Serial Monopoly on Blockchains

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Abstract

We study the following problem that is motivated by Blockchains where "miners" are serially given the monopoly for assembling transactions into the next block. Our model has a single good that is sold repeatedly every day where new demand for the good arrives every day. The novel element in our model is that all unsatisfied demand from one day remains in the system and is added to the new demand of the next day. Every day there is a new monopolist that gets to sell a fixed supply s of the good and naturally chooses to do so at the monopolist's price for the combined demand. What will the dynamics of the prices chosen by the sequence of monopolists be? What level of efficiency will be obtained in the long term?

We start with a non-strategic analysis of users' behavior and our main result shows that prices keep fluctuating wildly and this is an endogenous property of the model and happens even when demand is stable with nothing stochastic in the model. These price fluctuations underscore the necessity of an analysis under strategic behavior of the users, which we show results in the prices being stable at the market equilibrium price.

1 Motivation: Transaction Fees on Blockchains

Blockchain systems like Bitcoin [10] or Ethereum [2] sell "slots" on the blockchain to users who wish to put their transactions on it. Every period a "leader" (miner, validator, sequencer) is chosen to produce the next "block" in the blockchain, where the choice of the leader is done using some mechanism that need not concern us here such as proof-of-work or proof-of-stake. The size of each block is limited by the protocol in some way (e.g., bytes for Bitcoin or "gas" for Ethereum), and the leader gets to choose which transactions will fill the block up to that limit. The blockchain's transaction fee mechanism specifies how much the users of chosen transactions pay and how much the leader receives (in addition to a fixed "block reward") and needs to take into account that both the users and the leaders are strategic.

The mechanism used by the Bitcoin blockchain is simple *"pay your bid"*: users place bids for their transactions and the leader (miner) gets to choose an arbitrary subset of transactions and charges each of them exactly what was bid for it. Clearly a strategic leader will accept the highest bidding transactions (normalized to their "size") that fit within the block size limitations. A strategic user will obviously shade his bid by an amount that is not easy to calculate well.

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The Ethereum blockchain has a mechanism, known as EIP-1559 [3], that aims to be more straightforward for users to bid. The mechanism's most significant feature is that it uses *"dynamic posted prices"* where the ("base-gas") price for the next block is determined by the protocol as a function of previous blocks. Each user bids a maximum price and only transactions that bid at least the block's price will be included in the block, and they all pay the block's fixed gas price (rather than their bid). A significant additional feature of EIP-1559 is that all fees are "burnt" rather than going to the leader who only gets a small additional first-price-like "tip". Burning the fees is required to ensure that leaders are not motivated to collude with users. Conceptually, since the (base-gas) price of a block is deterministically determined by the protocol according to the history, neither the leader nor the users have any advantage in manipulation. The formula that determines the block prices increases them when there has recently been more demand than supply and decreases them in the opposite case thus managing to balance the satisfied demand with the average supply of slots in a block. The exact incentives in this mechanism and related ones are formally defined and studied in [11] and further in [4], an analysis of the block price dynamics appears in [9] and the general class of "dynamic posted prices" mechanisms is studied in [5]. We will not delve deeper into the details of this mechanism as for our purposes the simple conceptual description above suffices.

The fee-burning part of this mechanism may be viewed as undesirable as it reduces the Ethereum token supply which may or may not be desired from other points of view. A third mechanism – that does not require fee burning – suggested in [1] and in [8] is to use *monopolist pricing*: each leader is allowed to choose an arbitrary price for his block and can then collect all transactions that are willing to pay this price (in [1] this is called generalized second price). The rational leader will certainly choose the monopolist price that maximizes the product of the resulting block size and the price. Intuitively, as transactions are expected to be small relative to the total block size one may expect users to be "price-takers" and thus not to have any significant incentive to shade their bids. This mechanism was analyzed in [8, 12, 1], but again for our purposes this simple intuition suffices.

A major difference between the monopolist pricing mechanism and the two previous ones is in what they optimize for. The first two should reach (close to) the market equilibrium and thus optimize "social welfare" – the total value of accepted transactions subject to the blockchain capacity limitations[1]. The monopolist pricing optimizes the leader's revenue and may lead to unbounded losses of welfare. While not optimizing social welfare is certainly a weakness of this mechanism, as [8] argues, optimizing revenue may be an advantage for the security of the blockchain. In particular they note that mechanisms that reach market equilibrium have the problematic property that if the platform's capacity suffices to handle *all* demand, then the prices would go down to 0 which may endanger the security of the blockchain.

All the discussion so far looked at a single block in isolation: it looked at the single leader of the block and the set of users for that block and assumed that they all were myopic i.e that their strategic considerations were only about the given block. This is the case both for the intuitive explanations above and for the formal analysis in the papers cited. While this assumption may be a good modeling choice for the leaders since in large systems we expect a single miner to only be chosen to be leader infrequently, it is not at all

[1]While the exact analysis may depend on the model, intuitively both "pay your bid" and "EIP-1559" should reach the market equilibrium.

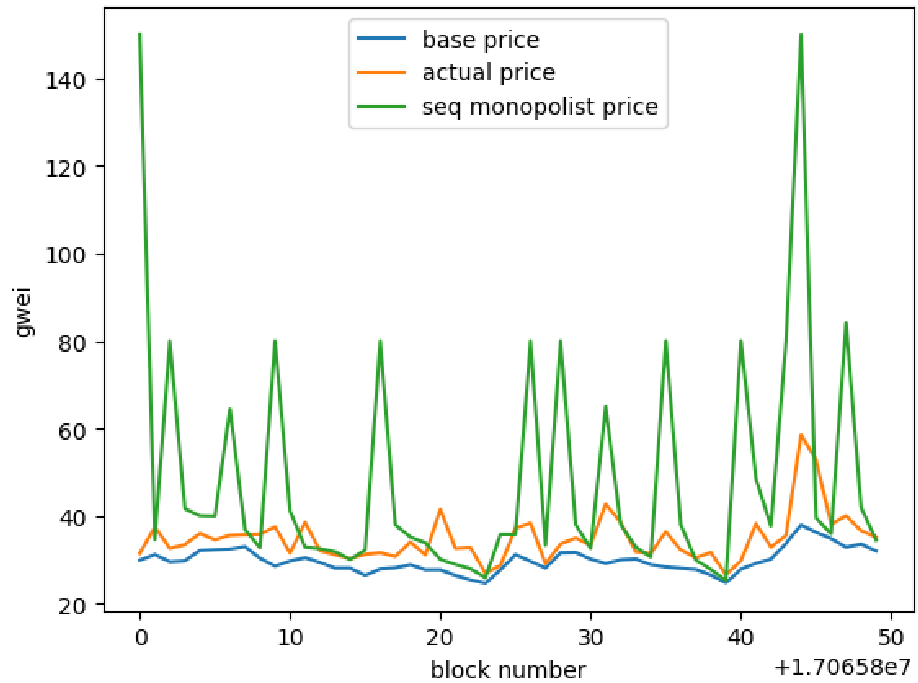

Figure 1: Simulation of serial monopoly on data from a sequence of 50 Ethereum blocks (10 minutes), comparing the series of monopolist prices to the true block gas prices with average tips (actual price) and without tips (EIP-1559 base price).

realistic for the users since a transaction that is not accepted to one block remains in the "mem-pool" and can be accepted into one of the next few blocks, potentially within less than a minute. Indeed [8] left the analysis of "patient users", as an open problem. Even ignoring the strategic behavior of users, just the fact that the unsatisfied transactions from one block remain as demand for future blocks, clearly leads to a nontrivial dynamic in the sequence of blocks.[2] Specifically, if a monopolist in some block has charged a high price, leaving much unsatisfied demand, then the next block will get this pent-up demand and thus see a heaver total demand at lower values which intuitively may cause the next block's price to be lower. Figure 1 shows a simulation of this dynamic on a data from a typical "uneventful" sequence of 50 Ethereum blocks. The serial monopoly dynamics extracted 12% more revenue on this sequence but lost 6.5% of total welfare (total sum of values of transactions, counted according to their bids). The heart of this paper is trying to analyze this behavior, understanding the fluctuations in prices, the welfare loss, and the implied strategic considerations of the users.

