Auctions with severely bounded communication

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Abstract

We study auctions with severe bounds on the communication allowed: each bidder may only transmit t bits of information to the auctioneer. We consider both welfare-maximizing and revenue-maximizing auctions under this communication restriction. For both measures, we determine the optimal auction and show that the loss incurred relative to unconstrained auctions is mild. We prove non-surprising properties of these kinds of auctions, e.g. that discrete prices are informationally efficient, as well as some surprising properties, e.g. that asymmetric auctions are better than symmetric ones.

1 Introduction

Recent years have seen the emergence of the Internet as a central platform of interaction between computers, humans, and firms. The different parties that interact on the Internet are in various levels and modes of cooperation and competition with each other. This happens in all levels of the interaction, from the lowest technical level of computer communication, routing, storage, and computing, and reaching to the highest level of electronic commerce in its many forms. Studying such types of computer systems that are distributed in terms of the participants' goals and incentives, naturally requires using a combination of techniques from economics, game theory and computer science. Indeed much recent work has been done on this borderline, see e.g. the surveys [20, 11].

Studying these types of distributed-incentive-computer-systems naturally leads to many new problems in each of the participating research fields – questions that arise due to considerations from the other fields. In particular, many new questions in economics arise due to the necessity of taking computational questions into account. This paper deals with such a question: how to design efficient auctions that are restricted to using a very small amount of communication.

Auctions have been suggested many times as an efficient mechanism for resource allocation in computer systems. See e.g. [20, 8, 25, 30] and the many references therein. The basic argument goes along these lines: when some computational resource needs to be allocated in a distributed system, we would like the system to allocate it in the most beneficial way. If we auction the resource among all the conflicting uses in an economically efficient way then we will do just that. Thus, for example, congestion over some communication link in a network can be handled by auctioning the bandwidth of the link. This type of idea has indeed been applied in various forms both for low level resources like network bandwidth [15, 29, 27, 12] or computing resources [33, 35, 34, 13, 22, 24], and for many e-commerce systems [2, 3, 1, 5, 4].

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Several researchers have considered the effect of various computational considerations on the design of auctions: online behavior [14, 6], unbounded supply [7, 10, 6], computational complexity in combinatorial auctions [32, 16, 21, 28, 37], timing uncertainty [26], and more. This paper studies the effect of severely restricting the amount of communication allowed in an auction of a single item. Each bidder is only allowed to send a single t-bit message to the auctioneer, who must then allocate the item and determine the price according to the messages received. I.e. each bidder has a set of k possible messages it can send (where $k = 2^t$). The simplest case is $t = 1$, and thus $k = 2$, i.e. each bidder sends a single bit of information, and the auctioneer must determine an allocation and pricing according to the bits received from the bidders. This is in stark contrast to the usual treatment in economics where we assume the communication is in terms of real numbers. The only treatment of similar issues we know of in the economic literature is [17], who considered similar questions in cases of restricting bid levels in oral auctions to discrete levels, and [36, 18] that analyze the inefficiency caused by discrete priority classes of customers.

The reader may ask why we bother studying such severe restrictions on communication, as a single real number does not seem like an excessive amount of information to transfer in any computer system. Well, there are several motivations for studying auctions with such severe restrictions on the communication. First, if auctions are to be used for allocation of low level resources in computer systems, then only a very small amount of computational effort can be spent on them. Thus, e.g, an auction for routing a single packet will need to require very little communication overhead, certainly not a whole real number. One would ideally like to "waste" only a bit or two on the bidding information. Preferably, the bidding information can be piggy-backed on some unused bits in the packet header of existing networking protocols (such as IP, TCP, or those used for QoS). Second, restrictions on communication can sometimes function as a proxy for other simplifications in the auction: low communication means low information revelation; low communication reduces the amount of required human input (and thus simpler user interface); low communication means a small number of payment amounts and thus may simplify handling them electronically, etc. Since we will design auctions that are very efficient despite using very low communication, we get all such properties as a bonus. Third, while single item auctions require only a single real number in communication, efficient combinatorial auctions require an exponential amount of communication [23] and are thus impossible computationally. Understanding the trade-offs between communication and allocative efficiency will thus allow enlarging the envelope of auctions that are efficient both in the economic sense and in the computational sense.

We consider both the question of optimizing total social welfare and the question of maximizing seller revenue (under individual rationality constraints) under the restriction of bounded communication. We completely characterize the optimal auctions in the case of 2-bidders, a characterization that holds for all notions of equilibria. We describe two families of auctions called "priority games" and "modified priority games" each having a dominant-strategy equilibrium. Each of these families is parametrized by certain threshold vectors. We derive the optimal values for these parameters, for which we prove:

Theorem 1.1. *For any pair of distributions on the valuations of two bidders, and any bound k on the number of possible messages allowed to each bidder, a priority game with the derived parameters achieves maximal social welfare. The loss in social welfare compared to auctions that are unconstrained in communication is $O(\frac{1}{k^2})$*

Theorem 1.2. *For any regular distribution on the valuations of two symmetric bidders, and any bound k on the number of possible messages allowed to each bidder, a modified priority-game with the derived parameters achieves maximal seller revenue (under individual rationality and Bayesian-Nash equilibrium constraints). The loss in seller revenue compared to auctions that are unconstrained in* *communication is* $O(\frac{1}{k^2})$.

The $O(\frac{1}{k^2})$ bound on the loss incurred is tight for some distributions, as we show by analysing the case of valuations uniformly distributed in $[0,1]$. In this case the loss of social welfare is exactly $\frac{1}{6\cdot(2k-1)^2}$ and the loss of seller revenue is $\Omega(\frac{1}{k^2})$.

Our analysis implies some expected as well as some unexpected results:

- **Low welfare and revenue loss:** Even severe bounds on communication result in a mild loss of efficiency. E.g. for the case of two bidders whose valuations are uniformly distributed in $[0,1]$, we obtain a 1-bit auction with expected welfare 0.648, compared to 0.667 which is what can be reached without any restriction on communication.

- **Asymmetry helps:** Asymmetric auctions may be more efficient than symmetric ones with the same communication bounds. E.g. for the case of two bidders whose valuations are uniformly distributed in $[0,1]$, symmetric 1-bit auctions can only achieve expected welfare of 0.625, compared to 0.648 for asymmetric ones. We also show that asymmetry helps achieving higher revenue.

- **Discrete Prices are Informationally Efficient:** We show that in the optimal auctions with k messages, bidders simply partition the valuation range to k continuous price ranges and bid their price range.

- **Dominant Strategy equilibrium incurs no additional cost:** The efficient auction we design has a dominant strategies equilibrium and yet is optimal among all auctions regardless of their definition of equilibrium.

We start by presenting, in section 2, a self-contained treatment of the simplest case: 2 bidders with uniformly distributed valuations, each allowed a single bit of communication. We continue with the general case: section 3 provides the model definition and introduces our notations, section 4 analyses welfare and revenue maximizing in auctions among two players. Finally, section 5 discusses the generalization to an arbitrary number of bidders.

2 2-players, 2 possible bids

We start with the description of the simplest case: games among 2 players where every player can send only a single bit to the auctioneer (or mechanism). Later on, we generalize these games to any number of possible bids and any number of players.

2.1 The model

The players in our model are risk-neutral, have independant private values for the item, and quasi-linear utilities. The valuation of player i is distributed in the range $[0,1]$ with a commonly-known distribution function f_i. In this section, we assume players' valuations are distributed uniformly. Throughout the paper, we deal with *ex-post Individually-Rational* (IR) mechanisms, i.e. games where the utility of zero is guaranteed for each player.

Our unique assumption is that every player has only two possible bids to choose from. Such mechanisms can be described with a 2x2 game matrix, where the 1st player (Alice) chooses a row, and the 2nd (Bob) chooses a column. Each entry of the matrix specifies the allocation and payments given a bids' combination. The mechanism can toss coins to determine the allocations. Figures 2.1 and 2.2 depicts examples for *2-players 1-bit* mechanisms.

