Mechanisms for Multi-Unit Auctions

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Abstract

We present an incentive-compatible polynomial-time approximation scheme for multi-unit auctions with general k-minded player valuations. The mechanism fully optimizes over an appropriately chosen sub-range of possible allocations and then uses VCG payments over this sub-range. We show that obtaining a *fully* polynomial-time incentive-compatible approximation scheme, at least using using VCG payments, is NP-hard. For the case of valuations given by black boxes, we give a polynomial-time incentive-compatible 2-approximation mechanism and show that no better is possible, at least using VCG payments.

1 Introduction

We consider multi-unit auctions of m identical items among n bidders. In our setting we view the number of items m as "large" and desire mechanisms whose computational complexity is polynomial in the *length* of m (in bits). Every bidder i has a valuation function $v_i : \{1..m\} \rightarrow \Re$, where $v_i(q)$ denotes his value for obtaining q items. We assume that v_i is weakly monotone (free disposal), and normalized ($v_i(0) = 0$). Our goal is the usual one of maximizing the social welfare $\Sigma_i v_i(q_i)$ where $\Sigma_{q_i} \leq m$.

In the general case, each v_i is represented by m real numbers, and in abstract settings may be accessed as a black box. In a concrete setting, we will assume that v_i is represented as a k-minded bid, i.e. given by a list: $(q_1, p_1)....(q_k, p_k)$, where $v_i(q) = \max_{\{j|q_j \leq q\}} p_j$. This corresponds to a XOR bidding language [8]. In general, k can be as large as m, and the case $k = 1$ is the single-minded case.

This problem has received much attention, starting from Vickrey's seminal paper [11] who described an incentive compatible mechanism for the case of "downward sloping valuations" in which the items can be optimally allocated greedily. The general case, however, is NP-hard, as the single-minded case is just a re-formulation of the weighted knapsack problem. Luckily, the knapsack problem has a fully-polynomial time approximation scheme (i.e. can be approximated to within a factor of $1 + \epsilon$ in time polynomial in $n, \log m, \epsilon^{-1}$), and it is not hard to see that the algorithm directly extends to the general case of multi-unit auctions.

This is where the well-known difficulty of constructing approximation mechanisms kicks in: it is not possible, in general, to convert an arbitrary approximation algorithm into an incentive compatible mechanism [9, 6]. While a significant amount of success has been obtained to date in doing this for various "single-parameter" settings such as single-minded bidders in multi-unit

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auctions, hardly any success has been obtained for more general valuations like k-minded bidders described above.

For the case of multi-unit auctions with single-minded bidders, [1] presents an incentive compatible fully polynomial time approximation scheme, improving upon a previous result of [7]. The only result known for k-minded bidders is a randomized 2-approximation mechanism that is truthful in expectation [5]. In this paper we present a polynomial time approximation scheme for the general case.

Theorem: For every fixed $\epsilon > 0$, there exists an incentive-compatible mechanism for multi-unit auctions with k-minded bidders whose running time is polynomial in n, $\log m$, and k. The dependence of the running time on ϵ is exponential.

The mechanism fully optimizes the social welfare over some carefully constructed sub-range of allocations and then simply uses VCG payments to ensure incentive compatibility. The main challenge is that of identifying an appropriate sub-range over which exact optimization is computationally possible, and yet such that this optimization is a valid approximation over the optimal allocation. It is known that such exact optimization over a sub-range is the only way to get an incentive-compatible mechanism using VCG payments [9, 2]. We will call this type of algorithms maximal-in-range (MIR) algorithms, or VCG-based algorithms.

We also generalize this mechanism to stronger bidding languages such as the one used in [3]. However, we prove two ways in which the mechanism can not be improved upon. First, we show that the dependence on ϵ cannot be made polynomial without destroying incentive compatibility. This is in contrast to the algorithmic possibility of fully polynomial time approximation, and to what is possible for single-minded bidders [1].

Theorem: No fully polynomial time incentive compatible approximation mechanism that uses VCG payments exists, unless P=NP.

Then we show that the dependence on k is necessary, and that no approximation scheme is possible in a general black-box model. This is shown in a general communication model, and even for two bidders.

Theorem: Every approximation mechanism among two bidders with general valuations that uses VCG payments requires exponentially many queries to obtain an approximation factor that is smaller than 2.

We do present an incentive compatible approximation mechanism in the general black box model that does obtain a factor of 2. This improves upon the randomized one of [5], that is only truthful in expectation.