In our model we have a series of monopolists where each of them is faced with all the pent-up demand from previous blocks as well as some new flow of demand and then gets to chose his monopolist price. We start by analyzing the dynamics assuming that bidders are non-strategic or, equivalently, act myopically as previously studied, even though their valuation is really patient, i.e. they get the value of their transaction even if it is scheduled at later blocks.[3]

[2]This is somewhat in the spirit of, e.g., [6, 7] in a different context.

[3]Our analysis focuses on "fully patient" bidders whose value for the transaction does not decay with time. This also models well scenarios where value decay happens at time scales that are significantly larger than the block times, a situation that seems to fit most transactions on the blockchain with notable exceptions being MEV and some DeFi. One may certainly also consider intermediate levels of patience

There are several reasons for exploring such myopic behavior by non-myopic users. First, this is a natural first step before continuing with a strategic analysis. This is especially true in this case where previous argumentation as well as intuition may suggest that users are close to being "price takers" and gain little, if at all, from strategizing. Second, this can explain simulations, such as that given in figure 1, that are run on existing data. Finally, this analysis will turn out to have implications for the strategic analysis that we will do later.

We will formally describe our model and results in the next subsection but, for now, let us state their intuitive implications for blockchains. Our main, and surprising, result is that monopolist pricing dynamics leads to never-ending price fluctuations and this happens endogenously without any stochastic element in the model and when the exogenous conditions are completely stable. In this respect serial monopoly completely sacrifices one of the main desiderata for a fee mechanism, that of price stability. Even worse, once prices fluctuate, users are motivated to shade their bids and wait for lower prices. This happens even when bidders are small and each one of them does not affect prices at all. Despite being a price taker within a single block, shading bids is highly beneficial across blocks.

We do find some silver lining here regarding the social welfare achieved. Recall that a main concern regarding monopolist pricing is that it sacrifices efficiency since leaders do not fill block completely. We do, however, show that the social welfare (total value of accepted transactions) achieved by the serial monopoly rule (still with users bidding non-strategically) is mathematically guaranteed to be at least one half of the optimal social welfare. As usual when one can prove a formal guarantee, things are better in most specific cases. E.g., for transaction values that are uniform in $[0, 1]$ we calculate the loss of social welfare to be only 6.25%. And, nicely, this happens while gaining on revenue.

As mentioned, this finding of price fluctuations strongly suggests that patient users should bid strategically and so calls for an analysis under strategic bids by the users. Continuing to analyze "fully-patient" users, but now acting strategically, we find that, in equilibrium, users shade their bids so that the system is always at the market prices without any price fluctuations. However, as this shading requires information about the market conditions, optimal bidding may be difficult.

We are thus back where we started, having reached essentially the same outcome and difficulties as the pay-your-bid mechanism. I.e., once strategic patience is taken into account the main motivations of [8] of near incentive-compatibility and of better revenue are lost. While one may view this as an overall negative conclusion given the original motivation, the outcome reached by serial monopoly should be the same as that reached by the pay-our-bid mechanism, and so it does look like a completely viable alternative for a fee mechanism. More refined comparison of the serial monopoly mechanism to the "vanilla" pay-your-bid mechanism may require experimentation and one may speculate that bidding can be easier for serial monopoly since, at equilibrium, my own payment does not depend on my own bid.

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where a transaction's value decays with time at some rate, and such cases would be expected to lie between the fully myopic and full patient extremes.

2 Serial Monopoly: Model and Results

We now formalize and analyze our model in abstract terms of a "serial monopoly" that may be of more general interest.

2.1 The (Non-Strategic) Model

So here is our model: At each time step $t = 1, 2, ...$ some demand for some homogeneous good arrives into the market. The daily demand is specified by a fixed demand function $Q(p)$ that specifies the demanded quantity at each price level p. For ease of exposition we will assume that Q is continuous and strictly decreasing.

Every day a new monopolist with a fixed daily supply s is chosen. This monopolist sees in front of him the total current demand $D^t()$ which is the sum of the pent-up demand from previous time steps and the new daily demand and chooses a price level p^t that maximizes his revenue. Specifically the price chosen by the monopolist at day t is the price p^t that maximizes the revenue $p \cdot min(s, D^t(p))$ and the quantity supplied is thus $q_t = D^t(p^t) \leq s$. The pent-up demand after this amount is supplied is given by a demand function $Z^t(p) = D^t(p) - q^t$ for $p \leq p_t$ and $Z^t(p) = 0$ for $p \geq p_t$. The total demand for the next time step is obtained by adding the daily demand $Q()$ to this pent-up demand. So, to summarize, here is the formal dynamics we study:

- A continuous and strictly decreasing demand function $Q()$ and a fixed supply amount s are given.
- There is initially no pent-up demand: $Z^0(p) = 0$ for all p.
- For every time step $t = 1...$:
  - The day t demand function is given by $D^t(p) = Z^{t-1}(p) + Q(p)$.
  - The day t monopolist price and quantity are given by $p^t = argmax_p(p \cdot min(s, D^t(p)))$ and $q_t = D^t(p^t)$.
  - The pent-up demand function after day t is given by $Z^t(p) = D^t(p) - q^t$ for $p \leq p^t$ and $Z^t(p) = 0$ for $p \geq p^t$.

This model simplifies matters as much as possible, in particular assuming (1) a fixed flow of demand, i.e. that the same demand distribution arrives every day, (2) that each monopolist has the same fixed supply amount, (3) that monopolists never repeat, i.e. are completely myopic and thus naturally behave as a simple monopolists, (4) the demand is "infinitely patient" so values do not decay with time and (5) that the daily demand is fixed, not chosen strategically, and fully known by the monopolist. Also note that the model is completely deterministic.

We would like to analyze what happens when this dynamics reaches an equilibrium: which prices and quantities would be reached and what is the resulting efficiency? We are in for an unpleasant surprise: the dynamics do not reach an equilibrium, but instead get some complex non-cyclic pattern of ever changing prices. Here is a typical example.

