Trading Probability for Fairness
(Extended abstract)

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Abstract

Properties of open systems can be modeled as objectives in two-player games. Depending on the interaction between the system and its environment, the games may be either turn-based (transitions of the system and its environment are interleaved) or concurrent (transitions are taken simultaneously). Depending on the properties we model, the objectives may be achieved in a finite game (safety properties) or an infinite game (general properties, specified with Büchi, co-Büchi, or parity fairness conditions). Finally, winning may be required surely (the objective should be achieved for sure) or almost surely (the objective should be achieved with probability 1). Almost-sure winning involves randomized strategies for the players and is harder to analyze. For example, while sure reachability games can be solved in linear time, the best algorithm for solving almost-sure reachability is quadratic.

In this paper it is shown that probabilistic concurrent games can be reduced to non-probabilistic turn-based games with a more general fairness condition. For example, finding the set of states that are almost-surely winning for player 1 in a concurrent reachability game can be reduced to finding the set of states that are surely winning for player 1 in a turn-based Büchi game. From a theoretical point of view, the reductions show that it is possible to trade the probabilistic nature of almost-sure winning by a more general fairness condition in a game with sure winning. The reductions improve our understanding of games and suggest alternative simple proofs of some known results such as determinacy of concurrent probabilistic Büchi games. From a practical point of view, our reductions turn solvers of non-probabilistic turn-based games into solvers of probabilistic concurrent games. An improvement in the well-studied algorithms for the former would immediately carry over to the latter. In particular, a recent improvement in the upper bound for non-probabilistic turn-based parity games allows us to improve the bound for solving probabilistic concurrent co-Büchi games from cubic to quadratic.

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1 Introduction

In *formal verification*, we check that a system meets a desired behavior by running an algorithm that decides whether a mathematical model of the system satisfies a formal specification that describes the behavior. We distinguish between *closed* and *open* systems [HP85]. A *closed system* is a system whose behavior is completely determined by the state of the system. An *open system* is a system that interacts with its environment and whose behavior depends on this interaction. While the formal verification of closed systems uses models based on labeled transition systems, the formal verification of open systems, and the related problems of control and synthesis, must use models based on *two-player games*, where one player represents the system, and the other player represents the environment [PR89, RW89, AL93, AHK97, KMTV00, KVW01]. At each round of the game, player 1 (the system) and player 2 (the environment) choose moves, and the choices determine the next state of the game. Specifications of open systems can be expressed as objectives in such games, and questions of deciding whether an open system satisfies a specification can be reduced to questions of deciding whether player 1 has a winning strategy in the game. The construction of winning strategies can also be used to *synthesize* correct systems and controllers from their specifications [PR89, RW89]. Thus, two-player games are of great importance when reasoning about open systems.

Different classes of games correspond to different types of interaction between the system and its environment, as well as to different specifications and satisfaction relations. We classify games using the following three basic properties[1].

- Is the game *turn-based* or *concurrent*? In a turn-based game, the states of the game are partitioned into states of player 1 and states of player 2. When the game is at a state of player i, only player i determines the next state. On the other hand, in a concurrent game, the two players choose moves simultaneously and independently, and both choices together determine the next state of the game. While turn-based games correspond to an interleaving semantics between the system and the environment, concurrent games correspond to synchronous interaction [AHM00, AHM01].

- Is the game *finite* or *infinite*? In a finite game, the objective of player 1 can be achieved after finitely many rounds. The basic finite game is *reachability*—can player 1 force the game to a given set of target states? Infinite games, on the other hand, may proceed ad infinitum[2]. The basic infinite game is *repeated reachability* (also known as *Büchi*)—can player 1 force the game to a given set of target states infinitely often? By translating specifications to automata, we can model with reachability games all safety specifications, and with Büchi games all ω-regular specifications. Note that player 2 wins the reachability game if he can keep the game out of the target set, and he wins the repeated reachability game if he can eventually keep the game out of the target set. Thus, in *co-Büchi games* the objective is to visit the set of target states only finitely often.

- Should the objective of the game be *surely* or *almost surely* achieved? Player 1 surely wins a game if, no matter how player 2 chooses his moves, he has a deterministic strategy to achieve the objective.

[1]An additional basic property is whether the games are of *complete* or *incomplete information* [Rei84]; that is, whether the players fully observe the states of the game. For example, when the system and the environment have internal hidden variables, the corresponding game is of incomplete information. In this paper we consider only games of complete information.

[2]Note that "infinite" does not describe the state space of the game but its duration.

On the other hand, Player 1 almost surely wins a game if, no matter how player 2 chooses his moves, he has a randomized strategy to achieve the objective with probability 1. We use almost-sure winning when we want to check whether a system satisfies its specification with probability 1.

In order to understand the different classes, consider a game in which, at each round, each player chooses a number in $\{0, 1\}$, and player 1 wins if eventually $c_1 = c_2$, where c_i is the number chosen by player i. Otherwise, player 2 wins. Note that this is a reachability game, where the target set consists of states with $c_1 = c_2$. If the game is turn-based, each round of the game involves two moves, one of each player. The player who moves second knows the choice made by the player who moves first. It is easy to see that this gives an advantage to the player who moves second. In our example, if player 2 chooses first, then player 1 can win after one round by choosing $c_1 = c_2$. On the other hand, if player 1 chooses first, then player 2 can win by always choosing $c_2 \neq c_1$. If the game is concurrent, in each round the players choose c_1 and c_2 simultaneously and independently. In this case, no player has a strategy to surely win. Still, player 1 almost surely wins with a strategy that chooses c_1 uniformly at random. Finally, if the objective of player 1 is to have $c_1 = c_2$ infinitely often, he can surely win in a turn-based game in which player 2 chooses first, and he can almost surely win in a concurrent game with a strategy that chooses c_1 uniformly at random.

For turn-based games, sure and almost-sure winning coincide, because randomized strategies are no more powerful than deterministic ones. Turn-based reachability games correspond to deciding *alternating reachability* in an AND-OR graph, which can be solved in linear time and is complete for PTIME [Imm81]. Algorithms for deciding turn-based Büchi games use the algorithm for alternating reachability as a subroutine. The subroutine may be called once for each state of the game, resulting in quadratic-time algorithms [EL86, VW86]. Intuitively, while the states from which player 1 wins in a turn-based reachability game can be expressed by a single least fixed point, the states from which player 1 wins in a turn-based Büchi game can only be expressed by a least fixed point nested inside a greatest fixed point. The exact complexity of turn-based Büchi games, somewhere between linear to quadratic time, is one of the most challenging open problem in the theory of verification. Note that dualizing a turn-based Büchi game results in a turn-based co-Büchi game, thus the complexity analysis above applies also to co-Büchi games.

As demonstrated in the game described above, sure and almost-sure winning do not coincide in concurrent games, where randomized strategies are more powerful than deterministic ones. Concurrent reachability games are studied in [AHK98a], where a quadratic algorithm for deciding almost-sure concurrent reachability games is presented. As in turn-based Büchi games, the set of states from which player 1 can almost surely win a concurrent reachability game can be expressed by a least fixed point nested inside a greatest fixed point, and the problem of whether a linear algorithm for finding this set exists is open. In fact, in quadratic time one can also decide an almost-sure concurrent Büchi game [AH00]. Thus, unlike the case of surely winning, reachability is not easier than Büchi when we consider almost-sure winning. Recall that the co-Büchi condition dualizes the Büchi condition. Consider again the game described above. If the objective of player 2 is to have $c_1 = c_2$ only finitely often, he can surely win in a turn-based game in which player 1 chooses first. In a concurrent game, however, player 2 cannot win, not even almost surely. In fact, as player 1 can almost surely win in a game where his objective is to have $c_1 = c_2$ infinitely often, player 2 cannot even guarantee that $c_1 = c_2$ only finitely often with any positive probability. In general, player 1 almost surely wins a game with a Büchi condition iff player 2 does not win the game with the dual co-Büchi condition with any positive probability. Thus, we cannot solve probabilistic co-Büchi games simply by dualizing Büchi games. The best known algorithm for solving almost-sure concurrent co-Büchi games involves the calculation of a fixed point expression of alternation depth two, and has cubic time complexity [AH00].

While being very different in their nature, the sure turn-based Büchi game and the almost-sure concurrent reachability game turned out to have similar solutions. Both games correspond to a nested fixed point, leading to a quadratic time upper bound and only a linear time lower bound. It is easy to translate a sure turn-based Büchi game $\mathcal{G}$ into an almost-sure concurrent reachability game $\mathcal{G}'$. Essentially, $\mathcal{G}'$ has a new state to which player 1 can move with probability $\frac{1}{2}$ from the target states of $\mathcal{G}$ [AHK98a]. The second direction, of translating concurrent games to turn-based games is considerably more difficult, as it has to translate a concurrent move of the two players into two or more moves taken sequentially, which gives an advantage to the player who does not move first—an advantage he did not have in the concurrent setting.

In this paper we study this second direction and describe translations of probabilistic concurrent games into non-probabilistic turn-based games. In order to overcome the difficulty mentioned above, we introduce *witness functions* for sure and almost-sure winning in Büchi and co-Büchi games. The witness functions relate the ability of player i to win with the ability to label the states of the game with weights so that some conditions on the labels are satisfied. The key idea is that the conditions on the labels of a state v and its successors can be expressed either in terms of probability distributions on the moves that each player can take from v, which enables us to relate the conditions with randomized strategies in a concurrent game, or in terms of the set of moves that each player can take from v, which enables us to relate the conditions with deterministic strategies in a turn-based game. In addition to serving as the theoretical base to our reduction, the witness functions are of independent interest: they improve our understanding of games, and suggest an alternative proof for some known results such as determinacy of almost-sure concurrent Büchi games [AH00]. We describe three linear reductions: of almost-sure concurrent reachability and Büchi games to sure turn-based Büchi games, and of almost-sure concurrent co-Büchi games to sure turn-based Parity(0,2) games[3].

Conceptually, our reductions imply that games with probabilistic winning conditions have the flavor of non-probabilistic richer winning conditions. It follows from our reductions that the sure turn-based Büchi game and the almost-sure concurrent reachability game have the same complexity: once an improvement of the quadratic bound would be found for one of them, it would immediately lead to an improved upper bound for the second, and similarly for an improvement of the linear lower bound. From a practical point of view, our reductions turn solvers of non-probabilistic turn-based games into solvers of probabilistic concurrent games. An improvement in the well-studied algorithms for the former would immediately carry over to the latter. In particular, it was recently shown that the turn-based Parity(0,2) game can be solved in quadratic time [Jur00]. Thus our second reduction suggests an algorithm for solving almost-sure concurrent co-Büchi games in quadratic time, improving the existing cubic algorithm [AH00]. In Section 8, we discuss further the practical aspects of our results.