Theorem: There exists an incentive compatible 2-approximation mechanism for multi-unit auctions among general valuations whose running time is polynomial in n and $\log m$. The access to bidders' valuations is through value queries: "what is $v_i(q)$?"[1]. The mechanism uses VCG payments.

The main open problem remains to improve the approximation ratio obtained, in an incentive compatible way. The only general method known for constructing such mechanisms is VCG, and our lower bounds state that this will not work here. Partial negative results on this problem appear in [4], which may support a conjecture that VCG-based mechanisms may be the only

[1]This is yet another improvement upon the mechanism of [5] which requires the stronger *demand* queries.

possible truthful mechanisms that obtain a reasonable approximation ratio, thus proving that our results are optimal.

Paper Organization

In Section 2 we present the PTAS for k-minded bidders, and the 2-approximation in the black-box model. Section 3 considers lower bounds for MIR algorithms in both models. In Section 4 we describe a general construction, and its algorithmic applications: truthful mechanisms for other models, and more powerful bidding languages.

2 The Basic Mechanisms

2.1 A Truthful PTAS for k-Minded Bidders

We will design an MIR algorithm for this problem, which directly yields an incentive-compatible VCG-based mechanism. We will define a range $\mathcal{R}$ of allocations, and see that if all bidders are k-minded then the algorithm outputs the best allocation in $\mathcal{R}$, and that the optimal solution in $\mathcal{R}$ can be found in polynomial time. Let us start with defining the subrange $\mathcal{R}$.

Definition 2.1 *We say that an allocation $(s_1, ..., s_n)$ is* t-round *if there exists a set T of bidders, $|T| \leq t$, such that for each bidder $i \notin T$, s_i is a multiple of $\left\lfloor \frac{m-l}{(n-t)^2} \right\rfloor$, where $l = \Sigma_{j \in T} s_j$.*

We let $\mathcal{R}$ be the set of all t-round allocations for some fixed t (that will depend only on the approximation guarantee). Next we prove that the value of the best allocation in $\mathcal{R}$ is close to the optimum:

Lemma 2.2 *Let $(a_1, ..., a_n)$ be an optimal t-round allocation, and $(o_1, ..., o_n)$ an optimal unrestricted allocation, then $\Sigma_i v_i(a_i) \geq (1 - \frac{1}{t+1}) \Sigma_i v_i(o_i)$.*

Proof: Let us start with an optimal unrestricted allocation $(o_1, ..., o_n)$, and use it to construct a t-round allocation with high value. Assume that all items are allocated (without loss of generality due to the monotonicity of the valuations), and that $v_1(o_1) \geq ... \geq v_n(o_n)$. Let $T = \{1, ..., t\}$ be the set of t bidders from Definition 2.1, and assign each bidder $i \in T$ a bundle of size o_i. As in Definition 2.1, let $l = \Sigma_{j \in T} o_j$.

Look at the bidder $j \notin T$ who got the largest number of items $o_j \geq \frac{m-l}{n-t}$. For each $i \notin T$, $i \neq j$, round *up* each o_i to the nearest multiple of $\left\lfloor \frac{m-l}{(n-t)^2} \right\rfloor$. Assign bidder j no items. This is a valid t-round allocation since we added at most $(n-t) \cdot \left\lfloor \frac{m-l}{(n-t)^2} \right\rfloor \leq \frac{m-l}{n-t}$ items by rounding up, but deleted at least $\frac{m-l}{n-t}$ items by removing o_j. As for the value of the solution, observe that each bidder $i \neq j$ gets a bundle no smaller than o_i. Also observe that $v_j(o_j) \leq \Sigma_{i \neq j} \frac{v_i(o_i)}{t+1}$, which gives the required approximation. $\square$

Our MIR approximation algorithm will try each subset of at most t bidders to be the set T of bidders. For each possible selection of T, the algorithm will consider all possible allocations to bidders in T according to the k bids each bidder submitted. That is, we will consider the allocation that assigns each bidder $i \in T$ exactly s_i items, if and only if $\Sigma_{i \in T} s_i \leq m$, and for each s_i there is a bid (q_j, p_j) in the k bids of bidder i, such that $s_i = q_j$.