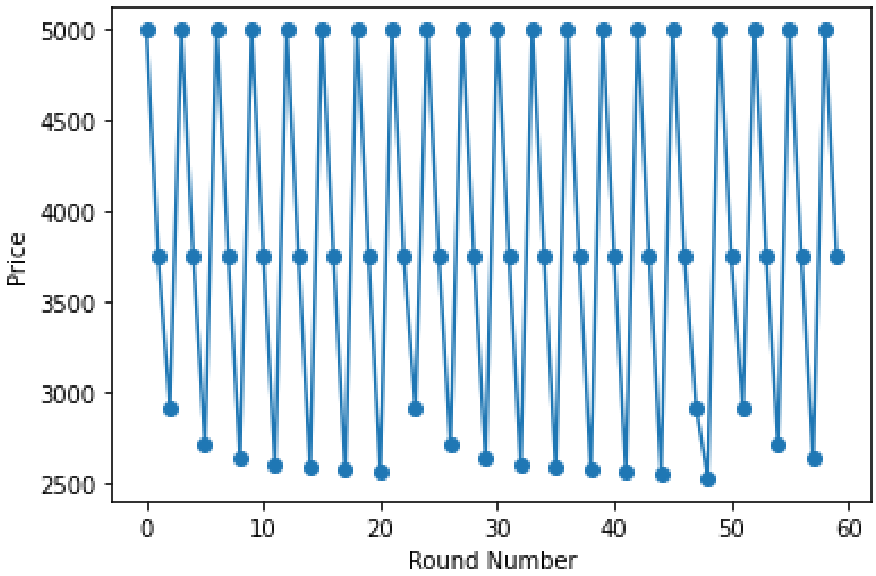

Figure 2: Serial monopoly prices for demand that is uniform on {1,2,...,10000}

2.2 Example

Assume that the daily demand is generated by user valuations that are uniform in $[0, 1]$, i.e. the daily demand function is given by $Q(p) = 1 - p$ (for $0 \le p \le 1$) and let us assume that the fixed daily supply is $s = 1$. The market equilibrium is at $price = 0$ and $quantity = 1$, which give a total social welfare of 1/2, and revenue of 0. The monopolist aims to maximize $p \cdot (1 - p)$ which happens for for $price = 0.5$ and $quantity = 0.5$ with sub-optimal social welfare of 3/8 and monopolist's revenue of 1/4. So let us follow the serial monopolists step by step.

1. Day 1: There is no pent-up demand so the first monopolist will indeed choose the monopoly price of $p^1 = 0.5$ with quantity $q^1 = 0.5$ obtaining revenue of 1/4. The pent-up demand after the first step is given by $1/2 - p$ for $p \le 1/2$ (and 0 for $p \ge 1/2$).

2. Day 2: The total demand at this stage is $3/2-2p$ for $p \le 1/2$ and is $1-p$ for $p \ge 1/2$. The monopolist now can do better than choosing the original monopoly price by choosing $p^2 = 3/8 = 0.375$ with quantity $q^2 = 3/4$ and revenue of $9/32 > 1/4$. The pent-up demand now is $3/4 - 2p$ for $p \le 3/8$ (and 0 for $p \ge 3/8$).

3. Day 3: the total demand is now given by $7/4 - 3p$ for $p \le 3/8$ (and the usual $1 - p$ for $p \ge 3/8$). The optimal price turns out to be $p^3 = 7/24 = 0.29...$ with quantity $q^3 = 7/8$. Pent-up demand is $7/8 - 3p$ for $p \le 7/24$.

4. Day 4: now we have a total demand of $15/8 - 4p$ for $p \le 7/24$ (and the usual $1 - p$ for $p \ge 7/24$). Surprisingly, the pent-up demand does not help the monopolist: the demand at price 7/32 is already 1 so the highest revenue obtainable in the range $p \le 7/24$ is at this price which would give revenue of only $7/32 < 1/4$ so the optimal revenue is obtained at the original monopolist price of $p^4 = 0.5$.

5. A simulation of the first 60 steps with discretized values appears in figure 1 where

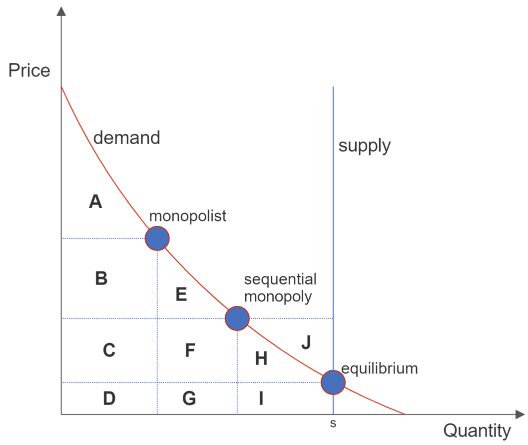

Figure 3: The Supply and demand curves with the market equilibrium, monopoly, and serial monopoly (price,quantity) points. Different areas that depict parts of the social welfare at these points are given labels.

we see that the prices keep irregularly oscillating. We wish to emphasize that the observed irregularity is *not* an artifact of the discretization or of the simulation.

2.3 Results

We first show that even though the prices keep oscillating irregularly, we can provide sufficient analysis of the price dynamics, in particular showing that there exists a price p^{ser} such that all demand above it is satisfied within a bounded time and all demand below it is never supplied at all. We show that this price is given by the formula $p^{ser} = p^{mon} \cdot q^{mon}/s$, where p^{mon} is the monopoly price of $Q()$ and q^{mon} the monopoly quantity. Thus, despite the lack of any convergence to equilibrium, p^{ser} can be viewed as the one "reached" by the dynamics in this sense of which demand gets supplied. The following theorem applies to any (strictly decreasing and continuous) demand function $Q()$ and supply amount s for which *the monopolist revenue is strictly higher than the market equilibrium revenue.*

Theorem 1. *The dynamics of the the daily prices p^t behave as follows:*

1. *They are sandwiched between the price $p^{ser} = p^{mon} \cdot q^{mon}/s$ and the monopolist price p^{mon}, i.e., for all t we have that $p^{ser} \leq p^t \leq p^{mon}$. In particular, no demand at prices lower than p^{ser} is ever supplied. Furthermore, these bounds are tight even in the limit and $p^{ser} = \liminf_{t\to\infty} p^t$ and $p^{mon} = \limsup_{t\to\infty} p^t$.*

2. *For every price $p > p^{ser}$ there exists a constant Δ_p such that in every consecutive Δ_p steps we have at least some $p^t \leq p$ and thus all demand at a price above p^{ser} is eventually supplied and furthermore this happens within a bounded time lag (where the bound Δ_p depends on the price).*

We then analyze the social welfare achieved by this dynamic. The "social welfare" is the sum (integral) of the values of the users whose demand is supplied. It is well known that monopolist pricing may cause an unbounded loss of social welfare. Surprisingly, we show that the social welfare achieved by serial monopolists approximates the optimum well! Since our analysis above shows that all demand above price p^{ser} is supplied, the average long term social welfare is depicted by the total area of regions A+B+C+D+E+F+G in figure 3. This welfare is obviously bounded by the optimal possible social welfare (depicted as the total areas of regions A+B+C+D+E+F+G+H+I in figure 3), but turns out to be not far from it.

Theorem 2. *The social welfare obtained by by the serial monopoly is at least one half of the optimal social welfare. This bound is tight as there exists some daily demand* $Q()$ *where the ratio is exactly* $1/2$.