[3]A Parity(0,2) winning condition consists of a partition of the state space into three sets P_0, P_1, and P_2, and the objective of player 1 is either to visit P_0 infinitely often, or visit P_2 infinitely often and P_1 only finitely often

2 Concurrent probabilistic games

2.1 Definitions

For a finite set X, a *probability distribution* on X is a function $\xi : X \to [0,1]$ such that $\sum_{x \in X} \xi(x) = 1$. We denote the set of probability distributions on X by $\mathcal{D}(X)$. For a probability distribution $\xi \in \mathcal{D}(X)$ we define $||\xi||$, the *support* of ξ, by $||\xi|| = \{ x \in X \ : \ \xi(x) > 0 \}$.

A two-player *concurrent probabilistic game structure* $G = (V, A, A_1, A_2, \delta)$ consists of the following components.

- A finite set V of vertices, and a finite set of actions A.
- Functions $A_1, A_2 : V \to 2^A$, such that for every vertex v, $A_1(v)$ and $A_2(v)$ are non-empty sets of actions available in vertex v to players 1 and 2, respectively.
- A probabilistic transition function $\delta : V \times A \times A \to \mathcal{D}(V)$, such that for every vertex v and actions $a \in A_1(v)$ and $b \in A_2(v)$, $\delta(v, a, b)$ is a probability distribution on the successor vertices.

At each step of the game, both players choose *moves* to proceed with. We consider two options here.

- *Pure action moves.* The set of moves is the set of actions $M = A$. The sets of moves available to players 1 and 2 in vertex v are $M_1(v) = A_1(v)$ and $M_2(v) = A_2(v)$, respectively.
- *Mixed (randomized) action moves.* The set of moves is the set of probability distributions on the set of actions $M = \mathcal{D}(A)$. The sets of moves available to players 1 and 2 in vertex v are $M_1(v) = \mathcal{D}\big(A_1(v)\big)$ and $M_2(v) = \mathcal{D}\big(A_2(v)\big)$, respectively. In this case we extend the transition function to $\delta : V \times M \times M \to \mathcal{D}(V)$, by $\delta(v, \alpha, \beta)(w) = \sum_{a \in A_1(v)} \sum_{b \in A_2(v)} \alpha(a) \cdot \beta(b) \cdot \delta(v, a, b)$.

We often write $\mathrm{Pr}_v^{\alpha,\beta}[w]$ for $\delta(v, \alpha, \beta)(w)$, and we define $\mathrm{Pr}_v^{\alpha,\beta}[W] = \sum_{w \in W} \mathrm{Pr}_v^{\alpha,\beta}[w]$, for $W \subseteq V$. Thus, $\mathrm{Pr}_v^{\alpha,\beta}[w]$ is the probability that the successor vertex is w, given that the current vertex is v and the players chose to proceed with α and β. Similarly, $\mathrm{Pr}_v^{\alpha,\beta}[W]$ is the probability that the successor vertex is a member of W.

A concurrent probabilistic game is played in the following way. If v is the current vertex in a play then player 1 chooses a move $\alpha \in M_1(v)$, and simultaneously and independently player 2 chooses a move $\beta \in M_2(v)$. Then the play proceeds to a successor vertex w with probability $\mathrm{Pr}_v^{\alpha,\beta}[w]$.

A *path* in G is an infinite sequence $v_0, v_1, v_2, \ldots$ of vertices, such that for all $k \geq 0$, there are moves $\alpha \in M_1(v_k)$ and $\beta \in M_2(v_k)$, such that $\mathrm{Pr}_{v_k}^{\alpha,\beta}[v_{k+1}] > 0$. We denote by Ω the set of all paths.

We say that a concurrent game structure $G = (V, A, A_1, A_2, \delta)$ is:

- *Turn-based*, if for all $v \in V$, we have either $|A_1(v)| = 1$ or $|A_2(v)| = 1$; i.e., in every vertex only one player may have a non-trivial choice;
- *Deterministic (non-probabilistic)*, if for all $v \in V$, $a \in A_1(v)$, and $b \in A_2(v)$, we have $||\delta(v, a, b)|| = 1$; i.e., in every move the next vertex is uniquely determined by the pure action moves chosen by the players. In this case we often write $\delta(v, a, b)$ for the unique $w \in V$, such that $\delta(v, a, b)(w) = 1$.

Strategies. A *strategy* for player 1 is a function $\pi_1 : V^+ \to M$, such that for a finite sequence $\overline{v} \in V^+$ of vertices, representing the history of the play so far, $\pi_1(\overline{v})$ is the next move to be chosen by player 1. A strategy must prescribe only available moves, i.e., $\pi_1(\overline{w} \cdot v) \in M_1(v)$, for all $\overline{w} \in V^*$, and $v \in V$. Strategies for player 2 are defined analogously. We write Π_1 and Π_2 for the sets of all strategies for players 1 and 2, respectively.

For an initial vertex v, and strategies $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$, we define $\mathit{Outcome}(v, \pi_1, \pi_2) \subseteq \Omega$ to be the set of paths that can be followed when a play starts from vertex v and the players use the strategies π_1 and π_2. Formally, $v_0, v_1, v_2, \ldots \in \mathit{Outcome}(v, \pi_1, \pi_2)$ if $v_0 = v$, and for all $k \geq 0$, we have that $\delta(v_k, \alpha_k, \beta_k)(v_{k+1}) > 0$, where $\alpha_k = \pi_1(v_0, \ldots, v_k)$ and $\beta_k = \pi_2(v_0, \ldots, v_k)$.

Once a starting vertex v and strategies π_1 and π_2 for the two players have been chosen, the probabilities of events are uniquely defined, where an *event* $\mathcal{A} \subseteq \Omega$ is a measurable set of paths. For a vertex v, and an event $\mathcal{A} \subseteq \Omega$, we write $\Pr_v^{\pi_1,\pi_2}(\mathcal{A})$ for the probability that a path belongs to $\mathcal{A}$ when the game starts from v, and the players use the strategies π_1 and π_2.

Winning criteria. A *game* $\mathcal{G} = (G, \mathcal{W})$ consists of a game structure G and a *winning criterion* $\mathcal{W} \subseteq \Omega$ (for player 1). In this paper we consider the following winning criteria.

- *Reachability.* For a set R of vertices we define the set of paths satisfying the reachability criterion as follows: $\text{Reach}(R) = \{ v_0, v_1, v_2, \cdots \in \Omega \ : \ \text{there is } k \geq 0, \text{ such that } v_k \in R \}$.

- *Büchi criterion.* For a set B of vertices, the Büchi criterion is defined by:

$$\text{Büchi}(B) = \{ v_0, v_1, v_2, \cdots \in \Omega \ : \ \text{for infinitely many } k \geq 0, \text{ we have } v_k \in B \}.$$

- *Co-Büchi criterion.* For a set C of vertices, the co-Büchi criterion is defined by:

$$\text{Co-Büchi}(C) = \{ v_0, v_1, v_2, \cdots \in \Omega \ : \ \text{for only finitely many } k \geq 0, \text{ we have } v_k \in C \}.$$

- *Parity criterion.* Let $P = (P_0, P_1, \ldots, P_d)$ be a partition of the set of vertices. The parity criterion is defined by:

$$\text{Parity}(P) = \{ \overline{v} \in \Omega \ : \ \min\big(\text{Inf}(\overline{v})\big) \text{ is even} \},$$

  where for a path $\overline{v} = v_0, v_1, v_2, \cdots \in \Omega$, we define

$$\text{Inf}(\overline{v}) = \{ i \in \mathbb{N} \ : \ \text{there are infinitely many } k \geq 0, \text{ such that } v_k \in P_i \}.$$

Note that a parity criterion $\text{Parity}(P_0, P_1)$ is equivalent to the Büchi criterion $\text{Büchi}(P_0)$, a parity criterion $\text{Parity}(\emptyset, P_1, P_2)$ is equivalent to the co-Büchi criterion $\text{Co-Büchi}(P_1)$, and a parity criterion $\text{Parity}(P_0, P_1)$ is equivalent to the reachability criterion $\text{Reach}(P_0)$, if we assume that the vertices in P_0 are absorbing (that is, once the game reaches a vertex in P_0, it visits only vertices in P_0 thereafter).

For uniformity we phrase all the results below in terms of parity games. By $\text{C}(0, j)$ we denote concurrent probabilistic parity games with a parity criterion $\text{Parity}(P_0, P_1, \ldots, P_j)$, and by $\text{C}(1, j)$ we denote concurrent probabilistic parity games with a parity criterion $\text{Parity}(\emptyset, P_1, P_2, \ldots, P_j)$. By $\text{D}(i, j)$ we denote $\text{C}(i, j)$

games with turn-based deterministic game structures. Thus, we write $\mathrm{C}(0,1)$ for concurrent probabilistic Büchi games, $\mathrm{C}(1,2)$ for concurrent probabilistic co-Büchi games, $\mathrm{D}(0,1)$ for turn-based deterministic Büchi games, etc.

Winning modes. Let $\mathcal{G} = (G, \mathcal{W})$ be a game. We say that a strategy $\pi_1 \in \Pi_1$ for player 1 is:

- a *sure winning* strategy for player 1 from vertex v in the game $\mathcal{G}(G, \mathcal{W})$, if for all $\pi_2 \in \Pi_2$, we have $\mathit{Outcome}(v, \pi_1, \pi_2) \subseteq \mathcal{W}$,
- an *almost-sure winning* strategy for player 1 from vertex v in the game $\mathcal{G}(G, \mathcal{W})$, if for all $\pi_2 \in \Pi_2$, we have $\mathrm{Pr}_v^{\pi_1,\pi_2}[\mathcal{W}] = 1$,
- a *positive-probability winning* strategy for player 1 from vertex v in the game $\mathcal{G}(G, \mathcal{W})$, if for all $\pi_2 \in \Pi_2$, we have $\mathrm{Pr}_v^{\pi_1,\pi_2}[\mathcal{W}] > 0$.

The same notions are defined similarly for player 2, with the set $\mathcal{W}$ in the winning condition replaced by $\Omega \setminus \mathcal{W}$.

For a class $\mathcal{C}$ of games, and a winning mode $\mu \in \{\mathrm{s}, \mathrm{a}, \mathrm{p}\}$, we write $\mathcal{C}_\mu$ for the class of games in which the goal of player 1 is to win with the mode μ, where "s" stands for sure win, "a" stands for almost-sure win, and "p" stands for positive-probability win. For example, $\mathrm{C(0,1)_a}$ are almost-sure win concurrent probabilistic Büchi games and $\mathrm{C(1,2)_p}$ are positive-probability win concurrent probabilistic co-Büchi games.

Determinacy. We say that a game $(G, \mathcal{W})$ is μ-determined if for every vertex v, either player 1 can μ-win from vertex v, or player 2 can $\tilde{\mu}$-win from v, where $\tilde{\mathrm{s}} = \mathrm{s}$, $\tilde{\mathrm{a}} = \mathrm{p}$, and $\tilde{\mathrm{p}} = \mathrm{a}$.