For each selection of T and allocation to the bidders in T according to their bids, the algorithm splits the remaining $m - l$ items into at most n^2 equi-sized bundles of size $\left\lfloor \frac{m-l}{(n-t)^2} \right\rfloor$, where l is the total number of items that bidders in T get. The maximum-in-range algorithm will optimally allocate these whole bundles among the bidders that are not in T. Finally, the algorithm outputs the best allocation among all allocations that were considered. All that is left is to show the following two lemmas:

Lemma 2.3 *For every fixed t the algorithm runs in time polynomial in n and $\log m$.*

Proof: There are less than $t \cdot n^t$ possible selections of sets T. For each selection of T there are at most k^t allocations to bidders in T that are considered. Finding the optimal allocation to the rest of the bidders is by dynamic programming. Without loss of generality, we assume that $T = \{n-t+1, ..., n\}$. We calculate the following information for every $1 \le i \le n-t$ and $1 \le q \le (n-t)^2$: $M(i,q)$ is the maximum value that can be obtained by allocating at most q regular bundles among bidders $1...i$. Each entry can be filled in polynomial time using the relations: $M(i,q) = \max_{q' \le q} v_i(q'b) + M(i-1, q-q')$. Overall we get that the algorithm runs in time polynomial in n and $\log m$, for every fixed t. $\square$

Lemma 2.4 *The algorithm finds an optimal t-round allocation.*

Proof: First, notice that the algorithm outputs a t-round allocation. Let us prove that it outputs an optimal one. Let $O = (o_1, ..., o_n)$ be an optimal t-round allocation, let T be the set from Definition 2.1, and let $l = \Sigma_{i \in T} o_i$. For each bidder $i \in T$ remove the maximal number q_i of items from o_i such that $v_i(o_i) = v_i(o_i - q_i)$. (It is possible that $q_i = 0$). Observe that there exists a pair (q_j, p_j) in i's XOR bids such that $q_j = o_i - q_i$. We now handle the bidders that are not in T. Each bidder $i \notin T$ currently hold a multiple of $b = \left\lfloor \frac{m-l}{(n-t)^2} \right\rfloor$ items in O, while in order to the allocation to be t-round we need the bidders not in T to receive multiples of $b' = \left\lfloor \frac{m-l'}{(n-t)^2} \right\rfloor$, for $l' = \Sigma_{i \in T}(o_i - q_i)$. However, notice that $b' \ge b$, and that the number of equi-sized bundles is at least the same. Hence, it is possible to assign bidders that are not in T bundles that are multiples of b in a way such that the value we get from these bidders is at least the same as in O. The lemma follows since the algorithm considers the newly constructed allocation. $\square$

We therefore have the following theorem:

Theorem 2.5 *There exists an incentive-compatible computationally-efficient VCG-based mechanism that guarantees a $(1 - \frac{1}{t+1})$-approximation for multi-unit auctions with k-minded bidders in time poly$(n, \log m, k)$, for every constant t.*

2.2 A 2-Approximation for Multi-Unit Auctions with Black-Box Access

We consider the multi-unit auction problem with general valuations given by black boxes. We will assume in our algorithm an "oracle access" to it that may be queried for $v_i(q)$, where q is the given bundle size[2].

We will design a 2-approximation MIR algorithm for this problem, which again yields an incentive-compatible VCG-based mechanism. Our MIR approximation algorithm will first split the items into n^2 equi-sized bundles of size $b = \lfloor \frac{m}{n^2} \rfloor$ as well as a single extra bundle of size r that holds the remaining elements (thus $n^2 b + r = m$). The maximum in range algorithm will optimally allocate these whole bundles among the n bidders. What we need to show is the following two simple facts:

[2]This is analogous to the weakest "value query" in combinatorial auction setting. Our lower bounds presented later will apply to all other query types as well.

Lemma 2.6 *An optimal allocation of the bundles can be found in time polynomial in n and $\log m$.*

Lemma 2.7 *Let $(a_1, ..., a_n)$ be an optimal allocation of the bundles that was found by the algorithm, and $(o_1, ..., o_n)$ an optimal unrestricted allocation, then $\Sigma_i v_i(o_i) \leq 2\Sigma_i v_i(a_i)$.*