A \ B	0	1
0	B wins and pays 0	B wins and pays 0
1	A wins and pays $\frac{1}{3}$	B wins and pays $\frac{2}{3}$

Figure 2.1: (g_1) A 2-player 1-bit game that achieves maximal expected welfare (efficiency)

A \ B	0	1
0	No allocation	B wins and pays $\frac{5}{8}$
1	A wins and pays $\frac{1}{2}$	B wins and pays $\frac{5}{8}$

Figure 2.2: (g_2) A 2-player 1-bit game that achieves maximal expected revenue

A Strategy s_i for player i is a function $s_i : [0,1] \rightarrow \{0,1\}$. A strategy determines the bid of player i according to his valuation v_i.

Each selfish bidder wants to maximize her expected utility. As the mechanism's designers, we will try to optimize "social" criteria such as expected *welfare* and *revenue.* The *expected welfare* (or efficiency) achieved by a mechanism is the expected valuation of the player that wins the item (if any). The *expected revenue* from a mechanism is the expected sum of bidders' payments.

2.2 Welfare and revenue optimizing mechanisms

In the case of unlimited communication between the bidder and the mechanism, when valuations are distributed uniformly, we know that the optimal expected welfare is $\frac{2}{3}$ (2nd-price auction, see [31]) and the optimal expected revenue is $\frac{5}{12} = 0.417$ (2nd-price auction with reservation price of $\frac{1}{2}$, see [9, 19]). In this section, we study how close can 2-player games, with 2 possible bids for each player, get to these values, with dominant-strategies equilibrium and ex-post individual-rationality.

Let g_1 denote the mechanism described in figure 2.1 and g_2 denote the mechanism in figure 2.2.

Theorem 2.1. *The mechanism g_1 has dominant strategies equilibrium, with ex-post individual-rationality and it achieves expected welfare of $\frac{35}{54} = \frac{2}{3} - \frac{1}{54} = 0.648$.*

The mechanism g_2 has dominant strategies equilibrium, with ex-post individual-rationality, and it achieves expected revenue of $\frac{25}{64} = 0.39$

Proof. (sketch) Consider the following strategy: "*bid* 1 *if your valuation is greater than* $\frac{1}{3}$*, else bid 0*". Clearly, this strategy is dominant for player A in g_1: when his valuation is smaller than $\frac{1}{3}$ he will gain negative utility if he bids "1". When his valuation is greater than $\frac{1}{3}$, bidding "0" gives him zero utility, but he can get positive utility by bidding "1". We call this kind of strategies *threshold-strategies.* Similarly, a threshold-strategy with $\frac{2}{3}$ is dominant for player B in g_1. The threshold-strategies with the values $\frac{1}{2}, \frac{5}{8}$ are dominant for A, B, respectively, in g_2.

Bidding "0", guarantees a zero payment for the players in both games. Thus, both games are ex-post individually-rational.

Now, we calculate the expected welfare in g_1 when both players play their dominant strategies described above. When a player uses a threshold strategy with x as a threshold, he bids "0" with probability x, and "1" with probability $1 - x$ (uniform distribution). The expected valuation of this player, given that he bids "0", is $\frac{x}{2}$, and it equals $\frac{1+x}{2}$ given that he bids "1". The expected welfare, is therefore:

$$\frac{1}{3}\frac{2}{3}\frac{(\frac{2}{3})}{2} + \frac{1}{3}(1-\frac{2}{3})\frac{(1+\frac{2}{3})}{2} + (1-\frac{1}{3})\frac{2}{3}\frac{(1+\frac{1}{3})}{2} + (1-\frac{1}{3})(1-\frac{2}{3})\frac{(1+\frac{2}{3})}{2} = \frac{35}{54}$$

Similarly, we can calculate the expected revenue in g_2, when both players play their dominant strategies: $0 + \frac{1}{2}(1-\frac{5}{8})\frac{5}{8} + (1-\frac{1}{2})\frac{5}{8}\frac{1}{2} + (1-\frac{1}{2})(1-\frac{5}{8})\frac{5}{8} = \frac{25}{64}$ ◻

Recall that with no communication limitations, the optimal welfare is $\frac{2}{3} = \frac{36}{54}$. We surprisingly see that despite severely limiting the communication from infinitely many bits to a single bit, the welfare loss is mild (only $\frac{1}{54}$).

2.3 Optimal mechanisms

Next, we claim that the mechanisms g_1 and g_2 (described in figures 2.1 and 2.2) achieve optimal welfare and revenue, respectively.

Theorem 2.2. *No 2-player 1-bit mechanism achieves strictly greater expected welfare, than the mechanism g_1 from figure 2.1 (even without restriction to mechanisms with dominant strategies equilibrium).*

Proof. (sketch) This theorem is proved in 3 steps:

Step 1: We show that every mechanism g can achieve its optimal welfare with a pair of threshold-strategies. I.e. there exists a pair of threshold-strategies such that no other strategies achieve strictly greater expected welfare in g.

Step 2: Consider mechanisms in which the item is always allocated to the player with the highest bid, and in case of equal bids, the item is always allocated using a pre-defined order on the players. We call this family of mechanisms *priority-games*. (For example, g_1 is a priority-game in which we always break ties in favour of B.) We show that for each priority-game there exists a pair of strategies (not necessarily in equilibrium) that achieves the maximal welfare among all 2-player 1-bit mechanism with any pair of strategies.

Step 3: From the previous steps we know that optimal welfare can be achieved in priority-games with threshold-strategies. Next, we express the expected welfare in a priority-game as a function of the threshold-values that the players use: $w(g,x,y) = xy\frac{x}{2} + x(1-y)\frac{1+y}{2} + (1-x)y\frac{1+x}{2} + (1-x)(1-y)\frac{1+y}{2}$

This function achieves unique maximum ($x, y \in [0,1]$) when $(x,y) = (\frac{1}{3}, \frac{2}{3})$. ◻

Theorem 4.3 is a generalization of theorem 2.2 (with a full proof).

Theorem 2.3. *No ex-post individually-rational mechanism achieves strictly greater revenue than g_2 (see figure 2.2) .*

Proof. Theorem 4.6 generalizes this result. ◻

Observe the following properties of g_1 and g_2, which demonstrate the properties of the optimal mechanisms in the general case:

- Both optimal-welfare and optimal-revenue are achieved when the players use threshold-strategies.
- The mechanism g_1 achieves the maximal welfare achievable by any 1-bit mechanism and any pair of strategies, without restrictions to any kind of equilibria. Nevertheless, we found a game with a dominant-strategies equilibrium that achieves this welfare. g_2 achieves maximal revenue among all the mechanism with Bayesian-Nash equilibrium with ex-post IR, but g_2 achieves this optimal revenue with dominant-strategies equilibrium.
- The welfare-maximizing threshold values $(x,y) = (\frac{1}{3}, \frac{2}{3})$ are what we call *mutually-centered*. I.e. x is the expected valuation of B given that B bids 0 (i.e. $x = E(v_B | 0 \le v_B \le y) = \frac{0+y}{2}$), and y is the expected valuation of A given that A bids 1 (i.e. $y = E(v_A | x \le v_A \le 1) = \frac{x+1}{2}$).

- The optimal mechanism is assymetric (priority-games are assymetric by definition). Actually we show in section 2.4 that symmetric mechanisms must achieve strictly smaller expected welfare.
- When optimizing revenue, the ex-post IR assumption is critical. We could alternatively assume ex-ante IR, where the players participate only if their *expected* utility is non-negative. In this case the mechanism "extracts" the whole welfare of the players (thus the optimal revenue is $\frac{35}{54}$).

2.4 Optimal symmetric games

In this section we show optimal 2-players 1-bit mechanisms, and prove that they incur greater welfare-loss and revenue-loss than asymmetric mechanisms. This section has been moved to appendix A.