Theorem 2.1 *1. All pure action turn-based deterministic parity games are s-determined [EJ91].*

2. All mixed action concurrent probabilistic parity games are a-determined [AH00].

3 Witnesses for turn-based deterministic games

In order to prove that a strategy is winning for a player in a parity game, one needs to argue that all infinite plays consistent with the strategy are winning for the player. A technically convenient notion of a witness has been used in [EJ91, Wal01, Jur00] to establish existence of a winning strategy by verifying only some finitary local conditions. We recall here the definitions and basic facts about witnesses (also called signatures [EJ91, Wal01], or progress measures [Jur00]) for the relevant special cases of D(0,1) and D(0,2) games.

For $n \in \mathbb{N}$, we write $[n]$ for the set $\{0, 1, 2, \ldots, n\}$, and $[n]_\infty$ for the set $\{0, 1, 2, \ldots, n, \infty\}$, where the element ∞ is bigger than all the others. Let $G = (V, A, A_1, A_2, \delta)$ be a game structure and let $\varphi : V \to [n]_\infty$. We define $\varphi_\infty = \{\, w \in V \,:\, \varphi(w) = \infty \,\}$, and for a vertex $v \in V$, we define $\varphi_{<v} = \{\, w \in V \,:\, \varphi(w) < \varphi(v) \,\}$, and $\varphi_{>v} = \{\, w \in V \,:\, \varphi(w) > \varphi(v) \,\}$.

3.1 Sure win witnesses for D(0,1) games

Let $\mathcal{G} = \big(G, \text{Parity}(P_0, P_1)\big)$ be a D(0,1) game, where $G = (V, A, A_1, A_2, \delta)$, and $\delta : V \times A \times A \to V$.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\exists a \in A_1(v).\, \forall b \in A_2(v). \quad \big(v \in P_0 \;\Rightarrow\; \delta(v,a,b) \notin \varphi_\infty\big) \;\wedge \big(v \in P_1 \;\Rightarrow\; \delta(v,a,b) \in \varphi_{<v}\big). \qquad (1)$$

We say that the function φ is a *(sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 1.

Witness for player 2. For a function $\psi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *ψ-progressive for player 2* if the following holds:

$$\exists b \in A_2(v).\, \forall a \in A_1(v). \quad \big(v \in P_0 \;\Rightarrow\; \delta(v,a,b) \in \psi_{<v}\big) \;\wedge \big(v \in P_1 \;\Rightarrow\; \delta(v,a,b) \notin \psi_{>v}\big). \qquad (2)$$

We say that the function ψ is a *(sure win) witness for player 2* if every vertex $v \in \psi_{<\infty}$ is ψ-progressive for player 2.

Lemma 3.1 (Sure winning strategies from sure witnesses)
Let $\varphi, \psi : V \to [n]_\infty$. If φ is a witness for player 1 and ψ is a witness for player 2, then player 1 has a winning strategy from every vertex $v \in \varphi_{<\infty}$, and player 2 has a winning strategy from every vertex $v \in \psi_{<\infty}$.

3.2 Sure win witnesses for D(0,2) games

Let $\mathcal{G} = \big(G, \text{Parity}(P_0, P_1, P_2)\big)$ be a D(0,2) game, where $G = (V, A, A_1, A_2, \delta)$, and $\delta : V \times A \times A \to V$.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\begin{aligned}\exists a \in A_1(v).\, \forall b \in A_2(v). \quad & \big(v \in P_0 \;\Rightarrow\; \delta(v,a,b) \notin \varphi_\infty\big) \;\wedge \big(v \in P_1 \;\Rightarrow\; \delta(v,a,b) \in \varphi_{<v}\big) \;\wedge \\ & \big(v \in P_2 \;\Rightarrow\; \delta(v,a,b) \notin \varphi_{>v}\big).\end{aligned} \qquad (3)$$

We say that the function φ is a *(sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 2.

Witness for player 2. For a pair of functions $\psi = (\psi^0, \psi^2)$, such that $\psi^0 : V \to [n]_\infty$, and $\psi^2 : V \to [n]$, we say that a vertex $v \in V$ is *ψ-progressive for player 2* if the following holds:

$$\begin{aligned}\exists b \in A_2(v).\, \forall a \in A_1(v). \quad & \big(v \in P_0 \;\Rightarrow\; \delta(v,a,b) \in \psi^0_{<v}\big) \;\wedge \big(v \in P_1 \;\Rightarrow\; \delta(v,a,b) \notin \psi^0_{>v}\big) \;\wedge \\ & \big(v \in P_2 \;\Rightarrow\; \delta(v,a,b) \in \psi_{<v}\big),\end{aligned} \qquad (4)$$

where we define

$$\psi_{<v} = (\psi^0, \psi^2)_{<v} = \big\{\, w \in V \;:\; \big(\psi^0(w), \psi^2(w)\big) <_{\text{lex}} \big(\psi^0(v), \psi^2(v)\big) \,\big\},$$

and $<_{\text{lex}}$ is the lexicographic ordering. We define $\psi_{<\infty} = \psi^0_{<\infty}$. We say that the function ψ is a *(sure win) witness for player 2* if every vertex $v \in \psi_{<\infty}$ is ψ-progressive for player 2.

Lemma 3.2 *Let $\varphi, \psi^0 : V \to [n]_\infty$, $\psi^2 : V \to [n]$, and $\psi = (\psi^0, \psi^2)$. If φ is a witness for player 1 and ψ is a witness for player 2 then: player 1 has a winning strategy from every vertex $v \in \varphi_{<\infty}$, and player 2 has a winning strategy from every vertex $v \in \psi_{<\infty}$.*

Theorem 3.3 *[EJ91, Wal01] If $\mathcal{G}$ is a deterministic turn-based parity game then there is a witness φ for player 1, and a witness ψ for player 2, such that $\varphi_{<\infty} \cup \psi_{<\infty} = V_\mathcal{G}$. Therefore, from every vertex one of the players has a winning strategy.*

3.3 Witnesses for concurrent probabilistic games

In Section 4 we define witnesses for both players in concurrent almost-sure win Büchi games. Then in Section 5 we use them to give a reduction from concurrent probabilistic almost-sure win Büchi games to turn-based deterministic Büchi games. As a by-product we get the following as a corollary of Theorem 3.3.

Theorem 3.4 *If $\mathcal{G}$ is a C(0,1)$_a$ game then there is a witness φ for player 1, and a witness ψ for player 2, such that $\varphi_{<\infty} \cup \psi_{<\infty} = V_\mathcal{G}$. Therefore, from every vertex, either player 1 has an almost-sure winning strategy, or player 2 has a positive-probability winning strategy.*

In Section 6 we define witnesses for both players in concurrent almost-sure win co-Büchi games. Then in Section 7 we use them to give a reduction from concurrent probabilistic almost-sure win co-Büchi games to D(0,2) games. As a by-product we get the following as a corollary of Theorem 3.3.

Theorem 3.5 *If $\mathcal{G}$ is a C(1,2)$_a$ game then there is a witness φ for player 1, and a witness ψ for player 2, such that $\varphi_{<\infty} \cup \psi_{<\infty} = V_\mathcal{G}$. Therefore, from every vertex, either player 1 has an almost-sure winning strategy, or player 2 has a positive-probability winning strategy.*

4 Witnesses for concurrent Büchi games

Proving that a player in a concurrent parity game has a winning strategy, in particular for non-sure winning modes, is often quite involved. In fact, the original papers [AHK98a, AH00] omit detailed proofs, and the full version [AHK98b] contains a gap (that can be bridged, see Appendix G) in the proof of correctness for limit-sure reachability. Instead of proving from first principles that certain strategies are winning for a player, we introduce, for various winning modes and criteria, the notions of witnesses, which are functions that assign natural numbers to vertices so that the assignment satisfies certain "local" constraints. We then prove that a witness for a player gives rise to a winning strategy for him. Once we show that witnesses are sufficient conditions for existence of winning strategies, we only need to focus on constructing witnesses, which is easier than analyzing probabilities of sets of infinite probabilistic plays induced by strategies, since only local finitary constraints need to be verified.

Let $\mathcal{G} = \big(G, \text{Parity}(P_0, P_1)\big)$ be a C(0,1) game, where $G = (V, A, A_1, A_2, \delta)$ is a game structure.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\exists_{\varepsilon>0}.\ \exists_{\alpha\in\mathcal{D}(A_1(v))}.\ \forall_{\beta\in\mathcal{D}(A_2(v))}.\ \begin{array}{l}\big(v \in P_0 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0\big) \wedge \\ \big(v \in P_1 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0 \ \wedge\ \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{<v}] \geq \varepsilon\big).\end{array} \tag{5}$$

We say that the function φ is an *(almost-sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 1. See Appendices A and B for a proof of the following Lemma.

Lemma 4.1 *If $\varphi : V \to [n]_\infty$ is a witness for player 1 in the C(0,1)$_a$ game then he has an (almost-sure) winning strategy from every vertex in $\varphi_{<\infty}$.*

Witness for player 2. For a function $\psi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *ψ-progressive for player 2* if the following holds:

$$\forall_{\delta>1}.\ \exists_{\beta\in\mathcal{D}(A_2(v))}.\ \forall_{\alpha\in\mathcal{D}(A_1(v))}.\ \begin{array}{l}\big(v \in P_0 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\psi_{<v}] > 0\big) \wedge \\ \big(v \in P_1 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\psi_{<v}] > 0 \ \vee\ \mathrm{Pr}_v^{\alpha,\beta}[\psi_{>v}] \leq 1/\delta\big).\end{array} \tag{6}$$

We say that the function $\psi : V \to [n]_\infty$ is a *(positive win) witness for player 2* if every vertex $v \in \psi_{<\infty}$is ψ-progressive for player 2. See Appendices A and C for a proof of the following Lemma.

Lemma 4.2 *If $\psi : V \to [n]_\infty$ is a witness for player 2 in the C(0,1)$_a$ game then he has a (positive-probability) winning strategy from every vertex in $\psi_{<\infty}$.*

5 Translation of C(0,1)$_\mathrm{a}$ games to D(0,1) games

The following "finitary" characterizations of vertices that are φ- and ψ-progressive for player 1 and player 2, respectively, are the key to our reduction of concurrent probabilistic to turn-based non-probabilistic games.

Action progressive vertices. Let $\varphi, \psi : V \to [n]_\infty$. We say that a vertex $v \in V$ is *action φ-progressive for player 1* if the following holds:

$$\begin{array}{l}\big(v \in P_0 \ \Rightarrow\ \exists_{a\in A_1(v)}.\ \forall_{b\in A_2(v)}.\ \mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0\big) \wedge \\ \big(v \in P_1 \ \Rightarrow\ \forall_{b\in A_2(v)}.\ \exists_{a\in A_1(v)}.\ \mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0 \ \wedge\ (\forall_{b'\in A_2(v)}.\ \mathrm{Pr}_v^{a,b'}[\varphi_\infty] = 0)\big).\end{array} \tag{7}$$

We say that a vertex $v \in V$ is *action ψ-progressive for player 2* if the following holds:

$$\begin{array}{l}\big(v \in P_0 \ \Rightarrow\ \forall_{a\in A_1(v)}.\ \exists_{b\in A_2(v)}.\ \mathrm{Pr}_v^{a,b}[\psi_{<v}] > 0\big) \wedge \\ \big(v \in P_1 \ \Rightarrow\ \exists_{b\in A_2(v)}.\ \forall_{a\in A_1(v)}.\ \mathrm{Pr}_v^{a,b}[\psi_{>v}] = 0 \ \vee\ (\exists_{b'\in A_2(v)}.\ \mathrm{Pr}_v^{a,b'}[\psi_{<v}] > 0)\big).\end{array} \tag{8}$$

Lemma 5.1 *Let $\varphi, \psi : V \to [n]_\infty$. 1. If a vertex $v \in V$ is action φ-progressive for player 1 then it is φ-progressive for him. 2. If a vertex $v \in V$ is action ψ-progressive for player 2 then it is ψ-progressive for him.*

See Appendix D for a proof of the Lemma.