The proofs are simple:
Proof: (of Lemma 2.6): The algorithm is by dynamic programming. We calculate the following information for every $1 \leq i \leq n$ and $1 \leq q \leq n^2$: $M(i, q)$ is the maximum value that can be obtained by allocating at most q regular bundles among bidders $1...i$, and $M^+(i, q)$ is the maximum value that can be obtained by allocating at most q regular bundles and the "remainder" bundle among bidders $1...i$. Each entry can be filled in polynomial time using the relations: $M(i, q) = \max_{q' \leq q} v_i(q'b) + M(i-1, q-q')$ and $M^+(i, q) = \max(\max_{q' \leq q} v_i(q'b) + M^+(i-1, q-q'), \max_{q' \leq q} v_i(q'b+r) + M(i-1, q-q'))$. □

Proof: (of Lemma 2.7): Let us start with an optimal unrestricted allocation $o_1...o_n$ where all items are allocated (without loss of generality since the valuations are monotone), and look at the bidder j that got the largest number of items $o_j \geq m/n$. There are now two possibilities: if $v_j(o_j) \geq \Sigma_{i \neq j} v_i(o_i)$ then by allocating all items to j (i.e. all regular-sized bundles as well as the remainder bundle) we get the required 2-approximation. Otherwise, round *up* each o_i to the nearest multiple of b (i.e. to full bundles), except for bidder j that gets nothing. This is a valid allocation since we added at most $nb \leq m/n$ items by rounding up, but deleted at least m/n items by removing o_j, and his value is certainly at least $\Sigma_{i \neq j} v_i(o_i)$ which gives the required approximation. □

We have thus proved:

Theorem 2.8 *There exists an incentive-compatible computationally-efficient VCG-based mechanism that gives a 2-approximation for multi-unit auctions with general valuations.*

3 Lower Bounds for VCG-Based Mechanisms

We now move on to show that both mechanisms essentially achieve the best approximation ratios possible. We say that an allocation $(s_1, ..., s_n)$ is *complete* if $\Sigma_i s_i = m$. Consider an MIR algorithm for n bidders that does not have full range of complete allocations. I.e., for some $0 \leq s_1, ..., s_{n-1} \leq m, \Sigma_{i<n} s_i \leq m$ it never outputs the allocation $(s_1, ..., m - \Sigma_{i<n} s_n)$. Now consider the set of valuations where for every bidder i $v_i(q) = 1$ if and only if $q \geq s_i$ (and 0 otherwise). The only allocation with value n is $(s_1, ..., m - \Sigma_{i<n} s_n)$ which is not in the range, while all other allocations have a value of at most $n - 1$.

From this we can easily get a lower bound any any computationally efficient MIR algorithm in the models considered in this paper. We start with a lower bound in the black-box model. The lower bound is on the number of queries that the bidders must be queried, and holds for any type of query – i.e., in a general communication setting.

Lemma 3.1 *An MIR algorithm for multi-unit auctions that achieves an approximation ratio better than 2 requires exponential communication. The result also applies for randomized settings.*

Proof: In the case of two bidders, an optimal algorithm is known to require exponential communication:

Lemma 3.2 ([10]) *Finding the optimal allocation in multi-unit auctions requires exponential communication, even if there are only two bidders and even for just finding the value of the allocation. This lower bound also applies for both randomized and nondeterministic settings.*

Thus, any MIR algorithm for 2 bidders that uses sub-exponential communication will be non-optimal and thus, as argued above, gives no better than a 2-approximation. The case of more than 2 bidders follows by setting all valuations but the first two to 0. □

The second result rules out the existence of FPTAS for k-minded bidders (the result essentially applies to all models that allow single-minded bidders, e.g., XOR bids, and the bidding language used in [3]):

Lemma 3.3 *Every polynomial-time MIR algorithm can not achieve an approximation ratio better than* $(1 - \frac{1}{n})$*, unless* $P = NP$*.*

Proof: Clearly, if the range of a polynomial-time MIR algorithm contains all complete allocations, then it optimally solves the multi-unit auction problem. By the standard reduction from knapsack to multi-unit auctions, it follows that for every polynomial-time MIR algorithm there exist large enough n and m for which the range of complete allocation is not full, unless $P = NP$. The lemma follows by the discussion above. □

This concludes the proof of the lower bounds for MIR mechanisms, except for one technical detail that should be explicitly mentioned. Our lower bounds were for MIR algorithms, while VCG-based mechanisms are only proved to give algorithms that are *equivalent* to MIR algorithms. However, both proofs holds even for finding the *value* of the optimal allocation and thus directly apply also to algorithms that are equivalent to MIR algorithms.