3 The general model

3.1 The players and the mechanism

We consider single item auctions among n risk-neutral players. Player i has a private valuation $v_i \in [0,1]$. The valuations are independently drawn from a distribution function f_i $\left(\forall_{v\in[0,1]}\, f_i(v) \geq 0\,,\right.$ $\left.\int_0^1 f_i(v)dv = 1\right)$ and the cumulative function is F_i. Throughout the paper we assume that all the distribution functions are continuous and always positive. Players want to maximize their utilities, which are *quasi-linear*. We assume a normalized model, i.e. players' valuations, for not winning the item, are zero. We also assume players utilities depend only on whether they win the item or not (no externalities).

In our model, each player i can send a message of $t_i = \lg(k_i)$ **bits** to the mechanism, i.e. player i can choose one of possible k_i **bids** (or actions). In most parts of the paper, we assume that all players have the same number of possible bids, k. The seller's valuation for the item is zero. Denote the possible set of bids for player i as $\beta_i = \{0, 1, 2, ..., k_i - 1\}$. In each auction, player i chooses a bid $b_i \in \beta_i$. Let $b = \{b_1, ..., b_n\}$ be a vector of players' bids. A mechanism should determine the allocation and payments given a vector of bids s:

Definition 1. A *mechanism* g is composed of a pair (a, p) where:

- $a : (\beta_1 \times ... \times \beta_n) \to [0,1]^n$ is the allocation scheme (not necessarily deterministic). We denote the i'th coordinate of $a(b)$ by $a_i(b)$, which is player i's probability for winning the item when the bidders' bids vector is b. Clearly, $\forall i\ \forall b\ a_i(b) \geq 0$ and $\forall b\ \sum_{i=1}^n a_i(b) \leq 1$.
- $p : (\beta_1 \times ... \times \beta_n) \to \Re^n$ is the payment scheme. $p_i(b)$ is player i's payment given a bids' vector b, which she pays only if she wins the item.

Note that we allow non-deterministic allocations, but we ignore non-deterministic payments (since we are interested in expected values, using lottery for the payments has no effect on our results).

Definition 2. In a *mechanism with k-possible bids*, for every player i, $|\beta_i| = k$. We denote the set of all the mechanisms with k-possible bids among n players by $G_{n,k}$. We denote the set of all n-player mechanisms in which $|\beta_i| = k_i$ for each player i, by $G_{n,(k_1,...,k_n)}$.

Throughout the paper, we deal with *ex-post Individually-Rational* (IR) mechanisms, i.e. mechanisms in which the utility of zero is assured for each player (including the seller). As a result: (1) A player will pay non-zero amount only when she wins the item. (2) The payment for each player is always in the range $[0,1]$.

Next, we define the notion of a strategy for a player, and show how players choose their strategies.

Definition 3. *A Strategy* s_i for player i in a game $g \in G_{n,k}$ describes how a player determines his bid according to his valuation, i.e. it is a function $s_i : [0,1] \to \{0,1,...,k-1\}$.

Denote $\varphi_k = \{s \,|s : [0,1] \to \{0,1,...,k-1\}\}$ (i.e. the set of all strategies for players with k possible bids).

Definition 4. A real vector $c = (c_0, c_1, ..., c_k)$ is a *vector of threshold-values* if $c_0 \leq c_1 \leq ... \leq c_k$.

Definition 5. A strategy $s_i \in \varphi_k$ is a *threshold-strategy* based on a vector of threshold-values $c = (c_0, c_1, ..., c_k)$, if $c_0 = 0$ and $c_k = 1$ and for every $c_i \leq v_i < c_{i+1}$ we have $s_i(v_i) = i$.

We use the notation: $s(v) = (s_1(v_1), ..., s_n(v_n))$, when s_i is a strategy for bidder i, $s = (s_1, ..., s_n)$ and $v = (v_1, ..., v_n)$. Note that $b = s(v)$ is a vector of bids. When $s = (s_1, ..., s_n)$ is a vector of strategies, and s_i a strategy for player i, s_{-i} denotes the strategies of the players except i, i.e. $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$. We sometimes use the notation $s = (s_i, s_{-i})$.

3.2 Optimality criteria

The players in our model choose strategies that maximize their utilities. We are interested in games where such strategies forms equilibria.

Definition 6. Let $u_i(g, s)$ be the expected utility of player i from game g when bidders use strategies s, i.e.

$$u_i(g, s) = E_{v\in[0,1]^n} \left(a_i\left(s(v)\right) \cdot \left(v_i - p_i\left(s(v)\right)\right)\right)$$

Definition 7. Strategy s_i for player i is *dominant* in mechanism $g \in G_{n,(k_1,...,k_n)}$ if regardless of the other players' strategies s_{-i}, i cannot gain higher utility by changing his strategy, i.e.

$$\forall \widetilde{s_i} \in \varphi_{k_i} \;\; \forall s_{-i} \;\; u_i(g, (s_i, s_{-i})) \geq u_i(g, (\widetilde{s_i}, s_{-i}))$$

We say that mechanism g has *a dominant-strategies equilibrium* if for every player i there is a strategy s_i which is a dominant.

Definition 8. Strategies $s = (s_1, ..., s_n)$ forms a *Bayesian-Nash equilibrium* in mechanism $g \in G_{n,(k_1,...,k_n)}$ if for every player i, s_i is the best response for the strategies s_{-i} of the other players, i.e.

$$\forall i \;\; \forall \widetilde{s_i} \;\; u_i(g, (s_i, s_{-i})) \geq u_i(g, (\widetilde{s_i}, s_{-i}))$$

Our goal is to find optimal communication-bounded mechanisms. Each selfish bidder wants to maximize her expected utility. As the mechanism designers, we will try to optimize "social" criteria such as *welfare* (efficiency) and *revenue*.

The *expected welfare* from a mechanism g, when bidders use strategies s, is the expected sum of bidders valuations. Because the item is indivisible, the expected welfare is actually the expected valuation of the player that wins the item (if any). Note that the expected welfare does not depend on the payments in the mechanism.

Definition 9. Let $w(g, s)$ denote the expected welfare in the n-player game g when bidders' strategies are s, i.e.

$$w(g, s) = E_{v\in[0,1]^n} \left(\sum_{i=1}^{n} a_i\left(s(v)\right) \cdot v_i\right)$$

and let $w_{n,k}^{opt}$ denote the maximal possible expected welfare from any n-player game where each player has k possible bids, with any vector of strategies allowed, i.e.

$$w_{n,k}^{opt} = \max_{g \in G_{n,k},\, s \in \varphi_1 \times ... \times \varphi_n} w(g,s)$$

Definition 10. Let $r(g,s)$ denote the expected revenue in the n-player game g when bidders' strategies are s, i.e.

$$r(g,s) = E_{v \in [0,1]^n} \left(\sum_{i=1}^{n} a_i \left(s(v) \right) \cdot p_i \left(s(v) \right) \right)$$

and let $r_{n,k}^{opt}$ denote the maximal possible expected revenue from any ex-post individually-rational, n-player k-possible-bids game and strategies s that forms a Bayesian-Nash equilibrium:

$$r_{n,k}^{opt} = \max_{\substack{g \in G_{n,k} \text{ is ex-post IR} \\ s \in \varphi_1 \times ... \times \varphi_n \text{ in Bayesian-Nash equilibrium}}} r(g,s)$$

Note that we define the optimal welfare as the maximal welfare achieved among all mechanisms and strategies, not necessarily in equilibria, and we define the optimal revenue as the maximal revenue achieved among all mechanisms with Bayesian-Nash equilibrium and ex-post IR. Yet, the optimal mechanisms (for both measures) that we present in this paper will have dominant-strategies equilibria.