We are now ready to reduce C(0,1)$_a$ games to D(0,1) games. The idea of the reduction is to replace each concurrent transition in the concurrent game by a small turn-based game in which each player aims at satisfying the condition in the definition of his action progressive vertex.

Let $G = (V, A, A_1, A_2, \delta)$ be a concurrent probabilistic game structure, and let $(G, \text{Parity}(P_0, P_1))_a$ be a C(0,1)$_a$ game. We define a D(0,1) game $(G', (P_0', P_1'))$ in the following way. The set of vertices of G' includes the set V of vertices of G. We describe the transition function of G' from every vertex $v \in V$, and from the extra vertices that a few-step play in G' from v to another vertex $w \in V$ can go through.

First, for every $v \in V$, $a \in A_1(v)$, $b \in A_2(v)$ we define one-step games $\mathcal{H}_0(v, a, b)$ and $\mathcal{H}_1(v, a, b)$ as follows: the unique initial vertex of game $\mathcal{H}_i(v, a, b)$ has priority i, and the following hold.

- In the initial vertex of game $\mathcal{H}_0(v, a, b)$ player 2 chooses a successor $w \in V$, such that $\delta(v, a, b)(w) > 0$;
- In the initial vertex of game $\mathcal{H}_1(v, a, b)$ player 1 chooses a successor $w \in V$, such that $\delta(v, a, b)(w) > 0$.

In the correctness proof of the reduction given below, the games $\mathcal{H}_0(v, a, b)$ and $\mathcal{H}_1(v, a, b)$ act as gadgets that allow:

- in game $\mathcal{H}_0(v, a, b)$: player 1 to "verify" the condition $\text{Pr}_v^{a,b}[\varphi_\infty] = 0$, and player 2 to "verify" the condition $\text{Pr}_v^{a,b}[\psi_{<v}] > 0$ (see Proposition E.1 in the Appendix E); and
- in game $\mathcal{H}_1(v, a, b)$: player 1 to "verify" the condition $\text{Pr}_v^{a,b}[\varphi_{<v}] > 0$, and player 2 to "verify" the condition $\text{Pr}_v^{a,b}[\psi_{>v}] = 0$ (see Proposition E.2 in the Appendix E.)

Next, we define the transition function of $\mathcal{G}'$ from every $v \in V$. Note that this transition function is simply a translation of the formula (7) from the definition of an action progressive vertex into a "formula evaluation" game, where player 1 is the "existential" player, and player 2 is the "universal" player.

- If $v \in P_0$ then the following game is played:
  1. in vertex $v \in P_0$, player 1 chooses a successor (v, a), where $a \in A_1(v)$;
  2. in vertex (v, a), player 2 chooses a one-step game $\mathcal{H}_0(v, a, b)$, where $b \in A_2(v)$.
- If $v \in P_1$ then the following game is played:
  1. in vertex $v \in P_1$, player 2 chooses a successor (v, b), where $b \in A_2(v)$;
  2. in vertex (v, b), player 1 chooses a successor (v, b, a), where $a \in A_1(v)$;
  3. in vertex (v, b, a), player 2 chooses either: the one-step game $\mathcal{H}_1(v, a, b)$, or the successor $(v, b, a, *)$;
  4. in vertex $(v, b, a, *)$ player 2 chooses a one-step game $\mathcal{H}_0(v, a, b')$, where $b' \in A_2(v)$.

Clearly, the game graph G' is turn-based and deterministic. The set P'_0 contains P_0 and all the initial vertices of the one-step games $\mathcal{H}_0(v, a, b)$. All the other vertices belong to P'_1.

Theorem 5.2 *Let $\mathcal{G}$ be a C(0,1)$_a$ game. For every vertex $v \in V_{\mathcal{G}}$, player 1 has an (almost-sure) winning strategy from v in $\mathcal{G}$ if and only if player 1 has a (sure) winning strategy from v in the D(0,1) game $\mathcal{G}'$.*

Proof idea: The idea of the proof is to argue that witnesses for either of the players in game $\mathcal{G}'$ give rise to witnesses for the same player in game $\mathcal{G}$. Then by the determinacy theorem for turn-based deterministic games (Theorem 3.3) and Lemma 4.2 we get Theorems 3.4 and 5.2. More precisely, it suffices to establish the following.

1. If $\varphi' : V' \to [n]_\infty$ is a witness for player 1 in the D(0,1) game $\mathcal{G}'$ then the restriction φ of φ' to V is a witness for player 1 in the C(0,1)$_\mathrm{a}$ game $\mathcal{G}$.
2. If $\psi' : V' \to [n]_\infty$ is a witness for player 2 in the D(0,1) game $\mathcal{G}'$ then the restriction ψ of ψ' to V is a witness for player 2 in the C(0,1)$_\mathrm{a}$ game $\mathcal{G}$.

See Appendix E for proofs of clauses 1. and 2. [Theorem 5.2] ■

6 Witnesses for concurrent co-Büchi games

Let $\mathcal{G} = \big(G, \text{Parity}(\emptyset, P_1, P_2)\big)$ be a C(1,2) game, where $G = (V, A, A_1, A_2, \delta)$ is a game structure.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\exists_{\varepsilon>0}.\ \exists_{\alpha\in\mathcal{D}(A_1(v))}.\ \forall_{\beta\in\mathcal{D}(A_2(v))}.\quad \begin{array}{l}\big(v \in P_1 \ \Rightarrow\ \Pr_v^{\alpha,\beta}[\varphi_\infty] = 0 \ \wedge\ \Pr_v^{\alpha,\beta}[\varphi_{<v}] \geq \varepsilon\big) \ \wedge \\ \big(v \in P_2 \ \Rightarrow\ \Pr_v^{\alpha,\beta}[\varphi_\infty] = 0 \ \wedge\ \Pr_v^{\alpha,\beta}[\varphi_{<v}] \geq \varepsilon \cdot \Pr_v^{\alpha,\beta}[\varphi_{>v}]\big).\end{array} \tag{9}$$

We say that the function φ is an *(almost-sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 1. See Appendices A and F for a proof of the following Lemma.

Lemma 6.1 *If $\varphi : V \to [n]_\infty$ is an (almost-sure win) witness for player 1 then he has an (almost-sure) winning strategy from every vertex in $\varphi_{<\infty}$.*

Witness for player 2. For a pair of functions $\psi = (\psi^0, \psi^2)$, such that $\psi^0 : V \to [n]_\infty$, and $\psi^2 : V \to [n]$, we say that a vertex $v \in V$ is *ψ-progressive for player 2* if the following holds:

$$\begin{array}{l}\exists_{m\in\mathbb{N}}.\ \forall_{\delta>1}.\ \exists_{\beta\in\mathcal{D}(A_2(v))}.\ \forall_{\alpha\in\mathcal{D}(A_1(v))}. \\ \quad \big(v \in P_1 \ \Rightarrow\ \Pr_v^{\alpha,\beta}[\psi^0_{<v}] > 0 \ \vee\ \Pr_v^{\alpha,\beta}[\psi^0_{>v}] \leq 1/\delta\big) \ \wedge \\ \quad \big(v \in P_2 \ \Rightarrow\ \Pr_v^{\alpha,\beta}[\psi^0_{<v}] > 0 \ \vee\ (\Pr_v^{\alpha,\beta}[\psi_{<v}] \geq 1/\delta^m \ \wedge\ \Pr_v^{\alpha,\beta}[\psi^0_{>v}] \leq (1/\delta) \cdot \Pr_v^{\alpha,\beta}\big[\psi_{<v}]\big)\big).\end{array} \tag{10}$$

We say that the function $\psi : V \to [n]_\infty$ is a *(positive win) witness for player 2* if every vertex $v \in \psi_{<\infty}$ is ψ-progressive for player 2. See Appendices A and G for a proof of the following Lemma.

Lemma 6.2 *If $\psi = (\psi^0, \psi^2)$ is a (positive win) witness for player 2then he has a (positive-probability) winning strategy from every vertex in $\psi_{<\infty}$.*

7 Translation of $C(1,2)_a$ games to D(0,2) games

The following "finitary" characterization of vertices that are φ-progressive for player 1 is the starting point of the idea behind our reduction of concurrent probabilistic co-Büchi games to turn-based non-probabilistic parity games.

Action progressive vertex. Let $\varphi : V \to [n]_\infty$. We say that a vertex $v \in V$ is *action φ-progressive for player 1* if the following holds:

$$\begin{aligned}&\big(v \in P_1 \;\Rightarrow\; \forall_{b\in A_2(v)}.\, \exists_{a\in A_1(v)}.\; \Pr_v^{a,b}[\varphi_{<v}] > 0 \;\wedge\; (\forall_{b'\in A_2(v)}.\; \Pr_v^{a,b'}[\varphi_\infty] = 0)\big) \;\wedge\\ &\Big(v \in P_2 \;\Rightarrow\; \exists_{\emptyset\neq X\subseteq A_1(v)}.\; (\forall_{a\in X}.\, \forall_{b\in A_2(v)}.\; \Pr_v^{a,b}[\varphi_\infty] = 0) \;\wedge\\ &\qquad\qquad\big((\forall_{a\in X}.\, \forall_{b\in A_2(v)}.\, \Pr_v^{a,b}[\varphi_{>v}] = 0) \;\vee\; (\exists_{a'\in X}.\, \Pr_v^{a',b}[\varphi_{<v}] > 0)\big)\Big).\end{aligned} \tag{11}$$

Lemma 7.1 *Let $\varphi : V \to [n]_\infty$. If a vertex $v \in V$ is action φ-progressive for player 1 then it is φ-progressive for him.*

See Appendix H for a proof of the Lemma.

We are now ready to reduce $C(1,2)_a$ games to D(0,2) games. Let $G = (V, A, A_1, A_2, \delta)$ be a concurrent probabilistic game graph, and let $\big(G, \text{Parity}(\emptyset, P_1, P_2)\big)$ be a $C(1,2)_a$ game. We define a D(0,2) game (G', P'_0, P'_1, P'_2) in the following way. The set of vertices of G' includes the set V of vertices of G. We describe the transition function of G' from every vertex $v \in V$, and the extra vertices that a game from v can go through.

