0$ to b_0. For every $1 \leq j \leq m$, there is an edge with cost 0 from l_i to c_j for every literal l_i appearing in c_j. We partition the $n+m$ players into n variable players, where the objective of Player i, for $1 \leq i \leq n$, is $\langle s, b_{i-1} \rangle$, and m clause players, where the objective of Player $n+j$, for $1 \leq j \leq m$ is $\langle s, c_j \rangle$. Finally, we set $e = (x_0', x_0)$. A scheme of the construction is given in Figure 6. Note that the formula φ influences only the edges from the literals to the clause vertices, and all other edges depend only on n and m.



■ **Figure 6** The NFG N.

The construction is polynomial, as the NFG N has $O(n+m)$ vertices and edges and costs are exponential in n, thus require only $O(n)$ bits to represent. Note that when costs are given in unary, the construction is exponential, thus, this result does not affect Θ_2^P-completeness in that case.

We say that a profile P defines a satisfying assignment if $\langle s, s'' \rangle \notin P$ and for every $0 \leq i \leq n-1$ it holds that the path from s' to x_i is in P iff the path from s' to $\neg x_i$ is not in P. That is, the variable players in P define an assignment by choosing between the path from s' to x_i and the path from s' to $\neg x_i$, and the clause players can reach their objective using only non-zero edges that are used by a variable player. Denote by f_P the assignment that is induced by the choices the variable players make in P, then φ is satisfied by f_P since every clause player chose a path that only use non-zero edges that a variable players uses. The path that the variable player chose induces a literal that is present in the clause, thus, it is satisfied. Note that every satisfying assignment f, using this profile construction, induces at least one profile that defines a satisfying assignment P, and it holds that $f = f_P$. For an assignment f, let $\lfloor f \rfloor_{10}$ denote the decimal value of f. For example, if $n = 5$, $f(x_0) = f(x_2) = 0$ and $f(x_1) = f(x_3) = f(x_4) = 1$, then $\lfloor f \rfloor_{10} = 2^1 + 2^3 + 2^4 = 26$.

We argue that for every profile P that defines a satisfying assignment f_P it holds that if

$f_P(x_0) = 1$ then for every $x \geq 0$ it holds that $cost(N[e \leftarrow x], P) = 2^{n+1} - n \cdot 2^n - \lfloor f_P \rfloor_{10} - \frac{1}{2} + x$, and otherwise $cost(N[e \leftarrow x], P) = 2^{n+1} + n \cdot 2^n - \lfloor f_P \rfloor_{10}$. The cost of P is determined by the variable players, as the clause players only use edges with non-zero cost that the variable players are using. Then, if $x_0 = 1$, the sum of costs of paths used in P from s' is $\sum_{1 \leq i \leq n-1} f_P(x_i)(2^n - 2^i) + 2^n - 2^0 - \frac{1}{2} + x = n \cdot 2^n - \sum_{0 \leq i \leq n-1} f_P(x_i) \cdot 2^i - \frac{1}{2} + x = n \cdot 2^n - \lfloor f_P \rfloor_{10} - \frac{1}{2} + x$. Otherwise, the sum of costs of paths used in P from s' is $\sum_{0 \leq i \leq n-1} f_P(x_i)(2^n - 2^i) = n \cdot 2^n - \sum_{0 \leq i \leq n-1} f_P(x_i) \cdot 2^i = n \cdot 2^n - \lfloor f_P \rfloor_{10}$ The only other non-zero edge that is used in this profile is $\langle s, s' \rangle$ with cost 2^{n+1}, thus the total cost of P is $cost(N[e \leftarrow x], P) = 2^{n+1} + n \cdot 2^n - \lfloor f_P \rfloor_{10}$ if $x_0 = 0$ and $cost(N[e \leftarrow x], P) = 2^{n+1} + n \cdot 2^n - \lfloor f_P \rfloor_{10} - \frac{1}{2} + x$.

Next, if φ is satisfiable, let f_{max} be a maximal lexicographic satisfying assignment. Denote by P_{max} a profile that defines a satisfying assignment such that $f_P = f_{max}$. By the observation above, note that at least one such profile exists. We argue that for $0 \leq x \leq \frac{1}{2}$, for every profile P it holds that $cost(N[e \leftarrow x], P_{max}) \leq cost(N[e \leftarrow x], P)$. We distinguish between the following cases:

- P defines a satisfying assignment f_P. If $f_{max} = f_P$, then the variable players in both profiles have the same strategies. Since in the case of a profile that defines a satisfying assignment it holds that the strategies of the clause players do not affect the cost of the profile, we have that for every $x \geq 0$ it holds that $cost(N[e \leftarrow x], P_{max}) = cost(N[e \leftarrow x], P)$. Otherwise, by maximality of f_{max}, it holds that $\lfloor f_{max} \rfloor_{10} \geq \lfloor f_P \rfloor_{10} + 1$. Then, for $0 \leq x \leq \frac{1}{2}$ we have that $cost(N[e \leftarrow x], P_{max}) \leq cost(N[e \leftarrow \frac{1}{2}], P_{max}) = 2^{n+1} + n \cdot 2^n - f_{max} \leq 2^{n+1} + n \cdot 2^n - (f_P + 1) = 2^{n+1} + n \cdot 2^n - f_P - 1 \leq 2^{n+1} + n \cdot 2^n - f_P - \frac{1}{2} + x \leq cost(N[e \leftarrow x], f_P)$.
- P does not define a satisfying assignment. Then, by definition either $\langle s, s'' \rangle \in P$, in which case $cost(N[e \leftarrow x], P) \geq 2^{n+1} + n \cdot 2^n \geq 2^{n+1} + n \cdot 2^n - \lfloor f_{max} \rfloor_{10} \geq cost(N[e \leftarrow x], P_{max})$ for every $0 \leq x \leq \frac{1}{2}$, or there is a variable x_i such that both the path from s' to x_i and the path from s' to $\neg x_i$ are in P. The minimal cost of such a profile for $0 \leq x \leq \frac{1}{2}$ is attained where for all $0 \leq i \leq n-1$, the path from s' to x_i is in P, and there is a single variable x_i such that the path from s' to $\neg x_i$ is in P. The sum of costs of the paths from s' to x_i for all $0 \leq x_i \leq n-1$ is $n \cdot 2^n - (2^n - 1) - \frac{1}{2} + x$. The path from s' to $\neg x_i$ adds 2^n to the total cost, and the edge $\langle s, s' \rangle$ adds an additional 2^{n+1} to the total cost of the profile. Thus, $cost(N[e \leftarrow x], P) = 2^{n+1} + n \cdot 2^n + \frac{1}{2} + x > cost(N[e \leftarrow x], P_{max})$.

Thus, for $0 \leq x \leq \frac{1}{2}$, we have that P_{max} is minimal in cost.

Assume first that φ is satisfiable and that in a maximal lexicographic assignment f_{max} it holds that $f_{max}(x_0) = 1$. Let P_{max} as above. Since for $0 \leq x \leq \frac{1}{2}$ we have that P_{max} is minimal in cost, $P_max \in SO(N[e \leftarrow x])$. Thus, for every such x we have that $cost_{SO}(N[e \leftarrow x]) = cost(N[e \leftarrow x], P_{max})$. In particulat, we have that $cost_{SO}(N[e \leftarrow 0]) = cost(N[e \leftarrow 0], P_{max}) = 2^{n+1} + n \cdot 2^n - \lfloor f_{max} \rfloor_{10} - \frac{1}{2} < 2^{n+1} + n \cdot 2^n - \lfloor f_{max} \rfloor_{10} = cost(N[e \leftarrow \frac{1}{2}], P_{max}) = cost_{SO}(N[e \leftarrow \frac{1}{2}])$, hence e SO-affects N, therefore $\langle N, e \rangle \in$Edge-SO-affects.