Definition 11. We say that a mechanism $g \in G_{n,k}$ *achieves expected welfare (revenue)* of α if there is a vector s of dominant-strategies for which the expected welfare (revenue) from g and s is α, i.e. $w(g,s) = \alpha$ ($r(g,s) = \alpha$). Mechanism $g \in G_{n,k}$ achieves *optimal excepted welfare (revenue)* if it achieves $w_{n,k}^{opt}$ ($r_{n,k}^{opt}$).

4 Optimal mechanisms for 2 players

In this section we present mechanisms with bounded communication that achieve optimal welfare and revenue.

Definition 12. The *priority-game* $PG_{n,k} \in G_{n,k}$ allocates the item to the player i that bids the highest bid (i.e. when $b_i > b_j$ for all j, the allocation is $a_i(b) = 1$ and $a_j(b) = 0$ for $j \neq i$), with ties consistently broken according to a pre-defined order on the players.

Definition 13. The *modified priority-game* $MPG_{n,k} \in G_{n,k}$ is a game with the same allocation as priority-games, except no allocation is done when all players bid 0.

We will assume (w.l.o.g) throughout this paper that in 2-player priority-games $B \succ A$, i.e. the mechanism allocates the item to A if she bids a higher bid than B, and otherwise to B.

Definition 14. A 2-players *priority-game based on the threshold-values* $x = (x_0 \leq ... \leq x_k)$ *and* $y = (y_0 \leq ... \leq y_k)$ is a mechanism that its allocation is as in a priority-game and given a pair of bids (i,j) of A, B respectively, A pays x_{j+1} whenever she wins, and B pays y_i when he wins. We denote this mechanism by $PG_k(x,y)$. The mechanism $PG_k(x,y)$ is presented in figure 4.1.

Definition 15. A 2-players *modified priority-game based on the threshold-values* $x = (x_0 \leq ... \leq x_k)$ *and* $y = (y_0 \leq ... \leq y_k)$ is a mechanism that its allocation is as in a modified priority-game and given a pair of bids (i,j) of A, B respectively, A pays x_{j+1} whenever she wins and when B wins he pays y_i when $i > 0$ and y_1 when $i = 0$. We denote this mechanism by $MPG_k(x,y)$. The mechanism $MPG_k(x,y)$ is presented in figure 4.2.

	0	1	2	..	k-2	k-1
0	$\mathbf{B}, 0$	$\mathbf{B}, 0$	$\mathbf{B}, 0$	...	$\mathbf{B}, 0$	$\mathbf{B}, 0$
1	A, x_1	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$	...	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$
2	A, x_1	A, x_2	$\mathbf{B}, y_2$	...	$\mathbf{B}, y_2$	$\mathbf{B}, y_2$
...	...	...	...	...	...	...
k-2	A, x_1	A, x_2	A, x_3	...	$\mathbf{B}, y_{k-2}$	$\mathbf{B}, y_{k-2}$
k-1	A, x_1	A, x_2	A, x_3	...	A, x_{k-1}	$\mathbf{B}, y_{k-1}$

Figure 4.1: A priority-game based on the threshold-values x, y. Note that the threshold- strategies based on x, y are dominant strategies, with ex-post IR. When x, y are mutually-centered, this mechanism achieves optimal welfare, among all the mechanisms and the possible-strategies.

	0	1	2	..	k-2	k-1
0	ϕ	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$	...	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$
1	A, x_1	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$	...	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$
2	A, x_1	A, x_2	$\mathbf{B}, y_2$	...	$\mathbf{B}, y_2$	$\mathbf{B}, y_2$
...	...	...	...	...	...	...
k-2	A, x_1	A, x_2	A, x_3	...	$\mathbf{B}, y_{k-2}$	$\mathbf{B}, y_{k-2}$
k-1	A, x_1	A, x_2	A, x_3	...	A, x_{k-1}	$\mathbf{B}, y_{k-1}$

Figure 4.2: A modified priority-game based on the threshold strategies x, y. Note that x, y are dominant strategies with ex-post IR. For a chosen values of x, y, this mechanism achieves optimal revenue.

It turns out that priority-games achieve optimal welfare, and modified priority-games achieve optimal revenue.

Proposition 4.1. *For every pair of threshold-values x, y, the threshold-strategies based on these threshold-values are dominant in both $PG_k(x, y)$ and $MPG_k(x, y)$, and these mechanisms are ex-post individually-rational.*

Definition 16. The threshold-values $x = (x_0, x_1, ..., x_{k-1}, x_k = 1)$, $y = (y_0, y_1, ..., y_{k-1}, y_k = 1)$ are *mutually-centered*, if the following constraints hold:

$$\forall 1 \leq i \leq k-1 \quad x_i = E(v_B \,|y_{i-1} \leq v_B \leq y_i) = \frac{\int_{y_{i-1}}^{y_i} f_B(v_B) \cdot v_B dv_B}{F_B(y_i) - F_B(y_{i-1})}$$

$$\forall 1 \leq i \leq k-1 \quad y_i = E(v_A \,|x_i \leq v_A \leq x_{i+1}) = \frac{\int_{x_i}^{x_{i+1}} f_A(v_A) \cdot v_A dv_A}{F_A(x_{i+1}) - F_A(x_i)}$$

Lemma 4.2. *For any pair of distribution functions on the players' valuations, and for any values of x_0 and y_0, there exist a unique pair of mutually-centered threshold-values x, y.*

4.1 Welfare-optimal 2 players mechanisms with k possible bids

In this subsection, we show the characterization of an efficient mechanism, for any pair of distribution functions on the players' valuations. We also give a tight upper bound on the welfare loss they incur.

Let $x^w = (x_0^w, ..., x_{k-1}^w, x_k^w = 1)$ and $y^w = (y_0^w, ..., y_{k-1}^w, y_k^w = 1)$ be mutually-centered threshold-values, where $x_0 = y_0 = 0$.

Theorem 4.3. *For any pair of distribution functions on the players' valuations, the mechanism $PG_k(x^w, y^w)$ achieves optimal welfare $(w^{opt}_{2,k})$, among all mechanisms in $G_{2,k}$. The optimal welfare is achieved with dominant-strategies equilibrium and ex-post IR and the welfare loss it incurs, compared with the optimal auction with no communication bounds, is $O(\frac{1}{k^2})$.*

Proof. In theorem B.12 in the appendix. □

We give an explicit solution for the case of uniformly-distributed valuations in $[0, 1]$. We show that the upper bound on the efficiency loss is tight, i.e. the efficiency loss in the worst case, is $\Omega(\frac{1}{k^2})$.

Theorem 4.4. *When players' valuations are uniformly distributed, the mechanism $PG_k(\overrightarrow{x}, \overrightarrow{y})$ achieves optimal welfare, among all mechanisms, where*

$$x = (0, \frac{1}{2k-1}, \frac{3}{2k-1}, ..., \frac{2k-3}{2k-1}, 1)$$

$$y = (0, \frac{2}{2k-1}, \frac{4}{2k-1}, ..., \frac{2k-2}{2k-1}, 1)$$

and the welfare loss it incurs is exactly $\frac{1}{6\cdot(2k-1)^2}$. The optimal welfare is achieved with dominant-strategies equilibrium and ex-post IR.

Proof. In theorem B.11 in the appendix. □

Note that the threshold strategies from theorem 4.4 are mutually-centered.

We saw in theorem 4.3 that for any pair of distribution functions we can construct a mechanisms that incurs a welfare loss of $O(\frac{1}{k})$. But can we design a mechanism that regardless of the distribution functions, will always incur a low welfare loss? The following theorem presents a mechanism with k-possible bids that incurs a welfare loss of $O(\frac{1}{k})$ regardless of the players' distribution functions, and we also show that no mechanism can do better.