4 A General Construction and Applications

We present a construction that generalizes the PTAS for k-minded bidders shown in Section 2. The construction takes a maximal-in-range algorithm for a constant number of bidders in some bidding language or model, and use it to construct a truthful mechanism for an unbounded number of bidders, while losing only an arbitrarily small constant in the approximation ratio.

We describe three applications of the construction. First, we reprove the PTAS for k-minded bidders of Section 2. Then, we consider the bidding language considered in [3]. In [3] an *approximately* truthful FPTAS for this bidding language was described, while we present a truthful VCG-based PTAS (Section 3 rules out the possibility of a VCG-based FPTAS). Finally, we present a truthful $(\frac{3}{4} + \epsilon)$-approximation mechanism for the case the valuations are sub-additive (a.k.a. complement free) and are accessed via a black box.

4.1 The Construction

Fix some bidding language or a model for multi-unit auction that allows value queries. In the construction below A denotes a maximal-in-range algorithm for t bidders and at most m items in this model. We denote the complexity of A by $A(t, m)$, its range by $\mathcal{R}_{a,t,m}$, and its approximation guarantee by α. We require the bidders to be able to answer value queries.

The Construction

1. Build the set Q of allocations as follows:

- Let $u = (1 + \frac{1}{t+1})$. Let $L = \{0, 1, u, u^2, ..., u^{\lfloor \log_u m \rfloor}, m\}$.
- For every set T of bidders, $|T| \leq t$, and $l \in L$:
  (a) Run A with $m - l$ items and the set T of bidders. Denote by s_i the number of items A allocates to each bidder $i \in T$.
  (b) Split the remaining l items into at most n^2 bundles, each consists of $\lfloor \frac{l}{n^2} \rfloor$ items.
  (c) Find the optimal allocation of the equal-size bundles among the bidders that are not in T. Denote by s_i the allocation to each bidder $i \notin T$.
  (d) Add $(s_1, ..., s_n)$ to Q.

2. Output the allocation with the highest welfare in Q.

Theorem 4.1 *The construction is maximal in range, and runs in time poly*$(\log m, n, A(m, t))$, *for every constant t. It outputs allocation with value of $(\alpha - \frac{3}{t+1})$ of the optimal allocation.*

Proof: We will make use of the following definition:

Definition 4.2 *An allocation is* $(\mathcal{R}, t, l)$-round *if:*

- *In each $R \in \mathcal{R}$ at most t bidders are allocated up to $m - l$ items.*
- *There exists a set T of bidders, $|T| \leq t$, such that the bidders in T are allocated according to some allocation in $\mathcal{R}$.*
- *Each bidder that is not in T receives an exact multiple of $\lfloor \frac{l}{n^2} \rfloor$ units.*

It is not hard to verify that the construction always outputs allocations that are in the following range :

$$\mathcal{R} = \{S | S \text{ is a } (\mathcal{R}_{A,k,m-l}, k, l)\text{-round allocation for } l \in L \text{ and } k \leq t\}$$

Lemma 4.3 *Let $(o_1, ..., o_n)$ be an optimal unrestricted allocation. There exists an allocation $(s_1, ..., s_n) \in \mathcal{R}$ for which $\Sigma_i v_i(s_i) \geq (\alpha - \frac{3}{t+1}) \cdot \Sigma_i v_i(o_i)$.*

Proof: Denote the value of the optimal solution by OPT. Without loss of generality, assume that $v_1(o_1) \geq ... \geq v_n(o_n)$. Let $l \in L$ be the largest such that $m - l \geq \Sigma_{i=1}^t o_t$. Let $(s_1, ..., s_t)$ be the allocation that A outputs running on bidders $1, ..., t$ bidders and $m - l$ items, and assign each bidder $1 \leq i \leq t$, s_i items. Observe that $\Sigma_{i=1}^t v_k(s_t) \geq \alpha \cdot \Sigma_{i=1}^t v_t(o_t)$. To handle bidders $t + 1$ to n, we will need to claim that reducing the number of items in a given instance does not hurt the quality of the optimal solution too much:

Claim 4.4 *Fix an instance of a multi-unit auction with n bidders and m items. Let $(o_1, ..., o_n)$ be an optimal solution, and denote its value by OPT_m. Then, the value of the optimal solution on $m' \leq m$ items and the same n bidders, $OPT_{m'}$, is at least $(\frac{m'}{m} \cdot OPT_m - v_{max})$, where $v_{max} = \max_i v_i(o_i)$.*