Next, assume that either φ is not satisfiable or that the maximal lexicographic assignment has $x_0 = 0$. We distinguish between the two cases:

- φ is unsatisfiable. Note that it follows that for every profile P, we have that P does not define a satisfying assignment. Then let P_{UNSAT} be the profile where for every player with objective $\langle s, t \rangle$, her strategy is $\{\langle s, s'' \rangle, (s'', t)\}$. For all $x \geq 0$ it holds that $cost(N[e \leftarrow x], P_{UNSAT}) = 2^{n+1} + n \cdot 2^n$. Note that P_{UNSAT} is the only valid profile that uses $\langle s, s'' \rangle$. For every other profile P, if both $\langle s, s' \rangle, \langle s, s'' \rangle \in P$ then $cost(N[e \leftarrow x], P_{UNSAT}) \leq 2^{n+2} + n \cdot 2^n \leq cost(N[e \leftarrow 0], P) \leq cost(N[e \leftarrow x], P)$. Otherwise, since P does not define a satisfying assignment, it must hold that $\langle s, s' \rangle \in P$, $\langle s, s'' \rangle \notin P$ and there is a variable x_i such that both the path from s' to x_i and the path from s' to $\neg x_i$ are in P. Using a similar argument as above, the minimal cost of such a profile, for all $x \geq 0$, is $2^{n+1}+n\cdot 2^n+\frac{1}{2}$. Hence, $cost(N[e \leftarrow x], P_{UNSAT}) < 2^{n+1}+n\cdot 2^n+\frac{1}{2} \leq cost(N[e \leftarrow x], P)$. Therefore, for all $x \geq 0$ we have that $P_{UNSAT} \in cost_{SO}(N[e \leftarrow x])$. Since $e \notin P_{UNSAT}$, we get that e does not SO-affect N.

- If φ is satisfiable, and the maximal lexicographic assignment f_{max} has $f_{max}(x_0) = 0$, then let P_{max} as above. As we previously saw, for $0 \leq x \leq \frac{1}{2}$, for every other profile P we have that $cost(N[e \leftarrow x], P_{max}) \leq cost(N[e \leftarrow x], P)$. Since $e \notin P_{max}$, for every $x \geq 0$ we have that $cost(N[e \leftarrow x], P_{max}) = cost(N[e \leftarrow 0], Pmax) \leq cost(N[e \leftarrow 0], P) \leq cost(N[e \leftarrow x], P)$. Since the cost of P_{max} is constant for all $x \geq 0$, we have that e does not SO-affect N.

In either case e does not SO-affect N, thus, $\langle N, e\rangle \notin$Edge-SO-affects.

A.2 The Θ_2^P lower bound in Theorem 8

A *vertex cover* (VC, for short) for G is a set $C \subseteq V$ such that for all edges $\langle v, v'\rangle \in E$, we have $\{v, v'\} \cap C \neq \emptyset$. We use a reduction from VC-compare, namely the problem of deciding, given two undirected graphs $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, whether the size of a minimal vertex cover of G_1 is less than or equal to the size of a minimal vertex cover of G_2. It is shown in [37] that the problem is Θ_2^P-complete. We first argue we can assume that $|V_1| = |V_2|$ and $|E_1| = |E_2|$. Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_1)$. Assume w.l.o.g that $|E_1| + k = |E_2|$, for some $k > 0$. By adding to G_1 two vertices that are connected by an edge, we adds a single edge to G_1 and increase its VC by 1. By adding to G_2 a vertex connected to $k+1$ other vertices to G_2 (as we argue below, we can assume that $|V_2| > k$), we add $k+1$ edges to E_2 and increase its VC by 1. Therefore, we can get two new graphs with an equal number of edges, with both VCs being increased by 1. Also, since adding isolated vertices does not change the number of edges nor the size of a VC, we can easily adjust the sizes of V_1 and V_2, namely make sure $|V_2| > k$ and $|V_1| = |V_2|$.

Given G_1 and G_2 with $|V_1| = |V_2| = n$ and $|E_1| = |E_2| = m$, we construct an NFG N and an edge e in it such that $\langle G_1, G_2\rangle \in$ VC-compare iff $\langle N, e\rangle \in$ Edge-SO-affects. We define $N = \langle k, V, E, c, \gamma\rangle$ as follows. First, $V = V_1 \cup V_2 \cup E_1 \cup E_2 \cup \{s_i\}_{i=1}^m \cup \{t_1, t_1', t_2, t_2', t\}$. That is, for each graph G_i, for $i \in \{1, 2\}$, the set V includes n vertices, termed vertex-vertices, and m vertices, termed edge-vertices. In addition, V includes m source vertices, a target vertex t, and four sub-target vertices t_1, t_1', t_2, and t_2'. Let $V_1 = \{v_1^1, \ldots, v_1^n\}$, $V_2 = \{v_2^1, \ldots, v_2^n\}$, $E_1 = \{e_1^1, ..., e_m^1\}$, and $E_2 = \{e_1^2, ..., e_m^2\}$. We define $E = \{\langle s_i, e_i^1\rangle | e_i^1 \in E_1\} \cup \{\langle s_i, e_i^2\rangle | e_i^2 \in E_2\} \cup \{\langle e_i^1, v_j^1\rangle | e_i^1 \in E_1$ and there exists $v_k^1 \in V_1$ such that $e_i^1 = \langle v_j^1, v_k^1\rangle\} \cup \{\langle e_i^2, v_j^2\rangle | e_i^2 \in E_2$ and there exists $v_k^2 \in V_2$ such that $e_i^2 = \langle v_j^2, v_k^2\rangle\} \cup \{\langle v_i^1, t_1\rangle | v_i^1 \in V_1\} \cup \{\langle v_i^2, t_2\rangle | v_i^2 \in V_2\} \cup \{\langle t_1, t_1'\rangle, \langle t_2, t_2'\rangle, \langle t_1', t\rangle, \langle t_2', t\rangle\}$.

The edges of N and their costs are as follows. For each $1 \leq i \leq m$, there is an edge with cost 0 from the source vertex s_i to the edge vertices e_i^1 and e_i^2. For every $e_i^1 = \langle v_{j_1}^1, v_{j_2}^1\rangle \in E_1$, there are edges with cost 0 from e_i^1 to v_{j_1} and to v_{j_2}, and the same for E_2. For every vertex $v^1 \in V_1$, there is an edge with cost 1 to t_1, and the same for V_2. There is an edge with cost n from t_1 to t_1', and an edge with cost $n+1$ from t_2 to t_2'. We then connect both t_1' and t_2' to t with cost 0. To complete the construction, we have m players. The objective of player $1 \leq i \leq m$ is $\langle s_i, t\rangle$. Finally, we set $e = \langle t_1', t\rangle$. A scheme of the construction is given in Figure 7. Note that it follows from the construction that for every $x \geq 0$, if a profile is in the SO of $N[e \leftarrow x]$, then either the strategies of all players use edges from G_1's side of the network, or the strategies of all players use edges from G_2's side. Otherwise, it must be that $\langle t_1, t_1'\rangle, \langle t_1', t\rangle$ and $\langle t_2, t_2'\rangle$ are in the profile, and therefore the cost of the profile is strictly

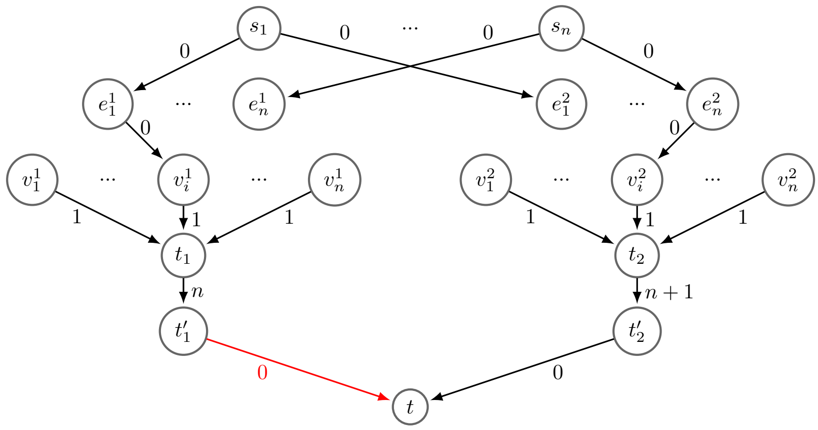


Figure 7 The NFG N.

greater than $2n+1+x$. However, the cost of a profile that uses edges from only one side of the network is bounded by $n+n+x \leq 2n+1+x$.