Theorem 4.5. *The mechanism $PG_k(x, y)$, where $x = y = (0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1)$, incurs an expected welfare loss $\leq \frac{1}{k}$ for any pair of distribution functions of the players' valuations. Moreover, for any mechanisms there exists a pair of distribution functions for which the expected welfare loss is $\Omega(\frac{1}{k})$.*

Proof. In theorem B.13 in the appendix. □

4.2 Revenue-optimal 2-players mechanisms with k possible bids

Most results in the literature on revenue-maximizing auctions, assume that the distribution functions of the players' valuations are *regular* (as defined below). When the valuations of all players are distributed with the same regular distribution-function, it is known that Vickrey's 2nd-price auction, with an appropriately chosen reservation price, is revenue-optimal ([31, 19, 9]).

Definition 17. ([19]) Let f be a distribution function on a finite range, and let F be its cumulative function. We say that f is *regular*, if the function

$$\widetilde{v}(v) = v - \frac{1 - F(v)}{f(v)}$$

is monotone, strictly increasing function of v. We call $\widetilde{v}$ *virtual utility* or *virtual surplus.*

The key observation of Myerson ([19]), which we also use, is that revenue maximization is equivalent to virtual-utility maximization.

For optimally chosen $\widetilde{x_0}$ and $\widetilde{y_0}$ (see appendix C.3), let $\widetilde{x} = (\widetilde{x_0}, ..., \widetilde{x_{k-2}}, 1)$ and $\widetilde{y} = (\widetilde{y_0}, ..., \widetilde{y_{k-2}}, 1)$ be mutually-centered threshold-values. Let $x^r = (x_0^r, ..., x_k^r)$ and $y^r = (y_0^r, ..., y_k^r)$ be threshold-values that satisfy $x_0^r = y_0^r = 0$, $x_k^r = y_k^r = 1$ and for every $1 \leq i \leq k-1$ $\widetilde{x_{i-1}} = \widetilde{v}(x_i^r)$.

Theorem 4.6. *When both players' valuations are distributed with the same regular distribution function, the mechanism $MPG_k(x^r, y^r)$ achieves optimal expected revenue among all the individually-rational mechanisms in $G_{2,k}$. It incurs a revenue loss, compared with the optimal auction with no communication limitations, of $O(\frac{1}{k^2})$.*

Proof. In theorem C.5 in the appendix. □

As in the case of welfare optimization, we give an explicit solution for the case of uniform distribution functions. this optimal mechanism incurs a revenue loss of $\Omega(\frac{1}{k^2})$, which is (due to theorem 4.6) asymptotically the worst case.

Theorem 4.7. *When players' valuations are distributed uniformly, the modified priority-game $MPG_k(x, y)$ achieves optimal expected revenue among all the individually-rational mechanisms in $G_{2,k}$, where*

$$x = (0, \frac{1}{2}, t + \frac{1 \cdot (1-t)}{2k-3}, t + \frac{3 \cdot (1-t)}{2k-3}, ..., t + \frac{(2k-5) \cdot (1-t)}{2k-3}, 1)$$
$$y = (0, t, t + \frac{2 \cdot (1-t)}{2k-3}, t + \frac{4 \cdot (1-t)}{2k-3}, ..., t + \frac{(2k-4) \cdot (1-t)}{2k-3}, 1)$$

and $t = \frac{-2\alpha + \sqrt{1+3\alpha}}{2(1-\alpha)}$ for $\alpha = \frac{1}{(2k-3)^2}$. This mechanism incurs revenue loss of $\Omega(\frac{1}{k^2})$.

Proof. In theorem C.7 in the appendix. □

Note that when the distribution functions are uniform, the transformation $\widetilde{v}^{-1}$ is linear, and thus the threshold values x and y from theorem 4.7, without the first zero element, are mutually-centered.

5 Results for any number of players

This section has been moved to appendix D.

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A Symmetric 2-players 1-bit mechanisms

The optimal mechanisms we have presented so far in this paper are asymmetric. Can we find symmetric mechanisms that achieve optimal results? In this section we will see that the answer is no: optimal symmetric games can achieve smaller welfare and revenue, though the differences are small. We show this only for the simplest case of 2-players 1-bit games.

	0	1
0	w.p. $\frac{1}{2}A$ wins, pays 0 w.p. $\frac{1}{2}$ B wins, pays 0	B wins and pays $\frac{1}{4}$
1	A wins and pays $\frac{1}{4}$	w.p. $\frac{1}{2}$ A wins, pays$\frac{1}{2}$ w.p. $\frac{1}{2}$ B wins, pays $\frac{1}{2}$

Figure A.1: (m_1) 2-players 1-bit symmetric mechanism that achieves optimal welfare

A.1 Welfare optimization

Let m_1 be the mechanism depicted in figure A.1. We show that m_1 achieves maximal revenue (for symmetric mechanisms), and this revenue is smaller than the optimal revenue in asymmetric mechanisms (0.648, see section 2.2).

Proposition A.1. *The mechanism m_1 achieves welfare of $\frac{5}{8}$ with dominant-strategies equilibrium and ex-post IR.*

Proof. (sketch) The threshold strategy $\frac{1}{2}$ is dominant for both players. The dominance here is less obvious than previous examples: for player A, if $v_A < \frac{1}{2}$ he would prefer bidding 0 either when B bids 0 (because $\forall_{v_A<\frac{1}{2}}$ $\frac{1}{2} \cdot v_A > v_A - \frac{1}{4}$) or when B bids 1 (because $\forall_{v_A<\frac{1}{2}}$ $0 > \frac{1}{2}(v_A - \frac{1}{2})$), and if $v_A \geq \frac{1}{2}$ he will similarly bid 1.

In addition, ex-post IR clearly exist (zero payment is guaranteed when players bid 0). Calculating the welfare can be done as in section. 2 □

Theorem A.2. *No symmetric 2-players 1-bit mechanism achieves welfare which is strictly higher than m_1 (with any strategies, not necessarily dominant).*

Proof. (sketch) We will start with the set of all the symmetric 2-players 2-possible-bids mechanisms, and reduce it to an optimal mechanism. As in section 2.3, we conclude that the following set of mechanisms will contain an optimal (symmetric) one:

	0	1
0	w.p. $\frac{1}{2}$ A wins and pays 0 w.p. $\frac{1}{2}$ B wins and pays 0	B wins and pays b
1	A wins and pays a	w.p. $\frac{1}{2}$ A wins and pays$\overline{a}$ w.p. $\frac{1}{2}$ B wins and pays $\overline{b}$

With threshold-strategies (x, y) the expected welfare is:

$$\begin{aligned} w(x,y) &= x \cdot y \cdot \left(\frac{1}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot \frac{y}{2}\right) + x \cdot (1-y) \cdot \frac{(1+y)}{2} + (1-x) \cdot y \cdot \frac{(1+x)}{2} \\ &\quad + (1-x) \cdot (1-y) \cdot \left(\frac{1}{2} \cdot \frac{(1+x)}{2} + \frac{1}{2} \cdot \frac{(1+x)}{2}\right) \end{aligned}$$

This function achieves a unique maximum ($x, y \in [0,1]$) when $(x, y) = (\frac{1}{2}, \frac{1}{2})$, and $w(g, \frac{1}{2}, \frac{1}{2}) = \frac{5}{8}$. When $a = b = \frac{1}{4}$ and $\overline{a}$=$\overline{b}$=$\frac{1}{2}$ we get the mechanism m_1 from figure **??** □

	0	1
0	No allocation	B wins and pays $\frac{1}{\sqrt{3}}$
1	A wins, pays $\frac{1}{\sqrt{3}}$	w.p. $\frac{1}{2}$ A wins, pays $\frac{1}{\sqrt{3}}$ w.p. $\frac{1}{2}$ B wins, pays $\frac{1}{\sqrt{3}}$

Figure A.2: (m_2) 2-players 1-bit symmetric mechanism that achieves optimal revenue

A.2 Revenue optimization

Let m_2 be the mechanism depicted in figure A.2. We show that m_2 achieves maximal revenue (for symmetric mechanisms), which is smaller than the optimal expected revenue in asymmetric mechanisms (0.39, see section 2.2).