This is a worst case result and for specific distributions the bounds are better. In particular, for the uniform distribution the ratio is 15/16 while for the "equal revenue" distribution, the classic example with a large gap between monopolist social welfare and optimal social welfare, serial monopoly turns out to yield optimal social welfare.

2.4 Strategic Analysis

All the analysis so far was for a fixed demand function and only analyzed the rational actions of the monopolists. If we imagine the demand coming from a continuum of utility-maximizing bidders (as in our motivating application) then this translates to these bidders acting in a myopic, non-strategic, way as in [8]. However the fact that, as we show, intra-block prices fluctuate wildly puts this modeling assumption in question as even price-taking bidders will likely be "patient", preferring to wait for cheaper blocks. Intuitively, patient users are motivated to strategically shade their bids down to (almost) p^{ser} which is the lowest price that they can get in any block. Once they all do so, the distribution that the monopolist sees in front of him is no longer $Q()$ but rather a strategically declared lower distribution and as he can only optimize his revenue relative to this distribution, intuitively leading to a lower serial price p'^{ser}, further reducing the bids of the users, etc.

This leads us to an equilibrium analysis of this system where both users and leaders are strategic. We stick with our simple non-stochastic deterministic model with price-taking bidders and have the daily demand distribution $Q()$ and supply amount s being common knowledge. We consider a game between multiple users and leaders: for each time step t we have a (new) continuum of infinitely-small users and a single (new) leader. Each of our users has a true value v, where v is chosen for each user in the continuum of users every time step according to the demand distribution Q. In each step t, each user with true value v declares a a value $\tilde{v}$ according to some strategic manipulation function that can be a function of his particulars: his value and the time, $\tilde{v} = m_t(v)$. The *declared demand* at time t, $\tilde{Q}^t$, is generated by the distribution of $\tilde{v}$ at time t. The leader at each time t may choose a price p^t and a *dominant strategy* is to choose the monopolist price for the *declared* current demand.[4]

We will focus on equilibria where the users' manipulation function is time-invariant, i.e. $m_t(v) = m(v)$ for some fixed manipulation function m, and where leaders all use their dominant (only non-weakly-dominated) strategy of monopoly pricing. This would be an equilibrium if for every bidder that is born in time t with value v, $m(v)$ is indeed a best reply to all the other users bidding according to m and all the leaders choosing the monopoly price.

[4]Since the leader may only collect a fee that is bounded by the *declared* value, the leader's knowledge of the true demand and even true values does not allow him to do any better.

It turns out that such an equilibrium of serial monopoly just goes back to the market equilibrium of the true demand, hence loosing the price fluctuations, optimizing social welfare, and giving up on any revenue optimization.

Theorem 3. *There exists an equilibrium in un-dominated strategies with time-invariant manipulation where $p^t = p^{eq}$ and $q^t = s$ for every t, where p^{eq} denotes the market equilibrium price, i.e. $Q(p^{eq}) = s$. In this equilibrium users bid $\tilde{v} = m(v) = min(v, p^{eq})$ and leaders charge monopoly prices.*

Furthermore, in every equilibrium in un-dominated strategies with time-invariant manipulation we have that $p^t = p^{eq}$ and $q^t = s$ for every t.

As in the non-strategic case, this analysis continued to consider bidders that "fully patient", i.e. assumed that user values do not decay with time. The next three sections are devoted, respectively, to the proofs of the three theorems, where most of the technical work is in the analysis of the dynamics in section 3.

3 Analysis of the Dynamics

This section gradually analyzes the (non-strategic) price dynamics in our model. All proofs of lemmas appear in the appendix.

3.1 The Evolution of Pent-up Demand

We will analyze the pent-up demand within intervals of prices, i.e., for $p' > p$, we are interested in $D^t(p) - D^t(p')$. The pent-up demand in the range $[p, p']$ evolves as follows: First, every time step a new amount of $Q(p) - Q(p')$ is added to the pent-up demand. Whenever some p^t is smaller than p then all of this demand is supplied and the pent-up demand for time $t + 1$ is zero. When p^t is larger than p' all of the pent-up demand just remains for the next step and whenever $p < p^t < p'$ then some of this demand is supplied and some is remains for the next step. Let us write this down formally.

When $p^{t-1} < p < p'$ we will have no pent-up demand $Z^{t-1}(p) = Z^{t-1}(p') = 0$ and thus $D^t(p) - D^t(p') = Q(p) - Q(p')$. When $p^{t-1} > p' > p$ we have that $Z^{t-1}(p) = D^{t-1}(p) - q^{t-1}$ and $Z^{t-1}(p') = D^{t-1}(p') - q^{t-1}$ so $Z^{t-1}(p) - Z^{t-1}(p') = D^{t-1}(p) - D^{t-1}(p')$ and $D^t(p) - D^t(p') = (D^{t-1}(p) - D^{t-1}(p')) + (Q(p) - Q(p'))$. Finally, when $p \le p^{t-1} \le p'$ we will have $Z^{t-1}(p') = 0$ while $Z^{t-1}(p) = D^{t-1}(p) - q^{t-1}$ and, since $D^{t-1}(p') \le q^{t-1} = D^{t-1}(p^{t-1})$, we have $Z^{t-1}(p) - Z^{t-1}(p') \le D^{t-1}(p) - D^{t-1}(p')$. Note that the last inequality holds in all three cases.

Summing this up over a prefix of times $\{1, 2, ...t\}$, or over a range of times $\{T+1, T+2, ...t\}$ we get the following lemma.

Lemma 4. *For every $p \le p'$ we have that:*

1. *For every t we have that $D^t(p) - D^t(p') \le t \cdot (Q(p) - Q(p'))$.*

2. *For all T and $t > T$ we have that: $D^t(p) - D^t(p') \le (t - T) \cdot (Q(p) - Q(p')) + (Z^T(p) - Z^T(p'))$.*

3. *For all T and $t > T$, if for all t' such that $T < t' < t$ we also have that $p^{t'} \ge p' > p$ then in fact we have equality $D^t(p) - D^t(p') = (t - T) \cdot (Q(p) - Q(p')) + (Z^T(p) - Z^T(p'))$.*

4. *For all T such that $p^T \leq p < p'$ (or $T = 0$) and all $t > T$ we have that $D^t(p) - D^t(p') \leq (t - T) \cdot (Q(p) - Q(p'))$.*

5. *For all T such that $p^T \leq p < p'$ (or $T = 0$), if for all t' such that $T < t' < t$ we also have that $p^{t'} \geq p' > p$, then $D^t(p) - D^t(p') = (t - T) \cdot (Q(p) - Q(p'))$.*

3.2 The Serial Monopoly Price and Quantity

Let us start by looking at the two basic (price,quantity) points of the daily market with demand $Q()$ and supply s: the market equilibrium and the monopolist pricing.

The market equilibrium point is when supply equals demand. i.e. at the price $p^{eq} = P(s) = Q^{-1}(s)$ (where $P()$ is the inverse function of $Q()$, at which point $q^{eq} = Q(p^{eq}) = s$, and the social welfare $SW^{eq} = \int_0^s P(q)dq$ is maximized. In figure 3 the social welfare at equilibrium is visualized the sum of the areas of regions A+B+C+D+E+F+G+H+I. The revenue at that point is $REV^{eq} = p^{eq} \cdot s$ which is visualized as the sum of regions D+G+I.