We are going to use one-step games $\mathcal{H}_0(v, a, b)$ and $\mathcal{H}_1(v, a, b)$ defined in Section 5. Moreover we define a very similar one-step game $\mathcal{H}_2(v, a, b)$, such that its unique initial vertex has priority 2, and in the unique initial vertex of $\mathcal{H}_2(v, a, b)$ player 2 chooses a successor $w \in V$, such that $\delta(v, a, b)(w) > 0$.

As in the case of Büchi games in Section 5, the one-step games $\mathcal{H}_i(v, a, b)$, for $i \in \{0, 1, 2\}$, serve as gadgets that allow the players to "verify" certain conditions occurring in the definition of an action progressive vertex. See Propositions I.1, I.2, and I.3 in the Appendix I for details.

Next, we define the transition relation from every $v \in V$.

- If $v \in P_1$ then the same game is played as for $v \in P_1$ in the reduction of $C(0,1)_a$ games to D(0,1) games described in Section 5.
- If $v \in P_2$ then the following game is played:
  1. in vertex $v \in P_2$, player 1 chooses a successor (v, a), where $a \in A_1(v)$;
  2. in vertex (v, a), player 2 chooses a successor (v, a, b), where $b \in A_2(v)$;
  3. in vertex (v, a, b), player 2 chooses either: the one-step game $\mathcal{H}_0(v, a, b)$, or the successor $(v, a, b, *)$;
  4. in vertex $(v, a, b, *)$, player 1 chooses either: the one-step game $\mathcal{H}_2(v, a, b)$, or the successor (v, b);

5. in vertex (v, b), player 1 chooses a successor (v, b, a'), where $a' \in A_1(v)$;
6. in vertex (v, b, a'), player 2 chooses either: the one-step game $\mathcal{H}_1(v, a', b)$, or the vertex (v, a').

The vertices in V keep their priority, i.e., P'_1 includes P_1, and P'_2 includes P_2. All the other new vertices different from the initial vertices of games $\mathcal{H}_i(v, a, b)$ have priority 2.

Theorem 7.2 *Let $\mathcal{G}$ be a C(1,2)$_a$ game. For every vertex $v \in V_{\mathcal{G}}$, player 1 has an (almost-sure) winning strategy from v in $\mathcal{G}$ if and only if player 1 has a (sure) winning strategy from v in the D(0,2) game $\mathcal{G}'$.*

See Appendix I for a proof of the Theorem. Since D(0,2) games can be solved in quadratic time [Jur00] and $\mathcal{G}'$ is linear in $\mathcal{G}$, we have the following.

Theorem 7.3 *C(1,2)$_a$ games can be solved in quadratic time.*

8 Discussion

Current algorithms for solving concurrent probabilistic games [AHK98a, AH00] are fairly complicated. On the other hand, the problem of solving turn-based games has been heavily studied and there are many algorithms available [EL86, CKS92, McN93, BCJ$^+$97, Sei96, LRS98, Jur00, VJ00]. So, from a practical point of view, our reductions allow to directly apply this work, and future related work, for solving concurrent probabilistic games: any algorithmic improvements for turn-based games would immediately carry over to concurrent probabilistic games. In particular, the recently algorithm for solving turn-based Parity(0,2) games in quadratic time [Jur00] gives, together with our reduction, an algorithm for solving almost-sure concurrent co-Büchi games in quadratic time, improving the existing cubic algorithm. We note that even though the proofs of correctness of the translations are involved, the translations themselves are fairly simple, so at a very low cost, one can turn a solver for turn-based parity games into a solver for almost-sure concurrent reachability, Büchi, and co-Büchi games.

In this paper we demostrated the reductions for the reachability, Büchi, and co-Buchi winning criteria. For turn-based Büchi games, special cases are known to be solvable in linear time. This includes *weak* games [MSS86], and games whose transitions form a tree with back edges [Niw96]. Using our reductions, we are able to define classes of concurrent probabilistic games for which the game can be decided in linear time.

We conjecture that our translations can be generalized to all almost-sure concurrent parity games. Formally, we suggest the following.

Conjecture 8.1 Let $\mathcal{G}$ be a $C(1, d)_a$ game. We can construct a $D(0, d)$ game $\mathcal{G}'$ such that for every vertex $v \in V_{\mathcal{G}}$, player 1 has an (almost-sure) winning strategy from v in $\mathcal{G}$ if and only if player 1 has a (sure) winning strategy from v in $\mathcal{G}'$.

Since $D(0, d)$ games can be solved in time $O(n^{\lfloor d/2 \rfloor + 1})$ [Jur00], this would imply improving the asymptotic time complexity of solving almost-sure concurrent parity games with d priorities from $O(n^{d+1})$ [AH00] to $O(n^{\lfloor d/2 \rfloor + 1})$.

Finally, let us note that the ability to reduce concurrent games to turn-based games does not mean that concurrent games are a superfluous model. Concurrent games are appropriate for modeling concurrent systems in which the underlying components interact synchronously [AHM00, AHM01]. While our reductions are convenient for solving questions about such systems, the turn-based systems we construct no longer model the original system in any natural sense.

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Appendix

A Two simplifying assumptions for games with witnesses

For the sake of the presentation of the proofs of Lemmas 4.1, 4.2, 6.1, and 6.2, we are going to adopt two simplifying assumptions about games with witnesses. These assumptions are supposed to slightly simplify notations in proofs. We claim that the proofs can be routinely generalized to arbitrary witnesses and leave it to the reader.

- We assume that the witness functions are 1-1 and hence we can always consider the set of vertices to be a set $[n]_\infty$ or $[n]_\infty \times [n]$, for some $n \in \mathbb{N}$, where we identify a vertex with the value assigned to it by the witness function.
- If $\psi : V \to [n]_\infty$ is a witness for player 2 in a C(0,1)$_\text{a}$ game then we assume that $\psi(v)$ is even if and only if $v \in P_1$. Similarly, if $\varphi : V \to [n]_\infty$ is a witness for player I in a C(1,2)$_\text{a}$ game then we assume that $\psi(v)$ is even if and only if $v \in P_2$. In other words, we require that vertices which are "good" for the co-Büchi player are assigned an even number by the witness function.

  In fact, from a witness θ we can easily obtain another witness satisfying the above condition by taking $\theta'(v) = 2 \cdot \theta(v) + \theta(v) \bmod 2$.

By applying these two assumptions to the cases of games with witnesses considered in Lemmas 4.1, 4.2, 6.1, and 6.2, we obtain what we call "generic" C(0,1)$_\text{a}$, C(1,2)$_\text{p}$, C(1,2)$_\text{a}$, and C(0,1)$_\text{p}$ games, studied in Appendices B, C, F, and G, respectively.

In Appendices B, C, F we consider game structures with the set of vertices $V = [n]_\infty$, for some $n \in \mathbb{N}$. For $i \in [n]_\infty$ we use the following notations: $(< i) = \{k \in [n] \,:\, k < i\}$, and $(> i) = \{k \in [n]_\infty \,:\, k > i\}$.

In Appendix G we consider game structures with the set of vertices $V = [n]_\infty \times [n]$, for some $n \in \mathbb{N}$. For $i, j \in \mathbb{N}$, we use the following notation: $\big(< (i,j)\big) = \big\{ (k,l) \in [n] \times [n] \,:\, (k,l) <_{\text{lex}} (i,j) \big\}$.

B Generic C(0,1)$_\text{a}$ games (proof of Lemma 4.1)

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent almost-sure win reachability game (CR$_\text{a}$ game) if $V = \{0, 1, 2, \ldots, n, \infty\}$, for some $n \in \mathbb{N}$, and

$$\exists_{\varepsilon>0}.\ \forall_{i\in\{1,2,\ldots,n\}}.\ \exists_{\alpha\in M_1(i)}.\ \forall_{\beta\in M_2(i)}.\ \left(\text{Pr}_i^{\alpha,\beta}[< i] \geq \varepsilon \ \wedge\ \text{Pr}_i^{\alpha,\beta}[\infty] = 0\right) \tag{12}$$

Lemma B.1 *From every state $i \neq \infty$, player 1 has a (memoryless) strategy to eventually reach vertex 0 with probability 1.*

Proof. Let player 1 use a (memoryless) strategy, such that in state i he chooses a move $\alpha_i \in M_1(i)$, for which $\text{Pr}_i^{\alpha_i,\beta}[< i] \geq \varepsilon$ and $\text{Pr}_i^{\alpha_i,\beta}[\infty] = 0$, for all $\beta \in M_2(i)$. Note that with this strategy a play never enters the state ∞. Moreover, from every state different from state 0, with probability at least ε^n, a play can reach state 0 in no more than n steps. Therefore, using standard arguments we get that state 0 is reached with probability 1. [Lemma B.1] ■

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent almost-sure win Büchi game (C(0,1)$_\text{a}$ game) if it is a generic CR$_\text{a}$ game such that

$$\exists_{\alpha \in M_1(0)} . \forall_{\beta \in M_2(0)} . \ \Pr_0^{\alpha,\beta}[\infty] = 0. \tag{13}$$

Lemma B.2 *From every state $i \neq \infty$, player 1 has a (memoryless) strategy such that with probability 1 vertex 0 is visited infinitely often.*

An analogous argument as for Lemma B.1 works here.

C Generic C(1,2)$_\text{p}$ games (proof of Lemma 4.2)

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent positive-probability win co-Büchi game (C(1,2)$_\text{p}$ game) if $V = \{0, 1, 2, \ldots, n, \infty\}$, for some $n \in \mathbb{N}$, and

$$\forall_{\delta > 1} . \forall_{i \in \{0,1,\ldots,n\}} . \exists_{\alpha \in M_1(i)} . \forall_{\beta \in M_2(i)} . \tag{14}$$
$$\big(\text{if } i \text{ is odd then } \Pr_i^{\alpha,\beta}[< i] > 0\big) \ \wedge \tag{15}$$
$$\big(\text{if } i \text{ is even then } (\Pr_i^{\alpha,\beta}[< i] > 0 \ \vee \ \Pr_i^{\alpha,\beta}[> i] \leq 1/\delta)\big). \tag{16}$$

Lemma C.1 *From every state $i \neq \infty$, player 1 has a (counting) strategy, such that with positive probability finitely many odd vertices are visited in the play.*

Proof. First we show that for every $\varepsilon > 0$, player 1 has a (counting) strategy to stay in state 0 forever with probability at least $1 - \varepsilon$. For $j \in \mathbb{N}$, let $\alpha_j \in M_1(0)$ that for all $\beta \in M_2(0)$, we have $\Pr_0^{\alpha_j,\beta}[> 0] \leq \varepsilon/(2^{j+1})$. If player 1 plays move α_j in step j then the probability that the play ever leaves state 0 is at most $\sum_{j=1}^{\infty} \varepsilon/(2^{j+1}) = \varepsilon$.

We prove by induction on i that player 1 has a strategy such that a play started from vertex i with positive probability visits only finitely many odd vertices. Suppose that the induction hypothesis holds for all $k < i$; we prove it for i. We consider two cases.