Proposition A.3. *The mechanism m_2 achieves revenue of 0.385 with dominant-strategies equilibrium and ex-post IR.*

Proof. (sketch) The threshold strategy $\frac{1}{\sqrt{3}}$ is clearly dominant for both players. The rest of the proof is similar to the proofs in section 2. □

Theorem A.4. *No symmetric mechanism with ex-post IR, achieves strictly greater revenue then m_2 (with any strategies)*

Proof. (sketch) Using the same considerations as in section 2.3, we can see that the following set of symmetric mechanisms contains one with optimal revenue (and ex-post IR):

	0	1
0	w.p. p A wins and pays 0 w.p. p B wins and pays 0	B wins and pays a
1	A wins and pays a	w.p. $\frac{1}{2}$ A wins and pays $\overline{a}$ w.p. $\frac{1}{2}$ B wins and pays $\overline{a}$

If $x > a$ or $x > \overline{a}$, the mechanism with $x = a = \overline{a}$ and $p = 0$ will achieve greater expected revenue with ex-post IR . Thus we can assume, $x = a = \overline{a}$ and $p = 0$. The expected revenue is:

$$R(a) = a(1-a)a + (1-a)aa + (1-a)(1-a)(\frac{1}{2}a + \frac{1}{2}a)$$

This function achieves unique maximum ($a \in [0,1]$) when $a = \frac{1}{\sqrt{3}}$ and $r(g, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = 0.385$ □

B Welfare optimization in 2-players mechanisms

B.1 Threshold-strategy optimality

Lemma B.1. *Given a mechanism $g \in G_{n,(k_1,\ldots,k_n)}$, there is a strategies-vector $\widetilde{s} \in \times_{i=1}^{n} \varphi_{k_i}$ such that for every player i, $\widetilde{s}_i$ is a threshold-strategy and*

$$w(g, \widetilde{s}) = \max_{s \in \times_{i=1}^{n} \varphi_k} w(g, s)$$

Proof. Given a strategies vector s^* which achieves optimal welfare in g (i.e. $w(g, s^*) = \max_{s \in \times_{i=1}^{k} \varphi_{k_i}} w(g, s)$), we will show that for every player i we can modify s_i^* to be a threshold-strategy, and the welfare will not decrease.

Assume s_i^* is not a threshold-strategy. Therefore, there must be $a, b, c \in [0,1]$, $a < b < c$ such that $s_i^*(a) = s_i^*(c) = m$ but $s_i^*(b) = t$ $(t \neq m)$. We will show that a strategies vector $\widetilde{s}$ identical to s^*, except $\widetilde{s_i}(b) = m$, holds $w(g, \widetilde{s}) \geq w(g, s^*)$.

Denote the probability that all players except i bids b_{-i} as $Pr(b_{-i})$. Thus, the expected welfare from a game g given that bidder i with valuation v_i bids m and that the other players use strategies s_{-i}^* is:

$$\sum_{b_{-i}} \left(a_i(m, b_{-i}) \cdot v_i + \sum_{j \neq i} a_j(m, b_{-i}) \cdot E(v_j \left| s_j^*(v_j) = b_j \right.) \right) \cdot Pr(b_{-i})$$

Note that this expected welfare is linear in v_i, and we denote it by $h(m) \cdot v_i + t(m)$ (the constants $h(m)$ and $t(m)$ depend on the bid m).

We know that $s_i^*(a) = m$. The strategies s^* achieve optimal welfare in g, therefore there is no other bid l such that if $s_i(a) = l$, the expected welfare will increase. Thus:

$$\forall l \neq m \quad h(m) \cdot a + t(m) \geq h(l) \cdot a + t(l) \tag{B.1}$$

Similarly, because $s_i^*(c) = m$, we can derive:

$$\forall l \neq m \quad h(m) \cdot c + t(m) \geq h(l) \cdot c + t(l) \tag{B.2}$$

We know that $a < b < c$, thus there is $\lambda \in [0,1]$ such that $\lambda a + (1 - \lambda)c = b$. We will multiply inequality B.1 with λ and B.2 with $(1 - \lambda)$, and sum the inequalities to get:

$$\forall l \neq m \;\; h(m) \cdot b + t(m) \geq h(l) \cdot b + t(l)$$

Thus, the expected welfare for player i, given $v_i = b$, is maximal when she bids m. Therefore, when modifying s_i^* such that $s_i^*(b) = m$ the total expected welfare will not decrease. We can repeat this process until there is no $a < b < c$ such that $s_i^*(a) = s_i^*(c) \neq s_i^*(b)$. Therefore, given a strategies-vector s^* such that $w(g, s^*) = \max_{s \in \times_{i=1}^{n} \varphi_{k_i}} w(g, s)$, we can change all strategies to threshold-strategies $\widetilde{s}$ and still have maximal welfare. □

B.2 Priority-games' optimality

B.2.1 Theorem's proof

Theorem B.2. *There is a priority-game $g \in G_{2,k}$ that achieves optimal welfare ($w_{2,k}^{opt}$).*

For proving theorem B.2 we have to prove some lemmas beforehand.

We have special interest in "simple" mechanisms where the item is always allocated (i.e. the item must be sold) or which the allocation is done without any lottery (deterministic games):

Definition 18. Mechanism $g \in G_{n,(k_1,\ldots,k_n)}$ is *deterministic* if for every combination of bids the allocation is fixed, i.e.

$$\forall b \in \beta_1 \times \ldots \times \beta_n \; \forall 1 \leq i \leq n \; a_i(b) \in \{0, 1\}$$

Definition 19. We say that in mechanism $g \in G_{n,(k_1,\ldots,k_n)}$ the *item must be sold* if for every combination of bids the item is always allocated, i.e.

$$\forall b \in \beta_1 \times \ldots \times \beta_n \; \sum_{i=1}^{n} a_i(b) = 1$$

Lemma B.3. *There is a mechanism $g^* \in G_{n,(k_1,...,k_n)}$, which is deterministic and in which the item must be sold, and a strategies' vector $s \in \times_{i=1}^{n} \varphi_{k_i}$, such that $g*$ and s achieve optimal welfare, i.e. $w(g^*, s) = w^{opt}_{n,(k_1,...,k_n)}$.*

Proof. In the next subsection (B.7). Proof's idea: For each bids' combination, we can assign the item with probability 1 to the player with the highest expected valuation, and the welfare will not decrease. ☐

Definition 20. A mechanism $g \in G_{n,(k_1,...,k_n)}$ is *monotone* if for any bids vector b and for any player i, the probability of a player to win the item cannot decrease, when he increases his bid, and all other player don't increase their bid (even if some of them decrease bids), i.e.

$$\forall_{b_i' \geq b_i} \forall_{b_{-i}' \leq b_{-i}} \quad a_i(b_i, b_{-i}) \leq a_i(b_i', b_{-i}')$$

The following lemma shows that we can modify any mechanism to be monotone, without decreasing the welfare.

Lemma B.4. *For any mechanism $g \in G_{n,(k_1,...,k_n)}$ and for any vector of threshold-strategies $s \in \times_{i=1}^{n} \varphi_{k_i}$, there is a monotone mechanism $g^* \in G_{n,(k_1,...,k_n)}$ such that $w(g^*, s) \geq w(g, s)$.*

Proof. We can assume, w.l.o.g, that for each bidder the thresholds are ordered from lowest to highest (i.e. if a bidder bids "m" for all $v \in [c_t, c_{t+1}]$ then she will bid "$m + 1$" for all $v \in [c_{t+1}, c_{t+2}]$). Thus, for any player i and any bid $m \in \{0, 1, ...k_i - 2\}$,

$$E(v_i | f_i(v_i) = m) \leq E(v_i | f_i(v_i) = m + 1) \tag{B.3}$$

Therefore, changing the mechanism as in the proof for lemma B.3, will modify the mechanism to be not only deterministic where the item must be sold, but also to be monotone. Given a bids vector (b_i, b_{-i}) for which player j has maximal expected valuation over all other players, if he increases his bid to $t \geq b_i$, he will still have the maximal expected valuation (according to equation B.3). For the same reason, if any other player decreases his bid, her average valuation will decrease too, so j still has the maximal expected welfare. ☐

The following lemma states that the optimal welfare strictly decreases, when the number of possible bids decreases. This property is intuitive, since with smaller number of bids, the mechanism's ability to differentiate the players' valuations decreases.