The monopolist chooses a price p^{mon} that maximizes $REV^{mon} = p^{mon} \cdot q^{mon}$, where $q^{mon} = Q(p^{mon})$ which on the graph is given by the sum of regions B+C+D. The social welfare at this point is $SW^{mon} = \int_0^{q^{mon}} P(q)dq$ which in figure reffig:sup-dem is given by the sum of regions A+B+C+D. As we assume that the monopolists revenue is strictly larger than the market equilibrium revenue $REV^{mon} > REV^{eq}$, we must have $p^{mon} > p^{eq}$, $q^{mon} < q^{eq}$, and $SW^{mon} < SW^{eq}$. The gap between REV^{mon} and REV^{eq} can be unbounded as the latter may even be 0 (if the demand is bounded by s, i.e. when $P(s) = 0$). The gap between SW^{mon} and SW^{eq} is also known to be potentially unbounded. Specifically, setting $H = P(0)/P(s)$, the gap can be large as $\ln(H)$, but no more.

We now define the "serial monopoly" point. While there is not going to be any convergence of the prices p^t, we will still be able to focus on a meaningful definition of (p^{ser}, q^{ser}) that captures useful information about the prices and quantities in the long term of the dynamic. We will define p^{ser} as the price at which selling all the supply would give the monopoly revenue.

Definition 3.1. The serial monopoly price and quantity are defined as $p^{ser} = p^{mon} \cdot q^{mon}/s$; $q^{ser} = Q(p^{ser})$.

Since $q^{mon} < s$ we have that $p^{ser} < p^{mon}$ and $q^{ser} > q^{mon}$. We also have $p^{ser} > p^{eq}$ and $q^{ser} < s$ since otherwise we would have $p^{eq} \cdot s = p^{ser} \cdot s = p^{mon} \cdot q^{mon}$, contradicting our assumption that the monopolist's revenue is strictly larger than the equilibrium revenue.

Comment: In the more general case where $Q()$ is not strictly decreasing, the definition of p^{ser} should be corrected to be the largest value p^{ser} such that $Q(p^{ser}) = Q(p^{mon} \cdot q^{mon}/s)$; we will continue to assume that Q is strictly decreasing so for us this complication is superfluous.

3.3 Basics of Price Dynamics

The first easy lemma states that p^{ser} is a clear lower bound for any p^t. This also provides the intuition for the particular choice of p^{ser}.

Lemma 5. *For every t we have that $p^t \geq p^{ser}$.*

Using lemma 4 (5) with $T = 0$ we see that all demand below p^{ser} remains as pent-up demand.

Corollary 6. *For every $p < p^{ser}$ and for all t, $D^t(p) - D^t(p^{ser}) = t \cdot (Q(p) - Q(p^{ser}))$.*

We next analyze the price movement, where the easy, but perhaps surprising observations is that prices always decrease, unless they "jump up" to the monopoly price (and then start decreasing again).

Lemma 7. *For every t either $p^t = p^{mon}$ or $p^t < p^{t-1}$.*

This in particular implies that prices never go above p^{mon}.

Corollary 8. *For all t: $p^t \leq p^{mon}$.*

The next lemma shows that whenever there is sufficient pent-up demand above p^{ser} then the price cannot go up to p^{mon} and furthermore the decrease in the prices is significant - when measured in terms of of the density of the demand.

Lemma 9. *Assume that for some $p > p^{ser}$ we have that $D^t(p) \geq s$ then $p^t < p^{t-1}$ and $(Q(p^t) - Q(p^{t-1})) \geq (t-1)^{-1} \cdot s \cdot (p - p^{seq})/p^{mon}$. Furthermore, if for some $T < t$ we had $p^T \leq p^t$ then actually $(Q(p^t) - Q(p^{t-1})) \geq (t-1-T)^{-1} \cdot s \cdot (p - p^{ser})/p^{mon}$.*

3.4 Sequences of decreasing price steps

The first simple lemma states that as long as prices remain above some threshold p' then the demand for all lower prices p just keeps being pent-up until it reaches as high a quantity as we desire which in our case is s.

Lemma 10. *For every $p < p'$ there exists Δ_0 such that for all T and all $\Delta \geq \Delta_0$ we have that either (a) there exists $T \leq t \leq T + \Delta$ with $p^t < p'$ or (b) $D^{T+\Delta}(p) \geq s$.*

We now reach the key part of the characterization showing that prices indeed approach p^{ser} (decreasing from above).

Lemma 11. *For every $p^* > p^{ser}$ there exists Δ such that for every T there exists some $T < t \leq T + \Delta$ with $p^t \leq p^*$.*

Let us say a word of intuition. Lemma 9 shows that when we are in a sequence of decreasing prices p^t, each time the rate of decrease – when measured in terms of $Q()$ – is proportional to $1/t$ (or even $1/(t-T)$ where T is the last time the price was below a threshold that we are aiming for). Thus, as the series $1/t$ diverges, we cannot have an infinite sequence of decreasing prices until we go below our desired threshold. The formal proof appears in the appendix.

3.5 Prices need to jump up

We start by stating the obvious fact that demand that was not supplied remains as pent-up demand:

Lemma 12. *For any time T we have that $\sum_{t=1}^{T-1} q^t \geq T \cdot Q(p) - D^T(p)$. If $p \leq \min_{t<T} p^t$ then we have equality.*

In particular, $\sum_{t=1}^{T-1} q^t = T \cdot Q(p^{ser}) - D^T(p^{ser})$. We are now ready to show that prices must jump up to p^{mon} infinitely often.

Lemma 13. *There exists infinitely many t such that $p^t = p^{mon}$.*

3.6 Putting it all Together

Let us now see how we have proved theorem 1.

Proof. (**of Theorem 1**) Lemma 5 and 8 provide the upper and lower bounds on p^t while lemmas 11 and 13 prove that these are tight as $t \to \infty$.

Lemma 11 provides the bound on the number of steps during which p^t can lie above p. Clearly once $p^t \leq p$ the demand at this price is completely supplied, $Z^t(p) = 0$. □

4 Welfare Analysis

As we have so far been able to prove the demand that is supplied by the serial monopoly dynamics is exactly that with $p > p^{ser}$, we get that the social welfare achieved is given by $\int_0^{q^{ser}} P(q)dq$, depicted in figure 3 by the sum of regions A+B+C+D+E+F+G. Using the way we defined p^{ser} we now get an approximate welfare result.