Let S_1 be a vertex cover of G_1. We denote by P_{S_1} the following profile. For every player $1 \leq i \leq m$, the strategy for player i is $\{(s_i, e_i^1), (e_i^1, v_j^1), (v_j^1, t_1), (t_1, t_1'), (t_1', t)\}$, where $v_j^1 \in S_1$. Since S_1 is a vertex cover for G_1, there must be such a vertex v_j^1 for every i. We use a similar notation for G_2. Furthermore, let P be a profile where all players only use G_1's side of the network. We denote by $S_P = \bigcup_{i=1}^{m} \{v | (e_i^1, v) \in P\}$. That is, the union of all vertices that the players chose in their strategies. Note that S_P is a vertex cover of G_1 since every player is associated with an edge, and each player selects a vertex that is adjacent to the edge she is associated with. We use similar notation for G_2. Note that by construction, if S is a vertex cover for G_1 then $\mathit{cost}(N[e \leftarrow x], P_S) = |S| + n + x$, and if S is a vertex cover of G_2 then $\mathit{cost}(N, P_S) = |S| + n + 1$.

Assume first that $\langle G_1, G_2 \rangle \in$ VC-compare, that is, the size of a minimal vertex cover of G_1 is less than or equal to the size of a minimal vertex cover of G_2. Let S_1, S_2 be minimal vertex covers of G_1, G_2, respectively. We argue that P_{S_1} is an SO profile of N. Assume towards contradiction that there is a profile P' such that $\mathit{cost}(N, P') < \mathit{cost}(N, P_{S_1})$. By the above observation, P' only use one side of the network. If P' only uses G_1's side, then $|S_{P'}| = \mathit{cost}(N, P') - n < \mathit{cost}(N, P) - n = |S_1|$, in contradiction to S_1 being a minimal vertex cover. Otherwise, if P' only uses G_2's side, then $|S_2| \leq |S_{P'}| = \mathit{cost}(N, P') - n < \mathit{cost}(N, P) - n = |S_1|$, in contradiction to the assumption.

Next, we define $d = |S_2| - |S_1| + 1$. We argue that $P_{S_2} \in \mathit{SO}(N[e \leftarrow d])$. Assume towards contradiction that there is a profile P such that $\mathit{cost}(N[e \leftarrow d], P) < \mathit{cost}(N[e \leftarrow d], P_{S_2})$. If P only uses edges from G_2's side of the network, then $|S_P| = \mathit{cost}(N, P) - n < \mathit{cost}(N, P_{S_2}) - n = |S_2|$, in contradiction to the minimality of S_2 in G_2. Otherwise, if P only uses edges from G_1's side of the network, then $\mathit{cost}(N[e \leftarrow d], P) = |S_P| + n + d \geq |S_1| + n + d = |S_2| + n + 1 = \mathit{cost}(N[e \leftarrow d], P_{S_2})$, in contradiction to the assumption. Therefore, $\mathit{cost}_{SO}(N[e \leftarrow 0]) = \mathit{cost}(N, P_{S_1}) = |S_1| + n \leq |S_2| + n < |S_2| + n + 1 = \mathit{cost}(N[e \leftarrow d], P_{S_2}) = \mathit{cost}_{SO}(N[e \leftarrow d]) \leq \mathit{cost}_{SO}(N[e \leftarrow \infty])$, hence e SO-affects N, and $\langle N, e \rangle \in$ Edge-SO-affects.

Next, assume that $\langle G_1, G_2 \rangle \notin$ VC-compare, that is, the size of a minimal vertex cover of G_1 is strictly larger than the size of a minimal vertex cover of G_2. Let S_1, S_2 be minimal vertex covers of G_1, G_2, respectively. We argue that $P_{S_2} \in \mathit{SO}(N)$. Assume towards contradiction that there is a profile P such that $\mathit{cost}(N, P) < \mathit{cost}(N, P_{S_2})$. If P only uses edges from

G_2's side of the network, then $|S_P| = cost(N, P) - n - 1 < cost(N, P_{S_2}) - n - 1 = |S_2|$, in contradiction to the minimality of S_2. If P only uses edges from G_1's side of the network, then since $|S_1| > |S_2|$ we have that $|S_1| \geq |S_2| + 1$. Therefore, $cost(N, P) = |S_P| + n \geq |S_1| + n \geq |s_2| + n + 1 = cost(N, P_{S_2})$, in contradiction to the assumption. Note that this holds regardless of the value of e, therefore for all $x \geq 0$ we have that $P \in SO(N[e \leftarrow x])$, hence $cost_{SO}(N[e \leftarrow 0]) = cost_{SO}(N[e \leftarrow \infty])$, hence e does not SO-affect N, and $\langle N, e\rangle \notin$ Edge-SO-affects.

A.3 Proof of Theorem 9

We start with membership of Edge-SO-optimization in NP. Given an NFG N, an edge e in N, and a threshold $\kappa \geq 0$, it can be verified in polynomial time that a witness P is a valid profile with $cost(N[e \leftarrow 0], P) \leq \kappa$. By Theorem 4, there is a value $x \geq 0$, such that $cost_{SO}(N[e \leftarrow x]) \leq \kappa$ iff $cost_{SO}(N[e \leftarrow 0]) \leq \kappa$. Hence, it is sufficient to consider $N[e \leftarrow 0]$. Furthermore, if $cost(N[e \leftarrow x], P) \leq \kappa$, then by minimality of the SO, it holds that $cost_{SO}(N[e \leftarrow x]) \leq cost(N[e \leftarrow x], P) \leq \kappa$. In the case of SO-optimization, the edge e is part of the witness.

Next, we show that the problems are NP-hard by reductions from the SO-cost problem. The reduction to Edge-SO-optimization is trivial: Given N and κ, we construct N' by adding to N an isolated edge e, which does not SO-affect N. It is easy to see that $\langle N, \kappa\rangle \in$ SO-cost iff $\langle N', e, \kappa\rangle \in$ Edge-SO-optimization. In the case of SO-optimization we cannot point to e, and the reduction is more complicated. Again we construct N' by adding to N an isolated edge e. In addition, we add to N a player that has to include e in her strategy, and we set the cost of e to $\kappa + 1$. Accordingly, if the cost of some edge can be changed in a way that causes the cost of the SO to go below κ, then this edge must be e, and so $\langle N, \kappa\rangle \in$ SO-cost iff $\langle N', \kappa\rangle \in$ SO-optimization.