Lemma B.5. *$w^{opt}_{2,(k,k)} > w^{opt}_{2,(k-1,k)}$ for every k.*

Proof. In the next subsection (B.8) ☐

Corollary B.6. *Let $g \in G_{2,k}$ be a deterministic, monotone game in which the item must be sold, and x, y be threshold strategies that achieve the optimal welfare (i.e. $w^{opt}_{2,k}$) with g. Then, in the matrix representation of g there are no rows or columns with identical allocation.*

Proof. From lemmas B.3, B.4 we know that there is a deterministic, monotone mechanism g, in which the item must be sold, that achieves $w^{opt}_{2,k}$. Lemma B.1 tells that there are threshold strategies that achieve $w^{opt}_{2,k}$ in g. Assume that player A has 2 different bids with the same allocation. g is monotone, thus these bids are consecutive. Clearly, a similar game with $k-1$ rows that unites the 2 identical rows, will have the same expected welfare, using the same threshold strategies (when uniting the thresholds of the identical rows). Thus, $w^{opt}_{2,(k-1,k)} \geq w^{opt}_{2,(k,k)}$. Contradiction to lemma B.5. ☐

We finally can prove theorem B.2. According to lemmas B.3, B.4 there is a deterministic, monotone game in which the item must be sold that achieves $w^{opt}_{2,k}$. In monotone, deterministic games in which the item must be sold, a row i looks like $[A,...,A,B...B]$ where r_i is the index of the first B in this row ($0 \leq r_i \leq k$). Corollary B.6 derives that there are no rows with the same allocation in the game's matrix, thus there are $k+1$ possible rows for the game matrix. However, we only have k rows in our game. Similarly, we have k (of possible $k+1$) columns in the game. Assume that the row $[B,B,...,B]$ is in g. Then, the column $[A,A...,A]$ is not in g. Therefore, our game matrix consist of all the columns except $[A,A,...,A]$, which compose the priority game where $B \succ A$. If the row $[B,B,...,B]$ is not in g, then the row $[A,...,A]$ must be in g, and g is the priority-game where $A \succ B$.

B.2.2 Proofs for lemmas

Lemma B.7. *There is a mechanism $g^* \in G_{n,(k_1,...,k_n)}$, which is deterministic and in which the item must be sold, and a strategies vector $s \in \times_{i=1}^{n}\varphi_{k_i}$, such that $g*$ and s achieve optimal welfare, i.e. $w(g^*,s) = w^{opt}_{n,(k_1,...,k_n)}$.*

Proof. Let $g \in G_{n,(k_1,...,k_n)}$ and $s \in \times_{i=1}^{n}\varphi_{k_i}$ such that $w(g,s) = w^{opt}_{n,(k_1,...,k_n)}$. Consider the bids vector $b^* = (b^*_1,...,b^*_n)$. Let j be the bidder with the maximal expected valuation when all bidders bid b^*, i.e.

$$j \in \arg\max_{i\in\{1,...,n\}} E(v_i|s(v_i) = b^*_i)$$

Let g^* be a mechanism identical to g except that we always assign the item to bidder j when the bids' vector is b^*(i.e. $a_j(b^*) = 1$ and $\forall_{i\neq j} a_i(b^*) = 0$). We will show that $w(g^*,s) \geq w(g,s)$ (recall that $Pr(b)$ is the probability that the players bid b, given a strategies vector s, and let a be the allocation scheme of g and a^* be the allocation scheme of g^*):

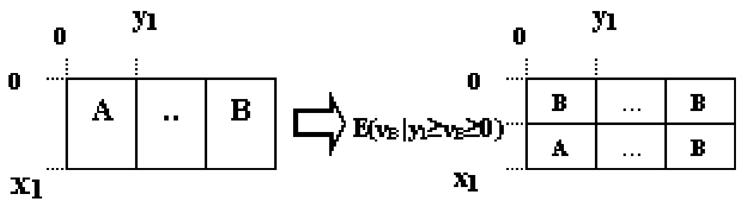
$$\begin{aligned}
& w(g,s) = E_{v\in[0,1]^n}\, w(g,s,v) = \sum_{b\in\beta_1\times...\times\beta_n} Pr(b)\cdot\left(\sum_{i=1}^{n} a_i(b)\cdot E(v_i|s(v_i)=b_i)\right) \\
= & \sum_{b\neq b^*}\left(Pr(b)\cdot\sum_{i=1}^{n} a_i(b)\cdot E(v_i|s(v_i)=b_i)\right) + Pr(b^*)\cdot\sum_{i=1}^{n} a_i(b^*)E(v_i|s(v_i)=b^*_i) \\
\leq & \sum_{b\neq b^*}\left(Pr(b)\cdot\sum_{i=1}^{n} a_i(b)\cdot E(v_i|s(v_i)=b_i)\right) + Pr(b^*)\cdot 1\cdot E(v_j|s(v_j)=b^*_j) \\
= & \sum_{b\neq b^*}\left(Pr(b)\cdot\sum_{i=1}^{n} a^*_i(b)\cdot E(v_i|s(v_i)=b_i)\right) + Pr(b^*)\cdot\sum_{i=1}^{n} a^*_i(b^*)E(v_i|s(v_i)=b^*_i) \\
= & \sum_{b}\left(Pr(b)\cdot\sum_{i=1}^{n} a^*_i(b)\cdot E(v_i|s(v_i)=b_i)\right) = E_{v\in[0,1]^m}\, w(g^*,s,v) = w(g^*,s)
\end{aligned}$$

We can perform similar modifications in g for each bids' vector b, and get a deterministic mechanism g^* where the item must be sold, and $w(g^*,s) \geq w(g,s) = w^{opt}_{n,(k_1,...,k_n)}$ thus $w(g^*,s) = w^{opt}_{n,(k_1,...,k_n)}$. ☐

Lemma B.8. $w^{opt}_{2,(k,k)} > w^{opt}_{2,(k-1,k)}$ *for every k.*

Proof. From lemmas B.3 and B.4 we can assume that there is a monotone, deterministic mechanism $g \in G_{2,k}$ in which the item must be sold such that there are threshold-strategies (x,y) for which

Figure B.1: Inserting a first row to a $(k-1)\times k$ game matrix



$w\left(g,(x,y)\right)=w_{2,k}^{opt}$. Because g is monotone, each row i in the game matrix (A is the rows player, B is the columns player) has index l_i such that it allocates the item to B if and only if $b_B \geq l_i$ (i.e. a row looks like: $[A,....,A,B,...B]$). The monotonicity also tells us that the index where the item starts to be allocated to B cannot decrease with the row number (i.e. $l_i \leq l_{i+1}$). B can start winning the item in $k+1$ possible indices inside a row (note that $[B,....B],[A,...A]$ are possible rows). Because we have only $k-1$ rows in g, there must be a type of row that is not one of the rows in g. We will modify g to $\widetilde{g}\in G_{2,(k,k)}$ by adding some missing row, and change the threshold strategy x to $\widetilde{x}$, such that the new mechanism and strategies achieve strictly greater expected utility than the originals. We will assume that the thresholds of each player are unique (i.e. $0<x_1<...<x_{k-1}<1$, $0<y_1<...<1$). (Otherwise, we have a bid that is never chosen, and we can omit it from the game, without welfare loss, and the rest of our proof will hold.)