Proof: Order bidders 1 to n by decreasing order of density $\frac{v_i(o_i)}{|o_i|}$. Let j be the largest integer such that $\Sigma_{i \leq j} o_i \leq m'$. Observe that $\Sigma_{i=1}^{j+1} v_i(o_i) \geq \frac{m'}{m} \cdot OPT_m$. Now observe that by removing bidder $j + 1$, the first j bidders form an allocation that uses at most m' items, and that $\Sigma_{i=1}^{j} v_i(o_i) \geq \frac{m'}{m} \cdot OPT_m - v_{max}$. $\square$

In the allocation we construct, bidders $t+1, ..., n$ will share l items. Let $j \in \arg\max_{i \geq t+1} v_i(o_i)$. Notice that $v_j(o_j) \leq \frac{OPT}{t+1}$. Denote the optimal allocation of these bidders when using only l items by $(o'_{t+1}, ..., o'_n)$. By the claim, $\Sigma_{i=t+1}^n o'_i \geq \frac{1}{u} \cdot \Sigma_{i=t+1}^n o_i - v_j(o_j)$.

Let y be the bidder who is allocated the bundle of the largest size in $(o'_1, ..., o'_n)$. Let $b = \lfloor \frac{l}{n^2} \rfloor$. Round up the assignment of each bidder $i \geq t+1, i \neq y$ to the nearest multiple of b. Assign bidder y no items. Notice that by monotonicity the value of every bidder i, $i \neq y$, $i = t+1, ..., n$ is at least the same, and thus the value these bidders hold together in the new allocation is at least

$$\Sigma_{i=t+1, i \neq y}^n o'_i - \frac{OPT}{t+1} \geq \frac{1}{u} \Sigma_{i=t+1}^n o_i - \frac{2 \cdot OPT}{t+1} \geq \Sigma_{i=t+1}^n o_i - \frac{3 \cdot OPT}{t+1}$$

Considering also the first t bidders, the total welfare of the allocation is at least $\alpha \cdot \Sigma_{i=1}^y v_i(o_i) + \Sigma_{i=t+1}^n o_i - \frac{3 \cdot OPT}{t+1}$. Hence, this allocation holds a value of at least $(\alpha - \frac{3}{t+1})OPT$. □

All that is left is to show that the construction runs in polynomial time:

Lemma 4.5 *The optimal allocation in $\mathcal{R}$ can be found in time $poly(\log m, n, A(m,t))$, for every constant t.*

Proof: Step 1c of the construction can be implemented using a dynamic programming similarly to Lemma 2.3; optimality of the allocation in $\mathcal{R}$ is clear. As for running time, the algorithm runs in time $poly(\log_{1+\frac{1}{t+1}} m \cdot n^k \cdot A(m,t))$, which is polynomial in the relevant parameters for every constant t. □

□

4.2 Applications of the Construction

A PTAS for k-Minded Valuations

We reprove the PTAS for k-minded bidders of Section 2. Notice that we can optimally solve a multi-unit problem with m items and n k-minded bidders by exhaustive search in time $poly((nk)^n, \log m)$, which is polynomial in $\log m$ and k for every constant n. By the construction (and since an optimal algorithm is in particular maximal in range), we get a PTAS for k-minded bidders: for every constant t, we get a $(1 - \frac{3}{t+1})$-approximation in time polynomial in n and $\log m$.

A PTAS for Marginal Piecewise Valuations

The following bidding langauge was used by [3]: a valuation v is determined by a list of at most k tuples $(u_1, m_1), ..., (u_k, m_k)$. The tuples determine the marginal utility of the jth item. In other words, to determine the value of a set of s items, we sum over all the marginal utilities. I.e., for each item j, $u_k \leq j \leq u_{k+1}$, let his marginal utility be $r_j = m_k$, and for every $s \leq m$ let $v(s) = \Sigma_{j=1}^s r_j$. (In fact, the above bidding language is more powerful than the one described in [3], which allows only marginal-decreasing piecewise valuations.)

We now show how to optimally solve a multi-unit auction problem in this setting with a constant number of bidders. A PTAS follows, just as in the k-minded case.

We say that bidder i is *precisely assigned* if he is allocated s_i items, and for some tuple j in his valuation $u_j = s_i$. The main observation here is that there is an optimal solution $(o_1, ..., o_n)$ for which at most one bidder is not precisely assigned. To see this, suppose that there are two bidders i and i' that are not precisely assigned. Then, we can move items to the bidder with the higher (or equal) marginal utility, without decreasing the value of the solution. We continue this process until for all bidders but at most one are precisely assigned.