Formally, let $N = \langle k, V, E, c, \gamma\rangle$ and $\kappa \geq 0$. We define $N' = \langle k+1, V', E', c', \gamma'\rangle$, where $V' = V \cup \{s, t\}$, $E' = E \cup \{\langle s, t\rangle\}$, $\gamma' = \gamma \cup \{\langle s, t\rangle\}$, and c' agrees with c on all edges in E and $c(\langle s, t\rangle) = \kappa + 1$. Assume first that $cost_{SO}(N) \leq \kappa$. Therefore, $cost_{SO}(N'[e \leftarrow 0]) = cost_{SO}(N) \leq \kappa$, and hence $\langle N', \kappa\rangle \in$ SO-optimization. Next, assume that $cost_{SO}(N) > \kappa$. Let $P' = \langle \pi_1, ..., \pi_{k+1}\rangle$ be a profile in N'. We denote by P the profile P' without the strategy of Player $k+1$; that is, $P = \langle \pi_1, ..., \pi_k\rangle$. Note that P is a profile in N. If we set the cost of e to be 0, then $cost(N'[(s, t) \leftarrow 0], P') = cost(N, P)$. By the minimality of the SO for N, we have that $cost(N, P) \geq cost_{SO}(N) > \kappa$, thus $cost(N', P') > k$. If we set the cost of an edge $e \neq (s, t)$ to be 0, then since $c((s, t)) = \kappa + 1$, we have that $cost(N'[e \leftarrow 0], P') \geq \kappa + 1 > \kappa$. Since this holds for all profiles P' and for all $e \in E'$, we have that $cost_{SO}(N'[e \leftarrow x]) \geq cost_{SO}(N'[e \leftarrow 0]) > \kappa$, for all $x \geq 0$. Hence $\langle N', \kappa\rangle \notin$ SO-optimization, and we are done.

A.4 DP-hardness proof in Theorem 11

For the construction described in the proof for Theorem 11, we argue that if φ_i for $i \in [1, 2]$ is satisfiable then $cost_{SO}(N_i) = 2(i+1) \cdot (n_i + 1)$, and otherwise $cost_{SO}(N_i) = 2(i+1) \cdot (n_i + 2)$.

Assume first that φ_i is satisfiable. Then, φ'_i is satisfiable. Let f^i be a satisfying assignment for φ'_i. We construct the following profile P. For each variable player j, her strategy is $\{(s_i, l'_j), (l'_j, l_j), (l_j, b^i_j)\}$, where $l_j = x^i_j$ if $f^i(x_j) = \mathbf{true}$ and $l_j = \neg x^i_j$ if $f^i(x_j) = \mathbf{false}$. Next, for each Clause Player k, her strategy is $\{(s_i, l'_j), (l'_j, l_j), (l_j, c^i_k)\}$, where l_j is a literal that is satisfied by f^i. Note that since f^i satisfies φ'_i, for each clause there must be at least one literal that is satisfied by f^i. Next, each variable player j has exactly two available strategies- $\{(s_i, x_j^i), ({x'_j}^i, x_j^i), (x_j^i, b_j)\}$ and $\{(s_i, \neg {x'_j}^i), (\neg {x'_j}^i, \neg x_j^i), (\neg x_j^i, b_j^i)\}$, each with cost $2(i+1)$. Since all variable players do not have an option to share edges with each other, it follows that for every profile, each variable player contributes $2(i+1)$ to the total cost of the profile. Therefore, for every profile, the cost of the profile is at least $2(i+1) \cdot (n_i+1)$. Since P attains the minimal cost of a profile in N_i, it is an SO profile with cost $2(i+1) \cdot (n_i+1)$.

Next, assume that φ_i is unsatisfiable. Therefore, φ'_i is unsatisfiable. Assume first that $cost_{SO}(N_i) < 2(i+1) \cdot (n_i+2)$. We construct the following profile P. For every clause player k except for the clause of $\neg z_i$, her strategy is $\{(s_i, z_i'), (z_i', z_i), (z_i, c_k^i)\}$. We set the strategy of the clause player of $\neg z_i$ to be $\{(s_i, \neg z_i'), (\neg z_i', \neg z_i), (\neg z_i, c_{\neg z_i})\}$. The strategy of every variable player is assigned at random. Every variable player (except for z_i) contributes $2(i+1)$ for the total cost of P. Since $(s_i, z_i'), (z_i', z_i), (s_i, \neg z_i'), (\neg z_i', \neg z_i) \in P$, and since all other clause players don't contribute any other non-zero edges to the profile, we have that the cost of P is $2(i+1) \cdot (n_i+2)$. Next, assume towards contradiction that $cost_{SO}(N_i) < 2(i+1) \cdot (n_i+2)$. Since all non-zero edge in E_i have cost $i+1$, and since every strategy that has $(s_i, {l'_j}^i)$ for some literal l_j^i must also include $({l'_j}^i, l_j^i)$ the cost of every profile must be divisible by $2(i+1)$. Since the cost of every profile is at least $2(i+1) \cdot (n_i+1)$, it follows that $cost_{SO}(N_i) = 2(i+1) \cdot (n_i+1)$. Let $P \in SO(N_i)$. We define an assignment f^P as follows. For each variable player j, if j's strategy in P is $\{(s_i, {x'_j}^i), ({x'_j}^i, x_j^i), (x_j^i, b_j^i)\}$ then $f^P(x_j^i) = \textbf{true}$. Otherwise, $f^P(x_j^i) = \textbf{false}$. Now, since the cost of P is $2(i+1) \cdot (n_i+1)$, we have that all clause players use non-zero edges that the variable players use (otherwise, the cost of P will be greater). That is, for each clause player k, her strategy in P is $\{(s_i, l_j'), (l_j', l_j), (l_j, c_k^i)\}$ where l_j is a literal appearing in c_k, and, the strategy of the variable player j is $\{(s_i, l_j'), (l_j', l_j), (l_j, b_j^i)\}$. Therefore, l_j is satisfied by f^P, and hence c_k is satisfied by f^P. Since this claim holds for all clause players, we have that f^P satisfies φ'_i, and therefore φ_i is satisfiable, in contradiction to the assumption.

It remains to show that φ_1 is satisfiable and φ_2 is not satisfiable iff there exists an edge $e \in E$ and a value $x \geq 0$ such that $cost_{SO}(N[e \leftarrow x]) = 4n_1 + 6n_2 + 16$. We distinguish between the following cases:

- If φ_1 and φ_2 are satisfiable, then $cost_{SO}(N) = cost_{SO}(N_1) + cost_{SO}(N_2) = 4n_1 + 6n_2 + 10 < 4n_1 + 6n_2 + 16$. Since by Theorem 4 the SO is monotone, it holds that the SO can be increased only by increasing the cost of some edge. Since every edge has a parallel edge with the same cost, increasing the cost of every edge does not change the cost of the SO, as there is an alternative path with a lower cost. Hence, the cost of the SO cannot be increased for all $e \in E$ and for all $x \geq 0$, in particular, for all $e \in E$ and for all $x \geq 0$ if holds that $cost_{SO}(N[e \leftarrow x]) \neq 4n_1 + 6n_2 + 16$.
- If φ_1 and φ_2 are unsatisfiable, then $cost_{SO}(N) = 4n_1 + 6n_2 + 14 < 4n_1 + 6n_2 + 16$. Using the same argument as above, for all $e \in E$ and for all $x \geq 0$ it holds that $cost_{SO}(N[e \leftarrow x]) \neq 4n_1 + 6n_2 + 16$.
- If φ_1 is unsatisfiable and φ_2 is satisfiable, then $cost_{SO}(N) = 4n_1 + 6n_2 + 20$. Note that the cost of the SO can be decreased by at most 3. First, the maximal cost of an edge in N is 3, the cost of every profile in $SO(N)$ can be decreased by at most 3. Since all non-zero edges are on a path with two edges, each with cost $i+1$, when decreasing the cost of one of the non-zero edges to 0, the total cost of the path is $i+1$. Therefore, since the variable players must chose a path of cost $2(i+1)$, except, perhaps, one players that choses a path of cost $i+1$, in the case where φ_i is satisfiable the total cost of every profile is reduced by at most $i+1$. In addition, using the same argument for the case where φ_i is unsatisfiable, there must be exactly n_i+2 players that choses a non-zero path, thus in this case as well, the total cost can be reduced by at most $i+1$, hence, the cost of the

SO can be reduced by at most 3. Therefore, for all $e \in E$ and for all $x \geq 0$ it holds that $cost_{SO}(N[e \leftarrow x]) \geq 4n_1 + 6n_2 + 17 > 4n_1 + 6n_2 + 16$.