- i is odd: Then by (15) player 1 can with positive probability force a move in one step to a state $k < i$ from where he has a positive-probability winning strategy.
- i is even: Then player 1 uses the following strategy in state i. In the j-th step he plays a move $\alpha_j \in A_1(i)$, such that for all $\beta \in A_2(i)$, we have either $\Pr_i^{\alpha_j,\beta}[< i] > 0$ or $\Pr_i^{\alpha_j,\beta}[> i] \leq \varepsilon/(2^{j+1})$. In case it never happens that $\Pr_i^{\alpha_j,\beta}[< i] > 0$ then similarly as in the base case, the probability of not staying in the even state i forever is at most $\sum_{j=1}^{\infty} \varepsilon/(2^{j+1}) = \varepsilon$. [Lemma C.1] ■

D Proof of Lemma 5.1

Proof. Suppose $v \in V$ is action φ-progressive for player 1. Consider the two cases.

- $v \in P_0$: By action φ-progressiveness of v it follows that there is $a \in A_1(v)$, such that for all $b \in A_2(v)$, we have $\Pr_v^{a,b}[\varphi_\infty] = 0$. If we take $\alpha \in \mathcal{D}(A_1(v))$, such that $\alpha(a) = 1$, then for every $\beta \in \mathcal{D}(A_2(v))$, we have $\Pr_v^{\alpha,\beta}[\varphi_\infty] = 0$.
- $v \in P_1$: By action φ-progressiveness of v it follows that for all $b \in A_2(v)$, there is $a_b \in A_1(v)$, such that both of the following hold:
  - $\Pr_v^{a_b,b}[\varphi_{<v}] > 0$, and
  - for all $b' \in A_2(v)$, we have $\Pr_v^{a_b,b'}[\varphi_\infty] = 0$.

  Let $\alpha \in \mathcal{D}(A_1(v))$ play all the actions in the set $A_B \subseteq A_1(v)$ uniformly at random, where $A_B = \{ a_b \ : \ b \in A_2(v) \}$, and

  From the second clause above we immediately get that for every $\beta \in \mathcal{D}(A_2(v))$, we have $\Pr_v^{\alpha,\beta}[\varphi_\infty] = 0$.

  Let $\varepsilon = \min_{b \in A_2(v)} \Pr_v^{a_b,b}[\varphi_{<v}]$. From the first clause above it follows that $\varepsilon > 0$. Then for all $\beta \in \mathcal{D}(A_2(v))$, we have

$$\Pr_v^{\alpha,\beta}[\varphi_{<v}] \geq \sum_{b \in A_2(v)} \alpha(a_b) \cdot \beta(b) \cdot \Pr_v^{a_b,b}[\varphi_{<v}] \geq \frac{\varepsilon}{|A_B|}.$$

In both cases we conclude that vertex v is φ-progressive for player 1.

Suppose $v \in V$ is action ψ-progressive for player 2. Consider the two cases.

- $v \in P_0$: By action ψ-progressiveness of v it follows that for all $a \in A_1(v)$, there is $b_a \in A_2(v)$, such that $\Pr_v^{a,b_a}[\psi_{<v}] > 0$. Let $\beta \in \mathcal{D}(A_2(v))$ play all the actions in $B_A \subseteq A_2(v)$ uniformly at random, where $B_A = \{ b_a \ : \ a \in A_1(v) \}$. Then for every $\alpha \in \mathcal{D}(A_1(v))$ we get

$$\Pr_v^{\alpha,\beta}[\psi_{<v}] \geq \sum_{a \in A_1(v)} \alpha(a) \cdot \beta(b_a) \cdot \Pr_v^{a,b_a}[\psi_{<v}] > 0.$$

- $v \in P_1$: By action ψ-progressiveness of v it follows that there is a $b \in A_2(v)$, such that for all $a \in A_1(v)$ we have either:
  - $\Pr_v^{a,b}[\psi_{>v}] = 0$, or
  - there is $b' \in A_2(v)$, such that $\Pr_v^{a,b'}[\psi_{<v}] > 0$.

  Let $\beta \in \mathcal{D}(A_2(v))$ play action b with probability $1 - (1/\delta)$ and all the other actions uniformly at random with the remaining probability. If for all $a \in ||\alpha||$, it holds that $\Pr_v^{a,b}[\psi_{>v}] = 0$, then we get $\Pr_v^{\alpha,\beta}[\psi_{>v}] \leq 1/\delta$. Otherwise, from the second clause above it follows that $\Pr_v^{\alpha,\beta}[\psi_{<v}] > 0$.

In both cases we conclude that vertex v is ψ-progressive for player 2. [Lemma 5.1] ■

E Proof of Theorem 5.2

Let $\varphi' : V' \to [n]_\infty$ be a witness for player 1, and let $\psi' : V' \to [n]_\infty$ be a witness for player 2 in the D(0,1) game $\mathcal{G}'$.

Proposition E.1 *Let h_0 be the initial vertex of the game $\mathcal{H}_0(v, a, b)$. Then the following hold.*

1. *If $\varphi'(h_0) < \infty$ then $\mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0$.*
2. *If $\psi'(h_0) < \psi(v)$ then $\mathrm{Pr}_v^{a,b}[\psi_{<v}] > 0$.*

Proposition E.2 *Let h_1 be the initial vertex of the game $\mathcal{H}_1(v, a, b)$. Then the following hold.*

1. *If $\varphi'(h_1) < \varphi(v)$ then $\mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0$.*
2. *If $\psi'(h_1) \leq \psi(v)$ then $\mathrm{Pr}_v^{a,b}[\psi_{>v}] = 0$.*

Proof of 1. We argue that if $v \in V$ is φ'-progressive for player 1 in game $\mathcal{G}'$ then it is action φ-progressive for player 1 in the C(0,1)$_\mathrm{a}$ game $\mathcal{G}$. Consider the two cases.

- $v \in P_0$: By φ'-progressiveness of v for player 1 it follows that there is $a \in A_1(v)$, such that for all $b \in A_2(v)$, we have $\varphi'(h_0) < \infty$, where h_0 is the initial vertex of the game $\mathcal{H}_0(v, a, b)$. Then by clause 1. of Proposition E.1 we have $\mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0$.
- $v \in P_1$: By φ'-progressivenes of v for player 1 it follows that for all $b \in A_2(v)$, there is $a \in A_1(v)$, such that both of the following hold:
  - $\varphi'(h_1) < \varphi(v)$, where h_1 is the initial vertex of the game $\mathcal{H}_1(v, a, b)$; hence by clause 1. of Proposition E.2 we get that $\mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0$.
  - for all $b' \in A_2(v)$, it holds that $\varphi'(h_0) < \infty$, where h_0 is the initial vertex of the game $\mathcal{H}_0(v, a, b')$; hence by clause 1. of Proposition E.1 we get that $\mathrm{Pr}_v^{a,b'}[\varphi_\infty] = 0$.

In both cases we get that vertex v is action φ-progressive for player 1 and hence by Lemma 5.1 vertex v is φ-progressive for player 1 in $\mathcal{G}$.

Proof of 2. The proof is analogous to the proof of 1.; use clauses 2. of Propositions E.1 and E.2.

[Theorem 5.2] ■

F Generic C(1,2)$_\mathrm{a}$ games (proof of Lemma 6.1)

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent almost-sure win co-Büchi game (C(1,2)$_\mathrm{a}$ game) if $V = \{0, 1, 2, \ldots, n, \infty\}$, for some $n \in \mathbb{N}$, and

$$\exists_{\varepsilon>0} \cdot \forall_{i \in \{0,1,\ldots,n\}} \cdot \exists_{\alpha \in M_1(i)} \cdot \forall_{\beta \in M_2(i)} \cdot \ \mathrm{Pr}_i^{\alpha,\beta}[\infty] = 0 \ \wedge \tag{17}$$

$$(\text{if } i \text{ is odd then } \mathrm{Pr}_i^{\alpha,\beta}[< i] \geq \varepsilon) \ \wedge \tag{18}$$

$$(\text{if } i \text{ is even then } \mathrm{Pr}_i^{\alpha,\beta}[< i] \geq \varepsilon \cdot \mathrm{Pr}_i^{\alpha,\beta}[> i]). \tag{19}$$

Lemma F.1 *From every state $i \neq \infty$, player 1 has a (memoryless) strategy such that with probability 1 odd vertices are visited only finitely many times.*

Proof sketch. For all $i \in \{0, 1, \ldots, n\}$, let α_i be an move $\alpha \in M_1(i)$ chosen as in (17). Let player 1 use the (memoryless) strategy which in a vertex $i \in \{0, 1, \ldots, n\}$ chooses the move α_i. Fix an arbitrary strategy for player 2. A play in which the players use these strategies can be viewed as a random walk W satisfying the condition (17)–(19).

From (19) it follows that for all $\beta \in M_2$, $\Pr_0^{\alpha_0,\beta}[> 0] = 0$, and hence a play started in vertex 0 stays there forever.

We argue that the following holds if a play is started from an arbitrary vertex in $\{1, 2, \ldots, n\}$:

with probability 1, the play either reaches vertex 0, or visits odd vertices only finitely many times. (20)

In order to establish (20) for the random walk W consider a random walk W' obtained by removing from W all the steps in which the vertex is even and the probability of going to a higher vertex in the next step is 0. Note that the modified random walk W' satisfies conditions (17)–(19) if W does. Moreover, if the random walk W' satisfies (20) then so does the original random walk W, hence it suffices to establish (20) for W'.

For that purpose let us further modify the random walk W' into a random walk W'', by setting the probability of staying in the same vertex in every step to 0, and scaling the probabilities of changing the vertex accordingly. In other words, if the current vertex is i and the probability of changing it to $j \neq i$ in W' is p_j, then the corresponding probability in the modified random walk W'' is defined to be $p_j/(\sum_{k \neq i} p_k)$. The important property of W'' is that if it satisfies (20) then so does W'. Hence it suffices to argue that W'' satisfies (20).

Observe that in the random walk W'' in every step the vertex decreases with probability at least $\varepsilon/(1+\varepsilon)$, and hence vertex 0 can always be reached with probability at least $\big(\varepsilon/(1+\varepsilon)\big)^r$ from a vertex r in at most r steps. It follows that vertex 0 is eventually reached with probability 1 in the random walk W''.

[Lemma F.1] ■

G Generic $C(0,1)_p$ games (proof of Lemma 6.2)

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent limit sure win reachability game (CR_a game) if $V = \{0, 1, 2, \ldots, n, \infty\}$, for some $n \in \mathbb{N}$, and

$$\begin{aligned} &\exists_{m>0} \,.\, \forall_{\delta>1} \,.\, \forall_{i \in \{1,2,\ldots,n\}} \,.\, \exists_{\alpha \in M_1(i)} \,.\, \forall_{\beta \in M_2(i)} \,.\\ &\quad (\Pr_i^{\alpha,\beta}[< i] \geq 1/\delta^m \ \wedge\ \Pr_i^{\alpha,\beta}[\infty] \leq (1/\delta) \cdot \Pr_i^{\alpha,\beta}[< i]) \end{aligned} \tag{21}$$

Lemma G.1 *For every $\varepsilon \in \mathbb{R}$, such that $0 < \varepsilon < 1$, from every state $i \neq \infty$, player 1 has a (memoryless) strategy to eventually reach vertex 0 with probability at least $1 - \varepsilon$.*

See Appendix G.1 for a proof of the Lemma.