Lemma 14. *For every demand function Q and supply level s we have that the welfare obtained by serial monopoly is at least half of the social welfare at equilibrium.*

Proof. In figure 3, the social welfare at equilibrium is given by the areas of regions A+B+C+D+E+F+G+H+I, while the social welfare at the serial monopoly point is given by regions A+B+C+D+E+F+G. It thus suffices to show that the area of H+I is bounded from above by A+B+C+D+E+F+G. In fact it is even true that C+D+F+G+H+I+J is bounded by B+C+D. That is true since the area of the former is $p^{ser} \cdot s$ while the latter is $p^{mon} \cdot q^{mon}$ which are equal. □

While the proof implies various stronger bounds such as $SW^{ser} \geq max(SW^{mon}, SW^{eq} - REV^{mon})$, the bound is tight as can be seen from the following example:

Example 1. *Assume that the demand is for $M-1$ units at price 1 plus an additional unit at price $M+1$ for some large M and assume that the supply is exactly $s = M$. (This example is discrete giving a non-continuous and only weakly decreasing demand function, but it is easy to add to it ϵ mass making it continuous and strictly decreasing as in our analysis.) The equilibrium point is at price $= 1$ and quantity $= M$ where the social welfare is $2M$. The monopolist would chose $p^{mon} = M+1$ and $q^{mon} = 1$ obtaining welfare of only $M+1$. The serial monopoly price would be chosen as $p^{ser} = (M+1)/M > 1$ and thus $q^{ser} = 1$ achieving social welfare of $M+1$ rather than the possible $2M$.*

So we now have essentially proved theorem 2:

Proof. (**of Theorem 2**) The combination of example 1 and lemma 14 is exactly the statement of the Theorem. □

In "typical" scenarios the loss is significantly lower as is demonstrated by the following examples.

Example 2. *Consider a demand that comes from the uniform distribution on $[0,1]$, i.e. $Q(p) = 1 - p$ for $0 \leq p \leq 1$ and supply of $s = 1$. The inverse function is given by $P(q) = 1 - q$, the equilibrium price is $p^{eq} = 0$ with $q^{eq} = 1$ giving total welfare of $\int_0^1 (1-q)dq = 1/2$. The monopoly price is $p^{mon} = 1/2$ with $q^{mon} = 1/2$ obtaining social welfare of only $\int_0^{1/2}(1-q)dq = 3/8$. The serial monopoly price would be $p^{ser} = 1/4$ with $q^{ser} = 3/4$, so the total welfare would be $\int_0^{3/4}(1-q)dq = 15/32$, which is only 6.25% lower than the optimal $1/2$.*

Example 3. *Consider a demand that comes from an "equal revenue" distribution:* $Q(p) = 1/p$ *for* $p \in [1, H]$ *(with* $Q(p) = 1$ *for* $p \le 1$ *and* $Q(p) = 0$ *for* $p > H$*) and supply* $s = 1$*. The inverse function is given by* $P(q) = 1/q$ *for* $q \in [1/H, 1]$ *(and* $P(q) = H$ *for* $q < 1/H$*). Equilibrium price is* $p^{eq} = 1$ *where* $q^{eq} = 1$ *at which the social welfare is* $\int_0^1 P(q)dq = \int_0^{1/H} H \cdot dq + \int_{1/H}^1 q^{-1}dq = 1 + \ln H$*. The monopoly price is* $p^{mon} = H$ *with* $q^{mon} = 1/H$ *(assuming ties are broken in the worst way, which could be ensured with a small perturbation) for which the social welfare is only 1. The serial monopoly price is* $p^{ser} = 1$ *identical to the equilibrium price and so obtains full welfare of* $1 + \ln H$*. (Here we had* $p^{ser} = p^{eq}$ *which can happen since in this example we have* $Rev^{eq} = Rev^{mon}$*. With a tiny perturbation we could have* $Rev^{eq} < Rev^{mon}$ *as assumed in our analysis of the dynamics which would then result in* $p^{ser} > p^{eq}$ *but arbitrarily close to it, still obtaining that the serial monopoly social welfare is arbitrarily close to the equilibrium welfare.)*

5 Strategic Equilibrium Analysis

In this section we attempt analyzing the dynamic behavior of this serial monopoly when the users are now strategic and "patient". I.e. while all of our previous analysis above assumed that each serial monopolist was faced with the true demand, we will now assume that the monopolists are faced with the demand that is declared by the users.

5.1 The Model

We stick with our simple non-stochastic deterministic model with *price-taking bidders* and with the continuous and strictly decreasing daily demand distribution Q and the supply s being common knowledge. We consider a game between users and monopolists: for each time step t we have a (newly born) continuum of infinitely-small users and a single (new) leader. Each of our users has a true value v, where, at each time step, v is chosen according to the demand distribution Q. Our game proceeds in time steps where at each time t we have two stages. In the first stage, each of the users "born" at this time declare a bid $\tilde{v}$ where each user with true value v bids according to some strategic manipulation function $\tilde{v} = m_t(v)$. We assume that users put their bid $\tilde{v}$ once when they first enter the market rather than being able to change their bid every step. Our users are fully patient and a user born at time t with value v gets utility $v - p^{t'}$ for the first $t' \ge t$ with $p^{t'} \le \tilde{v}$ (and 0 if no such t' exists). The *declared daily demand* $\tilde{Q}^t()$ is generated by the distribution of these $\tilde{v}$'s and is added to the (declared) pent up demand to obtain the total *declared* demand $\tilde{D}^t$ faced by the leader at time t. In the second stage of this time step, the (newly born) leader of time t gets to choose a price p^t for the current block and his reward for price p^t is $p^t \cdot min(s, \tilde{D}^t(p^t))$.[5]

[5] Technically, our model allows the leaders to choose a price p^t that leads to over-demand $D(p^t) > s$, and this definition of utility of the users satisfies all the – more than s quantity – of users who bid at least p^t which may not be realistic. We could define precisely a quantity of exactly s who get satisfied in such a case, but we do not have to worry about this here as our leaders will never (in any equilibrium) choose such p^t since their own utility could be strictly increased by choosing $p' > p^t$ with $D(p') = s$ which gets higher revenue.

5.2 An Equilibrium

Let us start by identifying a natural equilibrium of this game: First, the strategy of choosing $\tilde{p}^t$ to be the monopolist price for the declared total demand $\tilde{D}^t()$ at day t is clearly a dominant strategy for the leader of day t so our equilibrium will fix this strategy for the leaders[6]. We will have a fixed (time-independent) manipulation strategy for users: $\tilde{v} = m(v)$. Once this user time-independent strategy function m is chosen, the true demand distribution $Q()$ from which v is chosen defines a declared demand distribution $\tilde{Q}()$ induced by $m(v)$. The dynamics of the game then proceed as in our analysis in the previous sections but according to the declared distribution $\tilde{Q}()$ rather than the true distribution $Q()$.

Example 4. *The user's strategy functions $m(v) = min(v, p^{eq})$ are in equilibrium with the leader's monopolist pricing for each day, where p^{eq} is the equilibrium price of the true distribution $Q()$. In this equilibrium the daily prices and quantities do not fluctuate: $p^t = p^{eq}$ and $q^t = s$ for all t.*

To show that this an equilibrium we first show that $p^t = p^{eq}$ is indeed the monopoly price for every leader. Since with the declared demand induced by this m, the new daily declared demand at price p^{eq} exactly exhausts the daily supply of the monopolist, $\tilde{Q}(p^{eq} = s$, and all pent-up demand is at lower values, maximizing revenue according to $\tilde{D}^t$ is the same as maximizing revenue according to $\tilde{Q}$ which is p^{eq} for which the declared demand is clearly s. I.e. in this case p^{eq} is also the monopolist price of the declared $\tilde{Q}$ and thus there are no daily fluctuations in price but rather we always have $p^t = p^{eq}$. Now notice that the users' strategy function $m(v) = min(v, p^{eq})$ is indeed a best response since users with $v < p^{eq}$ are not being served and they can only be served by paying p^{eq} which they they are not willing to, while users with $v \geq p^{eq}$ are being served at price p^{eq} and that is optimal for them.