- If φ_1 is satisfiable and φ_2 is unsatisfiable, then $cost_{SO}(N) = 4n_1 + 6n_2 + 16$. Let $e \in E$, it holds that $cost_{SO}(N[e \leftarrow c(e)]) = 4n_1 + 6n_2 + 16$.

Hence, $\langle \varphi_1, \varphi_2 \rangle \in$ SAT-UNSAT iff there exists an edge $e \in E$ and a value $x \geq 0$ such that $cost_{SO}(N) = 4n_1 + 6n_2 + 16$.

A.5 Proof of Theorem 17

Assume first that there is a profile $P \in bNE(N[e \leftarrow 0])$ such that $e \notin P$ and for all $x \geq 0$ it holds that $cost_{bNE}(N[e \leftarrow x]) \geq cost_{bNE}(N[e \leftarrow 0])$. Since $P \in bNE(N[e \leftarrow 0])$ and $e \notin P$, we have by Lemma 16 that for all $x \geq 0$ it holds that P is an NE in $N[e \leftarrow x]$. Since $e \notin P$, we have that for all $x \geq 0$ it holds that $cost(N[e \leftarrow x], P) = cost(N[e \leftarrow 0], P) = cost_{bNE}(N[e \leftarrow 0])$. Therefore, by the minimality of the bNE, for all $x \geq 0$, it holds that $cost_{bNE}(N[e \leftarrow x]) \leq cost(N[e \leftarrow x], P) = cost_{bNE}(N[e \leftarrow 0])$. Since for all $x \geq 0$, we have that $cost_{bNE}(N[e \leftarrow x]) \geq cost_{bNE}(N[e \leftarrow 0])$, it follows that for all $x \geq 0$, we have that $cost_{bNE}(N[e \leftarrow 0]) \leq cost_{bNE}(N[e \leftarrow x]) \leq cost_{bNE}(N[e \leftarrow 0])$. Thus, for all $x \geq 0$, we have that $cost_{bNE}(N[e \leftarrow x]) = cost_{bNE}(N[e \leftarrow 0])$, hence e does not bNE-affect N.

For the other direction, assume that e does not bNE-affect N. Then, $cost_{bNE}(N[e \leftarrow x])$ is a constant function with value OPT. In particular, this means that for all $0 \leq x \leq \infty$, it holds that $cost_{bNE}(N[e \leftarrow 0]) = cost_{bNE}(N[e \leftarrow x])$. Therefore, it is enough to show that there is a profile P in $bNE(N[e \leftarrow 0])$ such that $e \notin P$. Assume towards contradiction that for all profiles $P \in bNE(N[e \leftarrow 0])$ it holds that $e \notin P$. We argue that there is a profile P such that for some $x > 0$, it holds that for all $0 < t \leq x$, the profile P is an NE and $e \notin P$. Assume towards contradiction that there is $\varepsilon > 0$ such that for all $0 \leq t \leq \varepsilon$, we have that $e \in P$ for all profiles $P \in bNE(N[e \leftarrow t])$. Therefore, for each such profile P, its cost is strictly monotonically increasing in the cost of e. By Lemma 14, P is an NE in at most a single segment. Since P is an NE in $N[e \leftarrow t]$ for $0 \leq t \leq \varepsilon$, for each such profile there is a single range $[a_P, b_P] \subseteq [0, \varepsilon]$ where it is an NE. Since the cost of each profile is strictly monotonically increasing, the cost of each profile is a function of the form $c_P + x$. Since e is constant, it must hold that for each profile P we have that $c_P + a_P = OPT$. Since there is a finite amount of profiles, and since the number of values in the range $[0, \varepsilon]$ is infinite for all $\varepsilon > 0$, it follows that there is a range $[l, r] \subseteq [0, \varepsilon]$ where for all $t \in [l, r]$ it holds that $bNE(N[e \leftarrow l]) = bNE(N[e \leftarrow t])$. Since the cost of every profile in $bNE(N[e \leftarrow l])$ is strictly monotonically increasing, and $cost_{bNE}(N[e \leftarrow l]) = OPT$, it follows that $cost_{bNE}(N[e \leftarrow r]) > OPT$, in contradiction to the fact that e does not bNE-affect N.

A.6 The Δ_2^P lower bound in Theorem 18

We use the same reduction as in the hardness result for Theorem 8, with a slight variation. Instead of having a single variable player per variable, we have two players with the same objectives.

Assume φ is satisfiable, and let f_{max} be a maximal satisfying assignment. We show that if $0 \leq x \leq \frac{1}{2}$, then there is a profile P_{max} that defines a satisfying assignment such that $f_{P_max} = f_{max}$, and that P_{max} is an NE in $N[e \leftarrow x]$. Furthermore, if $f_{max}(x_0) = 0$, then this claim holds for all $x \geq 0$. Let P_0 be the following profile. For every variable player with target b_i such that $i > 0$, if $f_{max}(x_i) = 1$ then her strategy is $\{(s, s'), (s', x_i), (x_i, b_i)\}$, and otherwise her strategy is $\{(s, s'), (s', \neg x_i), (\neg x_i, b_i)\}$. For the variable players of b_0 their strategy is $\{(s, s'), (s', x'_0), (x'_0, x_0), (x_0, b_0)\}$. For every clause player c_j, her strategy is

$\{(s, s'), (s', l), (l, c_j)\}$ where l is a literal present in c_j and that is satisfiable by f_{max}. Next, run BRD with P_0 as the initial value. For $t \geq 0$, denote by P_{t+1} the profile obtained by a single BRD step from P_t, and let P_{max} be the profile obtained from convergence. Note that it was shown in e.g. [38] that BRD converges in all NFGs, therefore P_{max} is both well-defined and is an NE. We argue that for all $t \geq 0$, it holds that for all variables $0 \leq i \leq n-1$, we have that the path from s' to $\neg f_{max}(x_i)$ is not in P_t, and $(s, s'') \notin P_t$. The proof is by induction over t. The base case is trivial for P_0. Assume that the claim holds for some $t \geq 0$, and let P_{t+1}. Assume towards contradiction that there is some variable i such that either the path from s' to $\neg f(x_i)$ or $(s, s'') \in P_{t+1}$. Then, there is a player j that has it in her strategy. By the induction assumption, it follows that j deviated from P_t to P_{t+1}, and that no other player has it in their strategy. Therefore, there is a variable player for b_i that have $(s, f_{max}(x_i))$ in their strategy. Hence, the cost of Player j's strategy in P_{t+1} is either greater than $2^n - 2^{n-1}$ (if her strategy has the path from s' to $\neg f_{max}(x_i)$ for some $0 \leq i \leq n-1$) or $2^{n+1} + n \cdot 2^n$ (if her strategy has (s, s'')). Since $i \leq n-1$, if $0 \leq x \leq \frac{1}{2}$ the cost of her strategy in P_t is at most $\frac{2^n - f_{max}(x_i)2^i - \frac{1}{2} + x}{2} \leq \frac{2^n}{2} = 2^n - 2^{n-1} \leq 2^{n+1} + n \cdot 2^n$, in contradiction to the definition of a BRD step. If $f_{max}(x_0) = 0$, then for every $x \geq 0$ the cost of her strategy in P_t is at most $\frac{2^n - f_{max}(x_i)2^i}{2} \leq \frac{2^n}{2} = 2^n - 2^{n-1} \leq 2^{n+1} + n \cdot 2^n$, in contradiction to the definition of a BRD step. Thus, the only non-zero paths in P_{max} are from s' to $f_{max}(x_i)$ for all $0 \leq i \leq n-1$, and hence $f_{P_{max}} - f_{max}$, and P_{max} is an NE for all $0 \leq x \leq \frac{1}{2}$, and is an NE for all $x \geq 0$ if $f_{max}(x_0) = 0$.