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent limit sure win Büchi game (C(0,1)_a game) if it is a generic CR_l game, such that

$$\forall_{\delta>1}.\ \exists_{\alpha\in M_1(0)}.\ \forall_{\beta\in M_2(0)}.\ \Pr_0^{\alpha,\beta}[\infty] \leq 1/\delta. \tag{22}$$

Lemma G.2 *From every state $i \neq \infty$, for every $\varepsilon > 0$, player 1 has a (counting) strategy such that with probability at least $1 - \varepsilon$ vertex 0 is visited infinitely often.*

See Appendix G.2 for a proof of the Lemma.

A concurrent game structure $G = (V, M, M_1, M_2, \delta)$ is a generic concurrent positive-probability win Büchi game (C(0,1)_p game) if $V = [n]_\infty \times [n]$, for some $n \in \mathbb{N}$, and

$$\exists_{m\in\mathbb{N}}.\ \forall_{\delta>1}.\ \forall_{i\in\{0,1,\dots,n\}}.\ \exists_{\alpha\in M_1(i)}.\ \forall_{\beta\in M_2(i)}. \tag{23}$$

$$\left(\Pr_{(i,0)}^{\alpha,\beta}[< i] > 0\ \vee\ \Pr_{(i,0)}^{\alpha,\beta}[> i] \leq 1/\delta\right)\ \wedge \tag{24}$$

$$\Big(\text{if } j > 0 \text{ then } \big(\Pr_i^{\alpha,\beta}[< i] > 0\ \vee\ (\Pr_i^{\alpha,\beta}[> i] \leq 1/\delta^m\ \wedge\ \Pr_{(i,j)}^{\alpha,\beta}[> i] \leq (1/\delta)\cdot \Pr_{(i,j)}^{\alpha,\beta}[< (i,j)])\big)\Big) \tag{25}$$

Lemma G.3 *From every state (i, j), such that $i \neq \infty$, player 1 has a (counting) strategy, such that with positive probability vertices in $\{0, 1, \dots, n\} \times \{0\}$ are visited infinitely often.*

See Appendix G.3 for a proof of the Lemma.

G.1 Proof of Lemma G.1

Proof. Define $\delta = 1/\varepsilon$, $\delta_1 = \delta$, and $\delta_{i+1} = \delta_i^{m+1}$, for all $i \in \{1, 2, \dots, n-1\}$. For all $i \in \{1, 2, \dots, n\}$, let $\alpha_i \in M_1(i)$ be a move α chosen as in (21) for $\delta = \delta_i$. For all $\beta \in M_2(i)$, we define $p_i^\beta = \Pr_i^{\alpha_i,\beta}[< i]$ and $q_i^\beta = \Pr_i^{\alpha_i,\beta}[\infty]$. Let player 1 use the (memoryless) strategy which in a vertex $i \in \{1, 2, \dots, n\}$ chooses the move a_i. Note that for all $\beta \in M_2(i)$, we have:

$$p_i^\beta \geq \frac{1}{\delta_i^m} \tag{26}$$

and

$$\frac{q_i^\beta}{p_i^\beta} \leq \frac{1}{\delta_i}. \tag{27}$$

By r_i denote the probability of reaching the vertex ∞ from a vertex in $\{i, i+1, \dots, n\}$ before a vertex in $\{0, 1, \dots, i-1\}$ is reached.

First we derive an upper bound on r_n. ¿From vertex n, for every move $\beta \in M_2(n)$ of player 2, either vertex ∞ can be reached in one step (with probability at most q_n^β), or in the first step vertex n is visited again (with probability at most $1 - p_n^\beta$.) Therefore, we have $r_n \leq q_n^\beta + (1 - p_n^\beta) \cdot r_n$, so by (27) we get:

$$r_n \leq \frac{q_n^\beta}{p_n^\beta} \leq \frac{1}{\delta_n}. \tag{28}$$

We prove by induction on $n-i$, that

$$r_i \leq \frac{n-i+1}{\delta_i}. \tag{29}$$

Suppose a play starts in a vertex in the set $\{i, i+1, \ldots, n\}$. Then there are the following options:

- vertex ∞ is reached before vertex in $\{0, 1, \ldots, i\}$ is reached (with probability at most r_{i+1}), or
- vertex i is reached and vertex ∞ is then reached in the next step (with probability at most q_i^β, where $\beta \in M_2(i)$ is the move played by player 2 when i is reached for the first time), or
- vertex i is reached and then a vertex in $\{i+1, \ldots, n\}$ is reached in the next step (with probability at most $1 - p_i^\beta$.)

Therefore we have $r_i \leq r_{i+1} + q_i^\beta + (1 - p_i^\beta) \cdot r_i$, and by the induction hypothesis (29), (26), and (27) we get:

$$r_i \leq \frac{r_{i+1} + q_i^\beta}{p_i^\beta} \leq \frac{n-i}{\delta_{i+1}} \cdot \delta_i^m + \frac{1}{\delta_i} = \frac{n-i}{\delta_i^{m+1}} \cdot \delta_i^m + \frac{1}{\delta_i} = \frac{n-i+1}{\delta_i}.$$

Hence, we have $r_1 \leq n/\delta_1 = n/\delta = n \cdot \varepsilon$, i.e., player 1 reaches vertex 0 with probability at least $1 - n \cdot \varepsilon$.

[Lemma G.1] ■

G.2 Proof of Lemma G.2

Proof. ¿From Lemma G.1 it follows that for every ε, such that $0 < \varepsilon < 1$, player 1 has a strategy σ_ε to reach vertex 0 with probability at least $1 - \varepsilon$, and move in one step from vertex 0 to vertex ∞ with probability at most ε. For all $k \in \mathbb{N}$, define $\varepsilon_k = 1 - (1-\varepsilon)^{1/2^{k+1}}$. Let player 1 use the strategy σ_{ε_k} between the k-th visit to vertex 0 and the $(k+1)$-st. Note that:

$$\prod_{k=1}^{\infty}(1 - \varepsilon_k) = \prod_{i=1}^{\infty}(1-\varepsilon)^{1/2^{k+1}} = (1-\varepsilon)^{1/2}.$$

Then the probability that vertex 0 is visited infinitely often is at least $\prod_{k=1}^{\infty}(1-\varepsilon_k)^2 = 1-\varepsilon$. [Lemma G.2] ■

G.3 Proof of Lemma G.3

Proof sketch. We say that the i-th layer consists of the vertices in the set $\{i\} \times \{0, 1, \ldots, n\}$. We prove the lemma for all vertices $(i, j) \in \{0, 1, \ldots, n\}^2$, by induction on i.

For $i = 0$, it suffices to observe that if we merge the vertices in layers from 1 to ∞ into a single vertex ∞, then we get a generic C(0,1)$_1$ game. Therefore, by Lemma G.2 we get that player 1 has a (counting) strategy from every state $(0, j)$, such that with positive probability the vertex $(0, 0)$ is visited infinitely often. In fact, he can make this probability arbitrarily close to 1.

For the inductive step, assume that player 1 has a positive probability winning strategy from all states in layers from 0 to $i - 1$. As in the base case, let us merge the vertices in layers from $i + 1$ to ∞ into the single vertex ∞. In this way the vertices in the i-th layer can be seen to form a generic $C(0,1)_1$ game. The only difference is that for certain moves probability of moving to a vertex in layers 0 to $i-1$ is positive. Lemma G.2 can be easily adapted to argue that either with positive probability a vertex in layers 0 to $i-1$ is reached, or with positive probability the state $(i, 0)$ is visited infinitely often (in fact, player 1 can make the latter probability arbitrarily close to 1.) [Lemma G.3] ■

H Proof of Lemma 7.1

Proof. Suppose $v \in V$ is action φ-progressive for player 1. Consider the two cases.

- $v \in P_1$: The same as in the proof of the first part of Lemma 5.1!
- $v \in P_2$: Let $X \subseteq A_1(v)$ be such that $X \neq \emptyset$ and the right-hand side of the second implication in (11) holds. Let $\alpha \in \mathcal{D}(A_1(v))$ play actions in X uniformly at random.

  From the first conjunct it follows that for all $\beta \in \mathcal{D}(A_2(v))$, it holds that $\mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0$.

  Define: $\varepsilon = \min_{a \in X, b \in A_2(v)} \{ \mathrm{Pr}_v^{a,b}[\varphi_{<v}] \ : \ \mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0 \}$. From the second conjunct it follows that, for all $a \in X$, and $b \in A_2(v)$, we have that either:

  - $\mathrm{Pr}_v^{a,b}[\varphi_{>v}] = 0$, or
  - $\mathrm{Pr}_v^{a_b,b}[\varphi_{<v}] \geq \varepsilon$, for some $a_b \in X$,

  and hence we get:

  $$\mathrm{Pr}_v^{a_b,b}[\varphi_{<v}] \geq \varepsilon \cdot \mathrm{Pr}_v^{a,b}[\varphi_{>v}]. \tag{30}$$

  Therefore, for all $\beta \in \mathcal{D}(A_2(v))$ we have:

  $$\begin{aligned}
  \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{<v}] &= \sum_{b \in A_2(v)} \sum_{a \in X} \alpha(a) \cdot \beta(b) \cdot \mathrm{Pr}_v^{a,b}[\varphi_{<v}] \\
  &\geq \sum_{b \in A_2(v)} \frac{1}{|X|} \cdot \beta(b) \cdot \mathrm{Pr}_v^{a_b,b}[\varphi_{<v}] \\
  &= \frac{1}{|X|^2} \cdot \sum_{b \in A_2(v)} \sum_{a \in X} \beta(b) \cdot \mathrm{Pr}_v^{a_b,b}[\varphi_{<v}] \\
  &\geq \frac{\varepsilon}{|X|^2} \cdot \sum_{b \in A_2(v)} \sum_{a \in X} \beta(b) \cdot \mathrm{Pr}_v^{a,b}[\varphi_{>v}] \\
  &= \frac{\varepsilon}{|X|} \cdot \sum_{b \in A_2(v)} \sum_{a \in X} \alpha(a) \cdot \beta(b) \cdot \mathrm{Pr}_v^{a,b}[\varphi_{>v}] \\
  &= \frac{\varepsilon}{|X|} \cdot \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{>v}].
  \end{aligned}$$

  In both cases we get that vertex v is φ-progressive for player 1. [Lemma 7.1] ■

I Proof of Theorem 7.2

Proof idea: The structure of the proof is analogous to that of Theorem 5.2. Specifically, it suffices to establish the following.

1. If $\varphi' : V' \to [n]_\infty$ is a witness for player 1 in the D(0,2) game $\mathcal{G}'$ then the restriction φ of φ' to V is a witness for player 1 in the C(1,2)$_\mathrm{a}$ game $\mathcal{G}$.
2. If $\psi' = (\psi'^0, \psi'^2)$ is a witness for player 2 in the D(0,2) game $\mathcal{G}'$ then the restriction ψ of ψ' to V is a witness for player 2 in the C(1,2)$_\mathrm{a}$ game $\mathcal{G}$.