5.3 Characterization

Technically, the equilibrium above is not the only Nash equilibrium point as, for example, we can have an equilibrium where everyone "pretends" that the supply is smaller than it really is $s' < s$, where all leaders only supply s' and all bidders bid according to $m'(v) = p'^{eq}$ if $v \geq p'^{eq}$ and $m'(v) = 0$ if $v < p'^{eq}$, where p'^{eq} is the equilibrium price of the daily demand $Q()$ with the smaller pretended supply s'. Note that with the given declared supply there is never more than s' demand at a strictly positive price and so the leaders are best responding, while with these leader strategies there is a "declared supply" of only s' and thus the users are best-replying. This equilibrium is certainly artificial since leaders are not using their dominant strategies.

We will thus restrict ourselves to characterizing equilibria in *un-dominated strategies*, in which leaders will always use their dominant strategy of choosing the monopolist price (of the declared total demand at their day). Furthermore we will concentrate on manipulation strategies that are *time-invariant*, i.e. where for all t the daily manipulation functions are the same $m_t(v) = m(v)$ for some fixed $m(v)$ (as in the example above)[7] The

[6] Since the leaders get paid according to the *declared* values, the leader's knowledge of the true demand does not allow him to do any better.

[7] This time-invariance allows us to easily employ our analysis of the myopic case in section 3, but our characterization does not *seem* to really require it.

strategy of users born at time t will be given by the mapping $\tilde{v} = m(v)$ thus the induced declared new demand distribution at time t is a fixed $\tilde{Q}$. I.e. be an equilibrium if for every bidder that is born in time t with value v, $m(v)$ is indeed a best reply to all the other users bidding according to m and all the leaders choosing the monopoly price. So we have:

Definition 5.1. An equilibrium in un-dominated strategies with time-invariant manipulation is a manipulation function $m()$ such that for every user born at time t with value v, the bid $m(v)$ is a best reply to all the other users bidding according to $m()$ and all the leaders using monopolist pricing.

We now start analyzing the properties of such equilibria.

Lemma 15. *In every equilibrium in un-dominated strategies with time-invariant manipulation $m()$, we have that that for all t: $p^t = \tilde{p}^{eq} = \tilde{p}^{mon}$ where $\tilde{p}^{eq}$ and $\tilde{p}^{mon}$ are defined with respect to the distribution $\tilde{Q}$ induced by m.*

Proof. Un-dominated strategies (monopoly pricing) for the leaders and time-invariant manipulations ($m()$) for the users lead us to exactly the myopic dynamics studied previously for the distribution $\tilde{Q}$ induced by $m()$. Let $(\tilde{p}^{ser}, \tilde{q}^{ser})$ be the serial monopoly price and quantity of the declared distribution. Our analysis in section 3 shows the dynamics with the declared demand will essentially sell the quantity $\tilde{q}^{ser}$ (per day) at prices that fluctuate between $\tilde{p}^{mon}$ and $\tilde{p}^{ser}$. At equilibrium, price fluctuations cannot occur since a bidder will never be willing to pay a higher price if he can later get a lower price. But as shown, such fluctuations will occur unless the equilibrium price $\tilde{p}^{eq}$ already gives the monopolist's revenue. It follows that at equilibrium $m()$ must induce a demand distribution $\tilde{Q}$ such that $\tilde{p}^{eq} = \tilde{p}^{mon}$ in which case we will have a fixed sale price $p^t = \tilde{p}^{eq}$ for all t. □

Lemma 16. *In every equilibrium in un-dominated strategies with time-invariant manipulation $m()$ we have that for all $v < \tilde{p}^{eq}$ we have $m(v) < \tilde{p}^{eq}$ and for all $v > \tilde{p}^{eq}$ we have $m(v) \geq \tilde{p}^{eq}$. Hence $\tilde{Q}(\tilde{p}^{eq}) = Q(\tilde{p}^{eq})$.*

Proof. A bidder with $v < \tilde{p}^{eq}$ would rather lose than pay $p^t = \tilde{p}^{eq}$ so, as his bid does not affect the price p^t, he will have to bid lower than $\tilde{p}^{eq}$ for that to happen. The same (but opposite) is true for $v > \tilde{p}^{eq}$. □

We are now ready to prove theorem 3.

Proof. (of theorem 3) The first part of the theorem was shown in the example in the previous subsection. We now prove the second part.

As shown in lemma 15, in any such equilibrium we have a fixed sale price $p^t = \tilde{p}^{eq}$ for all t and thus it suffices to show that $\tilde{p}^{eq} = p^{eq}$. Since $\tilde{p}^{eq}$ is the equilibrium price of $\tilde{Q}()$ then $\tilde{Q}(\tilde{p}^{eq}) = s$ and since, by the previous lemma, $\tilde{Q}(\tilde{p}^{eq}) = Q(\tilde{p}^{eq})$ we also have $Q(\tilde{p}^{eq}) = s$. But since p^{eq} is the market equilibrium price of Q we also have $Q(p^{eq}) = s$ and since Q was assumed to be strictly decreasing we must have thus $p^{eq} = \tilde{p}^{eq}$. □

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Appendix: Postponed proofs of lemmas

Proof. (of lemma 4) We can prove (1) and (2) by induction on t: when moving from step $t-1$ to t we have that $D^t(p) - D^t(p') = (Z^{t-1}(p) - Z^{t-1}(p')) + (Q(p) - Q(p')) \leq (D^{t-1}(p) - D^{t-1}(p')) + (Q(p) - Q(p'))$ and thus the LHS increases by at most $((Q(p) - Q(p'))$ which is exactly how the RHS increases. The base of the induction holds as for part A, $t = 1$, we have $Z^0(p) = Z^0(p') = 0$, while for part B, $t = T+1$, we have $D^t(p) - D^t(p') = (Q(p) - Q(p')) + (Z^T(p) - Z^T(p'))$.

For (3), note that when $p^{t-1} \geq p' > p$ we have that $Z^{t-1}(p) = D^{t-1}(p) - q^{t-1}$ and $Z^{t-1}(p') = D^{t-1}(p') - q^{t-1}$ so $D^t(p) - D^t(p') = (D^{t-1}(p) - D^{t-1}(p')) + (Q(p) - Q(p'))$ and thus we get equalities throughout the induction.