Now optimally solving a multi-unit auction problem with a constant number of bidders is obvious: select each of the n bidders in his turn to be the bidder that is not precisely assigned. In each iteration, denote by i the bidder that is not precisely assigned, and go over all allocations in which all other bidders are precisely assigned. Then, assign bidder i the remaining items. Observe that there are at most k possible sets that make a bidder precisely assigned. Thus, the algorithm runs in time $poly(n \cdot (n \cdot k)^{n-1}, \log m)$, which is polynomial in $\log m$ and k for every constant n.

A $(\frac{3}{4}+\epsilon)$-Approximation for Complement-Free Valuations

Here we assume that the valuations are given as a black-box (as in Section 2.2). In addition, we assume that for each valuation v and bundles $0 \leq s, t \leq m$ we have that $v(s) + v(t) \geq v(s+t)$.

Let us now describe an algorithm that provides a $\frac{3}{4}$-approximation in this setting when the number of bidders is constant. By the lemma, we get a $(\frac{3}{4}+\epsilon)$-approximation VCG-based mechanism for unbounded number of bidders, for every constant ϵ. The algorithm is quite simple: Fix some small enough constant $\delta > 1$. All bidders but at most one, can only receive a bundle s that is a power of δ (including the empty bundle). The bidder that does not get a bundle of size that is a power of δ gets the remaining items. We use exhaustive search to find the optimal allocation in this range.

To see that this algorithm indeed provides a $\frac{3}{4}$ approximation, look at an optimal solution $(o_1, ..., o_n)$. Without loss of generality, assume that $o_1 \geq ... \geq o_n$ (notice that unlike before now the bidders are ordered by their bundle size). Let O be the set of bidders with odd indices, and E be the set of bidders with even indices.

The analysis is divided into two cases. First suppose that $\Sigma_{i\in O} v_i(o_i) \geq \Sigma_{i\in E} v_i(o_i)$. Look at the following allocation: the bidders in E are rounded to the power of δ *that is near* $\frac{o_i}{2}$, the bidders in $O \setminus \{1\}$ are rounded up to the nearest power of δ, and bidder 1 gets the remaining items. Notice that this is a valid allocation since for small enough choice of δ we assign bidders in O no more items than what we removed from bidders in E. Also notice that this allocation is in the range. To see that in in this case we have a $\frac{3}{4}$ approximation, observe that bidders in O hold the at least the same value as in the optimal solution, since each bidder in O is allocated at least the same number of items as in the optimal solution. In addition, each bidder in E hold at least half of the items that he hold in the optimal solution. Thus, by subadditivity, we lost at most half of the value bidders in E hold in the optimal solution. In total we hold at least $\frac{3}{4}$ of the value of the optimal solution.

Let us now handle the case where $\Sigma_{i\in O} v_i(o_i) < \Sigma_{i\in E} v_i(o_i)$. Look at the allocation where all bidders in O are rounded up to the power of δ that is near $\frac{v_i}{2}$, and all bidders in E are rounded up to the nearest power of δ (except for one arbitrary bidder in E who gets the remaining items). This allocation is in the range, and the analysis is similar to the previous case, leaving us with a $\frac{3}{4}$ approximation also in the current case.

The running time of the algorithm is $poly(n \cdot (\log_\delta m)^{n-1})$, which is polynomial in $\log m$ for constant n and δ.

Notice that the approximation ratio achieved is almost the best possible, as any approximation better than $\frac{3}{4}$ by an MIR algorithm requires exponential communication: [10] shows that solving exactly multi-unit auctions with only two bidders requires exponential communications. The valuations used there can easily be made sub-additive by converting each valuation v to $v'(s) = v(s) + v(m)$. Thus, as in Section 2.2, the range of every MIR mechanism for two bidders cannot contain all complete allocations. Fix some MIR algorithm, and let $(s_1, m - s_1)$ be a complete allocation that is not in the range. Consider the following instance: bidder 1 values at least s_1 items with a value of 2, and smaller bundles with a value of 1, and bidder 2 values at least $m - s_2$ items with a value of 2, and smaller bundles with a value of 1. (Both bidders value the empty bundle with a value of 0.) Observe that the valuations of the two bidders are indeed complement free. Also observe that although the optimal welfare is 4, the mechanism cannot achieve any welfare better than 3.

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