Assume first that $\varphi \in$maximum-satisfying-assignment, that is, φ is satisfiable, and for the maximal lexicographic satisfying assignment f_{max} it holds that $f_{max}(x_0) = 1$. Hence, for every $0 \leq x \leq \frac{1}{2}$ there is a profile P^x_{max} that is an NE in $N[e \leftarrow x]$ that defines a satisfying assignment, and $f_{P^x_{max}} = f_{max}$. As shown in the proof for Theorem 8, for every profile P and $0 \leq x \leq \frac{1}{2}$, it holds that $cost(N[e \leftarrow x], P^x_{max}) \leq cost(N[e \leftarrow x], P)$, therefore $P^x_{max} \in bNE(N[e \leftarrow x])$, and for every such x it holds that $cost(N[e \leftarrow x], P^x_{max}) = cost_{bNE}(N[e \leftarrow x])$. In particular, we have that $cost_{bNE}(N[e \leftarrow 0]) = cost(N[e \leftarrow 0], P^0_{max})$ and $cost_{bNE}(N[e \leftarrow \frac{1}{2}]) = cost(N[e \leftarrow \frac{1}{2}], P^{\frac{1}{2}}_{max})$. In the proof for Theorem 8 we also showed that $cost(N[e \leftarrow x], P^x_{max}) = 2^{n+1} + n \cdot 2^n - \lfloor f_{max} \rfloor_{10} - \frac{1}{2} + x$, thus, $cost(N[e \leftarrow 0], P^0_{max}) < cost(N[e \leftarrow \frac{1}{2}], P^{\frac{1}{2}}_{max})$, hence $cost_{bNE}(N[e \leftarrow 0]) < cost_{bNE}(N[e \leftarrow \frac{1}{2}])$. Therefore, e bNE-affects N and $\langle N, e \rangle \in$Edge-bNE-affects.

Next, assume that $\varphi \notin$maximum-satisfying-assignment. We distinguish between the following cases:

- φ is unsatisfiable. Let P_{UNSAT} be the profile where for every player with objective $\langle s, t \rangle$, her strategy is $\{(s, s''), (s'', t)\}$. In the proof for Theorem 8 we showed that for every profile P and for every $x \geq 0$ it holds that $cost(N[e \leftarrow x], P_{UNSAT}) \leq cost(N[e \leftarrow x], P)$. We argue that P is an NE in $N[e \leftarrow x]$ for all $x \geq 0$. For every player the cost of her strategy is $\frac{2^{n+1} + n \cdot 2^n}{2n+m} < \frac{2^{n+1} + n \cdot 2^n}{2n} = \frac{2^n}{n} + 2^{n-1} < 2^{n+1}$, which is a lower bound to the cost of every strategy that is a deviation for her. Therefore, for all $x \geq 0$ we have that $P_{UNSAT} \in bNE(N[e \leftarrow x])$, therefore, $cost(N[e \leftarrow x], P_{UNSAT}) = cost_{bNE}(N[e \leftarrow x])$. Since for all $x \geq 0$ we have that $cost(N[e \leftarrow x], P_{UNSAT})$ is $2^{n+1} + n \cdot 2^n$, we have that $cost_{bNE}(N[e \leftarrow x])$ is constant, thus, e does not bNE-affect N.
- φ is satisfiable, and the maximal lexicographic assignment f_{max} has $f(x_0) = 0$. As shown above, for all $x \geq 0$ there is a profile P^x_{max} such that $f_{P^x_{max}} = f_{max}$ and P^x_{max} is an NE. In the proof for Theorem 8 we showed that for every profile P and for all $x \geq 0$ we have that $cost(N[e \leftarrow x], P^x_{max}) \leq cost(N[e \leftarrow x], P)$. Furthermore, we showed that for every such profile P^x_{max} and for all $x \geq 0$ we have that $cost(N[e \leftarrow x], P^x_{max}) = 2^{n+1} + n \cdot 2^n - \lfloor f_{max} \rfloor_{10}$. Therefore, for all $x \geq 0$ we have that $P^x_{max} \in bNE(N[e \leftarrow x])$

and hence $cost_{bNE}(N[e \leftarrow x]) = 2^{n+1} + n \cdot 2^n - \lfloor f_{max} \rfloor_{10}$. Therefore, e does not bNE-affect N.

Since in either case e does not bNE-affect N, we have that $\langle N, e\rangle \notin$ Edge-bNE-affects.

A.7 The Θ_2^P lower bound in Theorem 18

Assume first that $\langle G_1, G_2\rangle \in$ VC-compare. That is, if S_1, S_2 are two minimal VCs for G_1, G_2 respectively, then it holds that $|S_1| \leq |S_2|$. We define the following profile P_{S_1}. Each player i has the strategy $\{(s_i, e_i^1), (e_i^1, v_j^1), (v_j^1, t_1), (t_1, t_1'), (t_1', t)\}$, where $v_j^1 \in S_1$. If both vertices of e_i are in S_1, then select one at random. That is, each player choses G_1's side of the network, then it uses a vertex that is in a minimal vertex cover of G_1, and then choses the only available path from there to the target objective t. Note that since S_1 is a vertex cover for G_1, it holds that for every edge e_i^1 there is a vertex $v_j^1 \in S_1$ such that v_j^1 touches e_i^1 in G_1, therefore, there is an edge in N from e_i^1 to v_j^1. Finally, run BRD (Best-Response Dynamics) until convergence, and denote the result as P. Note that it was shown in e.g. [38] that BRD converges in all NFGs, therefore P is both well-defined and is an NE.

First, we argue that all strategies in P don't use G_2's side of the network. Assume towards contradiction that there is some step j in the run of BRD on P_{S_1} such that just before the jth step the current profile was P^j, and after the jth step the current profile is $P^{j'}$, such that all players in P^j don't have a strategy that uses G_2's side of the network, and in $P^{j'}$ there is a player i that uses G_2's side of the network. Since the initial value for BRD is P_{S_1}, and all the players in P_{S_1} don't use G_2's side of the network, under the contradiction assumption such j must exist. Let π_i^j be the strategy of Player i in P^j and let $\pi_i^{j'}$ be the strategy of Player i in $P^{j'}$. By definition of BRD, it holds that $cost_{N,P^j}(\pi_i^j) > cost_{N,P^{j'}}(\pi_i^{j'})$. However, since non of the players use G_2's side in P^j and since $P^{j'}$ is a result of a BRD step, we have that no player other than i uses G_2's side in $P^{j'}$. Therefore, it holds that $cost_{N,P^{j'}}(\pi_i^{j'}) = 1 + n + 1 > 1 + n \geq cost_{N,P^j}(\pi_i^j)$, hence we derive a contradiction.