Let $\varphi' : V' \to [n]_\infty$ be a witness for player 1, and let $\psi' = (\psi'^0, \psi'^2)$ be a witness for player 2 in the D(0,2) game $\mathcal{G}'$. Let φ and $\psi = (\psi^0, \psi^2)$ be restrictions of φ' and ψ', respectively, to the set V of vertices of game $\mathcal{G}$.

Similarly as in the case of Büchi games in Section 5, let us informally note that the one-step games $\mathcal{H}_i(v, a, b)$, for $i \in \{0, 1, 2\}$, acts as gadgets that allow:

- in game $\mathcal{H}_0(v, a, b)$: player 1 to "verify" the condition $\mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0$, and player 2 to "verify" the condition $\mathrm{Pr}_v^{a,b}[\psi_{<v}] > 0$; and
- in game $\mathcal{H}_1(v, a, b)$: player 1 to "verify" the condition $\mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0$, and player 2 to "verify" the condition $\mathrm{Pr}_v^{a,b}[\psi^0_{>v}] = 0$; and
- in game $\mathcal{H}_2(v, a, b)$: player 1 to "verify" the condition $\mathrm{Pr}_v^{a,b}[\varphi_{>v}] = 0$, and player 2 to "verify" the condition $\mathrm{Pr}_v^{a,b}[\psi_{<v}] > 0$.

The following propositions formalize this intuition.

Proposition I.1 *Let h_0 be the initial vertex of the game $\mathcal{H}_0(v, a, b)$. Then the following hold.*

1. *If $\varphi'(h_0) < \infty$ then $\mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0$.*
2. *If $\psi'(h_0) < \psi(v)$ then $\mathrm{Pr}_v^{a,b}[\psi^0_{<v}] > 0$.*

Proposition I.2 *Let h_1 be the initial vertex of the game $\mathcal{H}_1(v, a, b)$. Then the following hold.*

1. *If $\varphi'(h_1) < \varphi(v)$ then $\mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0$.*
2. *If $\psi'(h_1) \le \psi(v)$ then $\mathrm{Pr}_v^{a,b}[\psi^0_{>v}] = 0$.*

Proposition I.3 *Let h_2 be the initial vertex of the game $\mathcal{H}_2(v, a, b)$. Then the following hold.*

1. *If $\varphi'(h_2) \le \varphi(v)$ then $\mathrm{Pr}_v^{a,b}[\varphi_{>v}] = 0$.*
2. *If $\psi'(h_2) < \psi(v)$ then $\mathrm{Pr}_v^{a,b}[\psi_{<v}] > 0$.*

Proof of 1. We argue that if $v \in \varphi'_{<\infty}$ then it is action φ-progressive for player 1 in the C(1,2)$_\mathrm{a}$ game $\mathcal{G}$. Consider the two cases.

- $v \in P_1$: The same as in the proof of Theorem 5.2.
- $v \in P_2$: Let $X = \{\, a \in A_1(v) \ : \ \varphi'\big((v,a)\big) \leq \varphi(v) \,\}$, i.e., we have $a \in X$ if vertex (v,a) witnesses that vertex v is φ'-progressive for player 1. By φ'-progressiveness of v it follows that $X \neq \emptyset$, because v has priority 2. We claim that both of the following hold:

  1. $\forall_{a \in X} . \forall_{b \in A_2(v)} . \ \mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0$, and
  2. $(\forall_{a \in X} . \forall_{b \in A_2(v)} . \mathrm{Pr}_v^{a,b}[\varphi_{>v}] = 0) \ \vee \ (\exists_{a' \in X} . \mathrm{Pr}_v^{a',b}[\varphi_{<v}] > 0)$.

  Clause 1. follows from Proposition I.1 by noting that for all $a \in X$, φ'-progressiveness of vertex (v,a) implies that for all $b \in A_2(v)$, we have $\varphi'(h_0) \leq \varphi\big((v,a)\big) \leq \varphi(v) < \infty$, where h_0 is the initial vertex of the game $\mathcal{H}_0(v,a,b)$.

  Similar reasoning, i.e., several successive applications of the definition of φ'-progressiveness on the game graph of $\mathcal{G}'$, starting from vertices (v,a), for $a \in X$, and using Propositions I.2 and I.3, prove clause 2.

In both cases we get that vertex v is action φ-progressive for player 1 and hence by Lemma 7.1 vertex v is φ-progressive for player 1 in $\mathcal{G}$.

Proof of 2. Let $v \in \psi'_{<\infty}$, i.e., vertex v is ψ'-progressive for player 2 in the D(0,2) game $\mathcal{G}'$. The case when $v \in P_1$ can be handled in a similar way as in the proof of Theorem 5.2. We cover the more difficult case when $v \in P_2$. We argue that v is ψ-progressive for player 2 in the C(1,2)$_\mathrm{a}$ game $\mathcal{G}$.

Consider the following "ψ'-strategy subgraph" of the game graph of $\mathcal{G}'$, called "the (strategy) subgraph" below: vertex v is in the subgraph, vertices of player 1 have all their successors in the subgraph, and vertices of player 2 have exactly one successor in the subgraph, and ψ' is a witness for player 2 in the subgraph. Note that for every $a \in A_1(v)$, we have the unique $b_a \in A_2(v)$, such that vertex (v,a,b_a) is the successor of vertex (v,a) in the subgraph. Let $B' = \{\, b_a \ : \ a \in A_1(v) \,\}$. Let $B'' = \{\, b \in B' \ :$ vertex (v,b) is in the strategy subgraph $\}$.

Observe that the fragment of the strategy subgraph containing vertex v and the tuple-vertices whose first component is v is acyclic, since a cycle would contain only vertices with priority 2, and hence would be "losing" for player 2.

Let $m = |B'|$. We assign unique ranks from 1 to m to elements of B' as follows. The highest ranks are assigned arbitrarily to vertices in $B' \setminus B''$. The ranks of vertices in B'' are assigned in such a way that if there is a path from (v,b) to (v,b') in the strategy subgraph then rank$(b) >$ rank(b'). Such a rank assignement exists because, as mentioned above, the fragment of the strategy subgraph corresponding to vertex v is acyclic.

Fix $\delta > 2$. We define $\beta \in \mathcal{D}(B')$ by $\beta(b) = p_{\mathrm{rank}(b)}$, where:

$$p_1 + p_2 + \cdots + p_m = 1 \tag{31}$$

and:

$$p_i = \frac{p_{i-1}}{\delta}, \text{ for all } i \in \{2, 3, \ldots, m\} \tag{32}$$

A simple calculation shows that for $\delta > 2$ we have:

$$p_m \geq \frac{1}{\delta^m}. \tag{33}$$

If $\Pr_v^{\alpha,\beta}[\psi^0_{<v}] > 0$ then v is ψ-progressive for player 2 and we are done. Assume therefore that $\Pr_v^{\alpha,\beta}[\psi^0_{<v}] = 0$. It suffices to establish the following two lemmas.

Lemma I.4 *We have* $\Pr_v^{\alpha,\beta}[\psi_{<v}] \geq \lambda/\delta^m$*, for some constant* $\lambda > 0$.

Proof. Let $\lambda = \min_{a\in||\alpha||}\{\Pr_v^{a,b_a}[\psi_{<v}]\}$. Note that $\lambda > 0$, and for every $a \in ||\alpha||$, we have $\Pr_v^{a,b_a}[\psi_{<v}] \geq \lambda$. Therefore we get:

$$\Pr_v^{\alpha,\beta}[\psi_{<v}] \geq \sum_{a\in||\alpha||} \alpha(a) \cdot \beta(b_a) \cdot \Pr_v^{a,b_a}[\psi_{<v}] \geq \sum_{a\in||\alpha||} \lambda \cdot \alpha(a) \cdot \frac{1}{\delta^m} = \frac{\lambda}{\delta^m}.$$

[Lemma I.4] ■

Lemma I.5 *We have* $\Pr_v^{\alpha,\beta}[\psi_{<v}] \geq \mu \cdot \delta \cdot \Pr_v^{\alpha,\beta}[\psi^0_{>v}]$*, for some constant* $\mu > 0$.

Proof. First we establish the following instrumental lemma.

Lemma I.6 *For every* $b \in B'$*, and* $a \in ||\alpha||$*, if* $\Pr_v^{a,b}[\psi^0_{>v}] > 0$ *then there is* $b' \in B'$*, such that rank*$(b') <$ *rank*(b) *and* $\Pr_v^{a,b'}[\psi_{<v}] > 0$*, and hence we have*

$$\beta(b') \cdot \Pr_v^{a,b'}[\psi_{<v}] \geq \kappa \cdot \delta \cdot \beta(b) \cdot \Pr_v^{a,b}[\psi^0_{>v}], \tag{34}$$

for some constant $\kappa > 0$.

Proof. If $\Pr_v^{a,b}[\psi^0_{>v}] > 0$ then by Proposition I.2 the initial vertex of the one-step game $\mathcal{H}_1(v, a, b)$ is not in the strategy subgraph. Therefore, the vertex (v, a) must be the unique successor of vertex (v, b, a) in the strategy subgraph. Then the initial vertex of the game $\mathcal{H}_2(v, a, b_a)$ must be in the strategy subgraph, and so by Proposition I.3 we get that $\Pr_v^{a,b_a}[\psi_{<v}] > 0$. Observe that rank$(b_a) <$ rank(b) since there is a path from (v, b) to (v, b_a) in the strategy subgraph. [Lemma I.6] ■

¿From Lemma I.6 it follows that every non-zero term $\alpha(a) \cdot \beta(b) \cdot \Pr_v^{a,b}[\psi^0_{>v}]$ in the sum

$$\Pr_v^{\alpha,\beta}[\psi^0_{>v}] = \sum_{a\in||\alpha||,b\in B'} \alpha(a) \cdot \beta(b) \cdot \Pr_v^{a,b}[\psi^0_{>v}],$$

there is a term $\alpha(a) \cdot \beta(b') \cdot \Pr_v^{a,b'}[\psi_{<v}]$ in the sum

$$\Pr_v^{\alpha,\beta}[\psi_{<v}] = \sum_{a\in||\alpha||,b\in B'} \alpha(a) \cdot \beta(b) \cdot \Pr_v^{a,b}[\psi_{<v}],$$

such that

$$\beta(b') \cdot \text{Pr}_v^{a,b'}[\psi_{<v}] \geq \delta \cdot \beta(b) \cdot \text{Pr}_v^{a,b}[\psi^0_{>v}],$$

and hence

$$\text{Pr}_v^{\alpha,\beta}[\psi_{<v}] \geq \frac{\kappa}{|A_1(v)| \cdot m} \cdot \delta \cdot \text{Pr}_v^{\alpha,\beta}[\psi^0_{>v}].$$

Take $\mu = \kappa/(|A_1(v)| \cdot m)$. [Lemma I.5] ■ [Theorem 7.2] ■