For (4) and (5), just note that $p^T \leq p < p'$ (or $T = 0$) we actually have $Z^T(p) = Z^T(p') = 0$ and then apply (2) and (3) respectively. $\square$

Proof. (of lemma 5) The maximum revenue that is achievable from a price p is $p \cdot s$ and when $p < p^{ser}$ we have that $p \cdot s < p^{mon} \cdot q^{mon}$ and that revenue can be achieved at any step using the monopolist price. $\square$

Proof. (of lemma 7) For $p \geq p^{t-1}$ we have that $D^t(p) = Q(p)$ so the maximal revenue obtained by possible $p \geq p^{t-1}$ is exactly the monopolist's revenue that is obtained at $p^t = p^{mon}$ (we assume that ties in maximum revenue are broken consistently). So, unless $p^t = p^{mon}$ then we must obtain the maximum revenue in the range $p < p^{t-1}$. $\square$

Proof. (of lemma 9) We will prove the first part of the lemma. First we cannot have $p^t = p^{mon}$ as the revenue obtained from p would be higher: $p \cdot s > p^{ser} \cdot s = p^{mon} \cdot q^{mon}$.

As p^t gives better revenue than p i.e., we have that $p^t \cdot D^t(p^t) \geq p \cdot s = (p - p^{ser}) \cdot s + p^{ser} \cdot s = (p - p^{ser}) \cdot s + p^{mon} \cdot q^{mon}$.

Separating the total demand at time t to its two components we get $p^t \cdot D^t(p^t) = p^t \cdot Z^{t-1}(p^t) + p^t \cdot Q(p^t) \leq p^t \cdot Z^{t-1}(p^t) + p^{mon} \cdot q^{mon} \leq p^{mon} \cdot Z^{t-1}(p^t) + p^{mon} \cdot q^{mon}$.

Putting these together we get that $(p - p^{ser}) \cdot s \leq p^{mon} \cdot Z^{t-1}(p^t)$. Now $Z^{t-1}(p^t) = Z^{t-1}(p^t) - Z^{t-1}(p^{t-1}) \leq D^{t-1}(p^t) - D^{t-1}(p^{t-1}) \leq (t-1) \cdot (Q(p^t) - Q(p^{t-1})$ so it follows that $(p - p^{ser}) \cdot s \leq (t-1) \cdot p^{mon} \cdot (Q(p^t) - Q(p^{t-1})$ and thus $(Q(p^t) - Q(p^{t-1}) \geq (t-1)^{-1} \cdot s \cdot (p - p^{ser})/p^{mon}$.

The second part of the lemma is similar after taking into account that $D^{t-1}(p^t) - D^{t-1}(p^{t-1})$ is actually bounded by $(t-1-T) \cdot (Q(p^t) - Q(p^{t-1})$. $\square$

Proof. (of lemma 10) If we have that $p^t \geq p'$ for all $T \leq t \leq T + \Delta$, then using lemma 4(3) we have $D^{T+\Delta}(p) \geq D^T(p) - D^T(p') = \Delta \cdot (Q(p) - Q(p')) + (Z^T(p) - Z^T(p')) \geq \Delta \cdot (Q(p) - Q(p'))$. So just choose $\Delta_0 = s/(Q(p) - Q(p'))$. $\square$

Proof. (of lemma 11) Assume not, ant let T be some time step at which $p^t \leq p^*$ or $T = 0$ and let $p = (p^* + p^{ser})/2$ so $p^{ser} < p < p^*$ and $p^* - p^{ser} = 2 \cdot (p - p^{ser})$. By lemma 10 there exists Δ_0 after which $D^t(p) \geq s$ for all $t \geq T + \Delta_0$ until the first time that $p^t \leq p^*$. Fix any $\Delta > \Delta_0$ so that $p^t > p^*$ for all $T + \Delta_0 < t \leq T + \Delta$. using lemma 9 we get a decreasing sequence of prices $p^{T+\Delta_0} > p^{T+\Delta_0+1} > p^{T+\Delta_0+2} > \cdots p^{T+\Delta}$ with $(Q(p^{t+1}) - Q(p^t)) \geq (t-T)^{-1} \cdot s \cdot (p - p^{ser})/p^{mon}$. Summing up over all $T + \Delta_0 < t \leq T + \Delta$ we get $Q(p^{T+\Delta}) - Q(p^{T+\Delta_0}) \geq (\sum_{t=T+\Delta_0+1}^{T+\Delta} (t-T)^{-1}) \cdot s \cdot (p - p^{ser})/p^{mon}$. We now estimate $\sum_{t=T+\Delta_0+1}^{T+\Delta} (t-T)^{-1} = \sum_{i=\Delta_0+1}^{\Delta} i^{-1} \geq \ln(\Delta/(1+\Delta_0))$. So $Q(p^{T+\Delta}) - Q(p^{T+\Delta_0}) \geq \ln(\Delta/(1+\Delta_0)) \cdot s \cdot (p - p^{ser})/p^{mon}$. Since $Q(p^{T+\Delta}) - Q(p^{T+\Delta_0}) \leq Q(p^{ser}) - Q(p^{mon})$, whenever $Q(p^{ser}) - Q(p^{mon}) < \ln(\Delta/(1+\Delta_0)) \cdot s \cdot (p - p^{ser})/p^{mon}$ then we to get a contradiction. I.e. if we choose Δ so that $\ln(\Delta) > \ln(1+\Delta_0) + 2 \cdot (Q(p^{ser}) - Q(p^{mon})) \cdot p^{mon}/(s \cdot (p^* - p^{ser}))$ then at some step $T + \Delta_0 \leq t \leq T + \Delta$ we must have $p^t \leq p^*$. $\square$

Proof. (of lemma 12) The proof is by induction on T. For $T = 1$, the LHS is 0 and the RHS is 0. When moving from step $T-1$ to T, the LHS grows by exactly q^{T-1}. The first term on the RHS grows by $Q(p)$ and $Z^T(p) \geq D^{T-1}(p) - q^{T-1}$ and thus the second term on the RHS, $D^T(p) = Z^T(p) + Q(p)$, grows by at least $Q(p) - q^{T-1}$, as needed. When $p \leq p^{T-1}$ then we have that $Z^T(p) = D^{T-1}(p) - q^{T-1}$ and the second term grows by exactly the required amount. $\square$

Proof. (of lemma 13) Assume by way of contradiction that that there is some last time where $p^t = p^{mon}$ and thus by lemma 7 after this time the prices p^t are a monotone decreasing sequence and so by lemma 11 they approach p^{ser}. Let $q^{ser} < q^* < s$ (see section 3.2), let $p^* > p^{ser}$ be so that $p^* \cdot q^* < p^{ser} \cdot s = p^{mon} \cdot q^{mon}$ (such a value for p^* must exist since $Q()$ is continuous), and let T_0 be a point for which for every $t > T_0$ we have $p^t < p^*$ (which must exist according to lemma 11).

Since p^t optimizes revenue at time t we also must have $p^t \cdot q^t \geq p^* \cdot q^*$ and since $p^t < p^*$ for $t > T_0$ we must have $q^t > q^*$ for all $t > T_0$. It follows that the total supplied quantity up to some large time $T > T_0$ is $\sum_{t=1}^{T} q^t \geq (T - T_0) \cdot q^*$. We now apply lemma 12 to get $\sum_{t=1}^{T} q^t = (T + 1) \cdot Q(p^{ser}) - D^T(p^{ser}) \leq (T + 1) \cdot q^{ser}$. Putting these together we have that $(T - T_0) \cdot q^* \leq (T + 1) \cdot q^{ser}$ which is a contradiction for large enough T since $q^* > q^{ser}$. □