Next, denote by S the set of vertex vertices that are incident to an edge in P. We argue that $|S| = |S_1|$. Assume towards contradiction that $|S| \neq |S_1|$. We distinguish between the following cases. First, assume that $|S| < |S_1|$. Therefore, since P only uses G_1's side of the network, $S \subset E_1$. Therefore, every player i is associated with an edge e_i^1, and a vertex v_j^1, such that $(e_i^1, v_j^1) \in P$. Hence, each edge in G_1 can be covered by a vertex in S, hence S is a vertex cover of G_1, that is smaller in size that S_1, in contradiction to the minimality of S_1. Second, assume that $|S| > |S_1|$. Therefore, there is a vertex v_j^1 such that $v_j^1 \in S$ and $v_j^1 \notin S_1$. By definition of S, this implies that there is some i such that $(e_i^1, v_j^1) \in P$, and, for all players k it holds that $(e_k^1, v_j^1) \notin P_{S_1}$. It follows that there was a BRD step t such that before t the current profile was P^t and after the tth step the current profile was $P^{t'}$, such that $(e_i^1, v_j^1) \in P^{t'}$ and for all players k we have that $(e_k^1, v_k^1) \notin P^t$. Let π_i be the strategy of Player i in P^t and let π_i' be the strategy of Player i in $P^{t'}$. It follows that $cost_{N,P^t}(\pi_i) \leq 1 + n = cost_{N,P^{t'}}(\pi_i')$, in contradiction to the definition of BRD.

It follows that, by construction, P is an NE in N. We argue that P is an NE in $N[t_1', t \leftarrow 1]$. Let i be a player with strategy π_i in P, and let P' be a deviation profile for P and Player i such that the strategy for Player i in P' is π_i'. If π_i' uses G_2's side of the network, then since no player in P use G_2's side of the network, we have that $cost_{N[t_1',t\leftarrow 1],P}(\pi_i) \leq 1 + \frac{n+1}{m} < 1+n+1 = cost_{N[t_1',t\leftarrow 1],P'}(\pi_i')$. If π_i' uses G_1's side of the network, then since P is an NE in N, we have that $cost_{N[t_1',t\leftarrow 1],P}(\pi_i) = cost_{N,P}(\pi_i) + \frac{1}{m} \leq cost_{N,P'}(\pi_i') + \frac{1}{m} = cost_{N[t_1',t\leftarrow 1],P'}(\pi_i')$. Hence, P is an NE in $N[t_1', t \leftarrow 1]$.

Finally, note that by construction we have that $cost(N[t_1', t \leftarrow x], P) = |S_1| + n + x$. Let

P' be a profile in N. If P' uses both sides of the network, then $cost(N[t'_1, t \leftarrow x], P') \geq n + x + n + 1 \geq |S_1| + n + x = cost(N[t'_1, t \leftarrow x], P)$. If P' only uses G_2's side of the network, then $cost(N[t'_1, t \leftarrow x], P') \geq |S_2| + n + 1$. Since $|S_1| \leq |S_2|$, we have that for $0 \leq x \leq 1$ it holds that $|S_2| + n + 1 \geq |S_1| + n + x = cost(N[t'_1, t \leftarrow x], P)$. If P' only uses G_1's side of the network, then by the minimality of S_1 we get that $cost(N[t'_1, t \leftarrow x], P') \geq |S_1| + n + x = cost(N[t'_1, t \leftarrow x], P)$. Therefore, we get that $P \in bNE(N), bNE(N[t'_1, t \leftarrow 1])$, and therefore $cost_{bNE}(N) < cost_{bNE}(N[t'_1, t \leftarrow 1])$. Hence, $\langle N, e, \kappa \rangle \in$ *Edge-bNE-affects*.

Next, assume that $\langle G_1, G_2 \rangle \notin$ VC-compare. That is, if S_1, S_2 are two minimal VCs for G_1, G_2 respectively, then it holds that $|S_1| > |S_2|$. We define the profile P_{S_2} in the same way that we defined P_{S_1} above, and we define P to be the result of BRD with the initial value P_{S_2}. Using the same logic, it can be shown that all strategies in P don't use G_1's side of the network, and that the set of vertex vertices that are incident to some edge in P is a minimal vertex cover for G_2. Therefore, by construction, P is an NE in N. We argue that for all $x \geq 0$, P is an NE in $N[t'_1, t \leftarrow x]$. Let i be a player with strategy π_i in P, and let P' be a deviation profile for P and Player i such that the strategy for Player i in P' is π'_i. If π'_i uses G_1's side of the network, then since no player in P use G_1's side of the network, we have that $cost_{N[t'_1, t \leftarrow x], P}(\pi_i) \leq 1 + \frac{n+1}{m} < 1 + n + x = cost_{N[t'_1, t \leftarrow x], P'}(\pi'_i)$. If π'_i uses G_2's side of the network, then since P is an NE in N, we have that $cost_{N[t'_1, t \leftarrow x], P}(\pi_i) = cost_{N,P}(\pi_i) \leq cost_{N,P'}(\pi'_i) + \frac{x}{m} = cost_{N[t'_1, t \leftarrow x], P'}(\pi'_i)$. Hence, P is an NE in $N[t'_1, t \leftarrow x]$.

Finally, note that by construction we have that $cost(N[t'_1, t \leftarrow x], P) = |S_2| + n + 1$. Let P' be a profile in N. If P' uses both sides of the network, then $cost(N[t'_1, t \leftarrow x], P') \geq n + x + n + 1 \geq |S_2| + n + 1 = cost(N[t'_1, t \leftarrow x], P)$. If P' only uses G_1's side of the network, then since $|S_1| > |S_2|$ we have that $cost(N[t'_1, t \leftarrow x], P') \geq |S_1| + n + x \geq |S_2| + 1 + n + x \geq |S_2| + n + 1 = cost(N[t'_1, t \leftarrow x], P)$. Since $|S_1| \leq |S_2|$, we have that for $0 \leq x \leq 1$ it holds that $|S_2| + n + 1 \geq |S_1| + n + x = cost(N[t'_1, t \leftarrow x], P)$. If P' only uses G_2's side of the network, then by the minimality of S_2 we get that $cost(N[t'_1, t \leftarrow x], P') \geq |S_2| + n + 1 = cost(N[t'_1, t \leftarrow x], P)$. Therefore, we get that for all $x \geq 0$ $P \in bNE(N[t'_1, t \leftarrow x])$, and since the cost of P is independent of the cost of (t'_1, t), $e = (t'_1, t)$ does not bNE-affect N. Hence, $\langle N, e, \kappa \rangle \notin$ *Edge-bNE-affects*.

A.8 Proof of Theorem 19

We start with membership in NP. Given N, e and κ as above, a witness is a profile P and a value $x \in R$. We first show that there always exists such x that is polynomial in input. Let $\mu = \inf\{argmin_{t \in \mathbb{R}} cost_{bNE}(N[e \leftarrow t])\}$. That is, if OPT is the minimal value of the cost of the bNE, and $S = \{t | cost_{bNE}(N[e \leftarrow t]) = OPT\}$, then $\mu = inf(S)$. Since $cost_{bNE}(N[e \leftarrow x]) < \kappa$ we get that $cost_{bNE}(N[e \leftarrow \mu]) < \kappa$. We continue to show that μ is representable by polynomially-many bits. Let $P \in bNE(N[e \leftarrow \mu])$. By Lemma 14, it follows that there is a segment $[a_P, b_P]$ that is the maximal range where P is an NE. By definition, P is an NE iff for every player i with strategy π_i, it holds that Player i has no incentive to deviate to an alternative strategy π'_i. The cost of every strategy π of a profile P_0 in $N[e \leftarrow x]$ is given by $\sum_{e' \in \pi \setminus \{e\}} \frac{c(e')}{used_{P_0}(e')} + \mathbb{1}_{e \in \pi} \frac{x}{used_{P_0}(e)}$. We denote by $c_{P_0,\pi} = \sum_{e' \in \pi \setminus \{e\}} \frac{c(e')}{used_{P_0}(e')}$. Since for every profile P_0 and for every edge e' it holds that $used_{P_0}(e') = O(k)$, it holds that for every strategy π the denominator of $c_{P_0,\pi}$ is bounded by $O(k^m)$, where m is the number of edges in N, which can be represented using $O(mlogk) = O(mk)$ bits. Since the objective of every player and the cost of every edge are given as input, it is safe to assume that $O(mk)$ is polynomial in input. The numerator is therefore bounded by $O(k \sum_{e' \in E} c(e))$, which again can be represented in polynomially many bits. Thus, $c_{P_0,\pi}$ can be represented in polynomially-many bits as the quotient of numbers representable by polynomially-many bits. Then, the requirement that no player has an incentive to deviate in $N[e \leftarrow x]$ induces an inequality of the form $c_{P,\pi_i} + \mathbb{1}_{e\in\pi_i}\frac{x}{used_P(e)} \leq c_{P',\pi_i'} + \mathbb{1}_{e\in\pi_i'}\frac{x}{used_{P'}(e)}$ where P' is a profile constructed from P using Player i's deviation π_i'. By using sub operator we get $\frac{x}{used_P(e)} \cdot \mathbb{1}_{e\in\pi_i} - \frac{x}{used_{P'}(e)} \cdot \mathbb{1}_{e\in\pi_i'} \leq c_{P',\pi_i'} - c_{P,\pi_i}$. The term $c_{\pi_i',P'} - c_{\pi_i,P}$ is representable by polynomially many bits as the difference of numbers that are representable by polynomially-many bits. Next, if both $\mathbb{1}_{e\in\pi_i}, \mathbb{1}_{e\in P'}$ are 0, then the inequality either evaluates to True or False regardless of x. If it does not hold, then P is never an NE in contradiction to the definition of P, and otherwise we can say that it holds for $0 \leq x \leq \infty$, both are polynomial in input. This is also the case if both $\mathbb{1}_{e\in\pi_i}, \mathbb{1}_{e\in P'}$ are 0 and $used_P(e) = used_P(e')$. Otherwise, if exactly one of them is 1, assume that $\mathbb{1}_{e\in\pi_i} = 1$. Then $\frac{x}{used_{P'}(e)} \cdot \mathbb{1}_{e\in\pi_i'} = 0$. Therefore, we have that $used_P(e) \geq 1$ and hence the inequality is equivalent to $x \leq used_P(e)(c_{\pi_i',P'} - c_{\pi_i,P})$. Since both $used_P(e)$ and $c_{\pi_i',P'} - c_{\pi_i,P}$ are representable by polynomially-many bits, so is their product. The argument is similar for the case where $\mathbb{1}_{e\in\pi_i'} = 1$ with the exception of reversing the inequality. Finally, if both $\mathbb{1}_{e\in\pi_i}, \mathbb{1}_{e\in\pi_i'} = 1$ and $used_P(e) \neq used_{P'}(e)$, then the inequality is equivalent to $x \leq \frac{(c_{\pi_i',P'} - c_{\pi_i,P}) \cdot used_P(e) \cdot used_{P'}(e)}{used_P(e) - used_{P'}(e)}$, which is again representable by polynomially many bits as the quotient of numbers that are representable by polynomially-many bits. The range $[a_P, b_P]$ where P is an NE is the solution set of this set of inequalities, and is therefore their intersection. Since all inequalities are week, every inequality represents a closed set, thus the solution set is a closed set, and in particular both a_P and b_P are the edge points of one of the inequalities, thus, both are representable by polynomially-many bits.

We argue that $\mu = inf_{P\in bNE(N[e\leftarrow\mu])} a_P$. Denote $a = inf_{P\in bNE(N[e\leftarrow\mu])} a_P$. By definition, for every $P \in bNE(N[e \leftarrow \mu])$ it holds that $cost(N[e \leftarrow \mu], P) = OPT$. Therefore, for every $P \in bNE(N[e \leftarrow \mu])$, since P is an NE in $N[e \leftarrow \mu]$, it follows that $a_P \leq \mu$. Hence $a \leq a_P \leq \mu$. Furthermore, since for every such profile P it holds that $a_P \leq \mu$, we get that $cost(N[e \leftarrow a_P], P) \leq cost(N[e \leftarrow \mu], P) = cost_{bNE}(N[e \leftarrow \mu]) = OPT$. By minimality of OPT, we get that for every such profile P it holds that $cost(N[e \leftarrow a_P], P) = OPT$. By definition of μ, for every profile P it holds that if there is a value t such that $cost(N[e \leftarrow t], P) = OPT$, then $\mu \leq t$. Hence, for every profile $P \in bNE(N[e \leftarrow \mu])$ we have that $\mu \leq a_P$. Next, since there is a finite number of profiles, the set $bNE(N[e \leftarrow \mu])$ is finite. Hence, the set $\{a_P | P \in bNE(N[e \leftarrow \mu])\}$ is finite, and therefore, $a \in \{a_P | P \in bNE(N[e \leftarrow \mu])\}$. That is, the infimum and the minimum coincide. Therefore, since for every such profile P we have that $\mu \leq a_P$, in particular $\mu \leq a$. Thus, $a \leq \mu$ and $\mu \leq a$, hence $\mu = a$. Now, since $a \in \{a_P | P \in bNE(N[e \leftarrow \mu])\}$, and we argued that for every $P \in bNE(N[e \leftarrow \mu])$ we have that a_P can be represented by polynomially many bits, μ can be represented by polynomially many bits, as required.

Note that there can be exponentially many strategies per player, thus calculating μ can be computationally hard and in particular not polynomial. However, we are not required to be able to calculate μ efficiently. It is enough to bound the representation size of the result, then the witness is polynomially bounded by input.

So, the witness x is polynomial in the input. Next, we argue that given a profile P, we can verify that P is an NE in polynomial time. For each player i, fix the strategies of each of the other players, then search for a lightest path from s_i to t_i. If the cost of the strategy of Player i in P is higher than the cost of the path we found, then it is a beneficial deviation, hence P is not an NE. Therefore, it can be verified it polynomial time that P is an NE in $N[e \leftarrow x]$ and that $cost(N[e \leftarrow x], P) \leq \kappa$. In the case of bNE-optimization, the witness also contains the edge e.

For hardness, we reduce bNE-cost to both problems. We use the same reductions as in

Theorem 9. In the case of edge-bNE-optimization, the argument to bNE is trivially extended. For bNE-optimization, assume first that $\langle N, \kappa \rangle \in$bNE-cost, that is, $cost_{bNE}(N) \leq \kappa$. Therefore, $cost_{bNE}(N'[e \leftarrow 0]) = cost_{bNE}(N) \leq \kappa$, hence $\langle N', \kappa \rangle \in$bNE-optimization.

Next, assume that $\langle N, \kappa \rangle \notin$bNE-cost. Therefore, $cost_{bNE}(N) > \kappa$. It holds for every $e' \neq e$ and for every value $x \geq 0$ that $cost_{bNE}(N'[e' \leftarrow x]) = cost_{bNE}(N[e' \leftarrow x]) + \kappa + 1 > \kappa$. Furthermore, it holds that $cost_{bNE}(N'[e \leftarrow x]) = cost_{bNE}(N) + x \geq cost_{bNE}(N) > \kappa$. Therefore, $\langle N, \kappa \rangle \notin$bNE-optimization.