Case 1. We can insert the new line as a first line in the game's matrix representation (i.e. when both players bid "0", A wins the item). See figure

We know that g is welfare-optimal, and that A wins the item when both players bid 0, so $E_{v_A}(v_A|0\leq v_A\leq x_1)\geq E_{v_B}(v_B|0\leq v_B\leq y_1)$ (otherwise allocating the item to B increases the expected welfare). Because the valuations' probabilities is always positive, $0<E_{v_A}(v_A|0\leq v_A\leq x_1)<1$. In the game matrix, we will add a first row identical to the first row in g, except allocating the item to B when they both bid 0. Let $x_1^{'}=E_{v_B}(v_B|0\leq v_B\leq y_1)-\varepsilon$ for some $0<\varepsilon<E_{v_B}(v_B|0\leq v_B\leq y_1)$. Let $\widetilde{x}=(0,x_1^{'},x_1,x_2,...,x_{k-2},1)$. When A,B uses $(\widetilde{x},y)$ as threshold strategies in $\widetilde{g}$, we will see that the expected welfare is strictly greater than the welfare when they use (x,y) in g (which is the optimal welfare for $k\times k$ games).

The expected welfare is a weighted sum of the expected welfare in all entries. Clearly the expected welfare in all entries where $b_A\in\{2,...,k_1-1\}$ is not changed, because each entry's probability and the allocation inside it haven't changed. Each entry $(b_A,b_B)=(0,i)$, $i>0$, in the first row of g, is split in $\widetilde{g}$ to two entries which together add the same expected welfare as in entry $(0,i)$ in g. We first show this property when g allocates the item in entry $(0,i)$ to B:

$$\begin{aligned}
& \begin{pmatrix} welfare\, from\, entry \\ (0,i)\; in\; \widetilde{g} \end{pmatrix} + \begin{pmatrix} welfare\, from\, entry \\ (1,i)\; in\; \widetilde{g} \end{pmatrix} \\
= \;& (F_A(\widetilde{x_1})-F_A(0))\cdot(F_B(y_{i+1})-F_B(y_i))\cdot E_{v_B}(v_B|y_i\leq v_B\leq y_{i+1}) \\
& +(F_A(\widetilde{x_2})-F_A(\widetilde{x_1}))\cdot(F_B(y_{i+1})-F_B(y_i))\cdot E_{v_B}(v_B|y_i\leq v_B\leq y_{i+1}) \\
= \;& (F_A(\widetilde{x_2})-F_A(0))\cdot(F_B(y_{i+1})-F_B(y_i))\cdot E_{v_B}(v_B|y_i\leq v_B\leq y_{i+1}) \\
= \;& (F_A(x_1)-F_A(0))\cdot(F_B(y_{i+1})-F_B(y_i))\cdot E_{v_B}(v_B|y_i\leq v_B\leq y_{i+1}) \\
= \;& \begin{pmatrix} welfare\, from\, entry \\ (0,i)\; in\; g \end{pmatrix}
\end{aligned}$$

If g allocates the item in entry $(0,i)$ to A :

$$\begin{aligned}
& \begin{pmatrix} welfare\, from\, entry \\ (0,i)\ in\ \widetilde{g} \end{pmatrix} + \begin{pmatrix} welfare\, from\, entry \\ (1,i)\ in\ \widetilde{g} \end{pmatrix} \\
= \;& (F_A(\widetilde{x_1}) - F_A(0)) \cdot (F_B(y_{i+1}) - F_B(y_i)) \cdot E_{v_A}(v_A | 0 \leq v_A \leq \widetilde{x_1}) \\
& + (F_A(\widetilde{x_2}) - F_A(\widetilde{x_1})) \cdot (F_B(y_{i+1}) - F_B(y_i)) \cdot E_{v_A}(v_A | \widetilde{x_1} \leq v_A \leq \widetilde{x_2}) \\
= \;& (F_B(y_i) - F_B(y_{i-1})) \cdot (F_A(\widetilde{x_1}) - F_A(0)) \cdot \frac{\int_0^{\widetilde{x_1}} f_A(v) v dv \int}{(F_A(\widetilde{x_1}) - F_A(0))} \\
& + (F_B(y_{i+1}) - F_B(y_i)) \cdot (F_A(\widetilde{x_2}) - F_A(\widetilde{x_1})) \cdot \frac{\int_{\widetilde{x_1}}^{\widetilde{x_2}} f_A(v) v dv \int}{(F_A(\widetilde{x_2}) - F_A(\widetilde{x_1}))} \\
= \;& (F_B(y_{i+1}) - F_B(y_i)) \cdot \int_0^{\widetilde{x_2}} f_A(v) v dv \\
= \;& (F_B(y_{i+1}) - F_B(y_i)) \cdot (F_A(\widetilde{x_2}) - F_A(0)) \cdot E_{v_A}(v_A | 0 \leq v_A \leq \widetilde{x_2}) \\
= \;& (F_B(y_{i+1}) - F_B(y_i)) \cdot (F_A(x_1) - F_A(0)) \cdot E_{v_A}(v_A | 0 \leq v_A \leq x_1) \\
= \;& \begin{pmatrix} welfare\, from\, entry \\ (0,i)\ in\ g \end{pmatrix}
\end{aligned}$$

Note that because $x_1' < E_{v_B}(v_B | 0 \leq v_B \leq y_1)$ and the always-positive probabilities, we have:

$$E_{v_B}(v_B | 0 \leq v_B \leq y_1) > x_1' = \widetilde{x_1} > E_{v_A}(v_A | 0 \leq v_A \leq \widetilde{x_1})$$

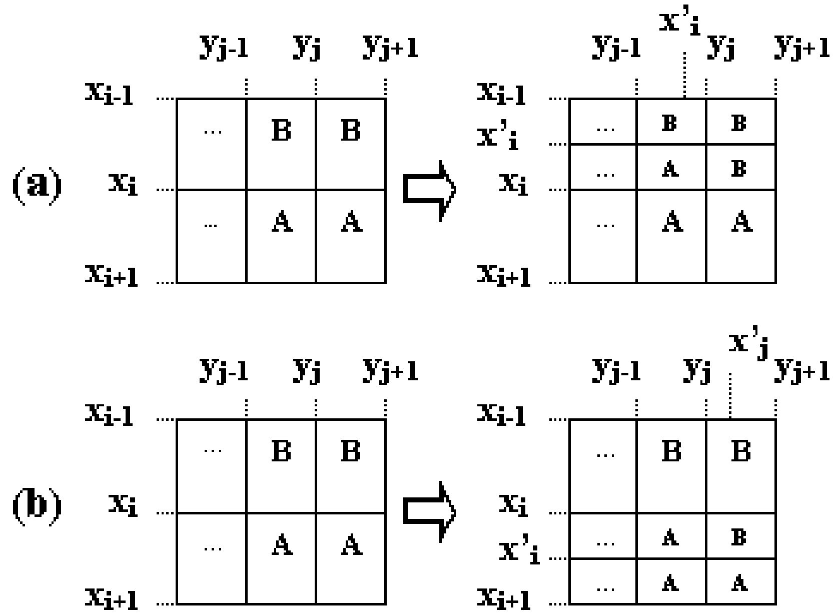
Thus, the contribution of entries $(0,0)$ and $(1,0)$ to the expected welfare in $\widetilde{g}$ is strictly greater than the expected welfare in entry $(0,0)$ in g:

$$\begin{aligned}
& \begin{pmatrix} welfare\, from\, entry \\ (0,0)\ in\ \widetilde{g} \end{pmatrix} + \begin{pmatrix} welfare\, from\, entry \\ (1,0)\ in\ \widetilde{g} \end{pmatrix} \\
= \;& (F_A(\widetilde{x_1}) - F_A(0)) \cdot (F_B(y_1) - F_B(y_0)) \cdot E_{v_B}(v_B | 0 \leq v_B \leq y_1) \\
& + (F_A(\widetilde{x_2}) - F_A(\widetilde{x_1})) \cdot (F_B(y_1) - F_B(y_0)) \cdot E_{v_A}(v_A | \widetilde{x_2} \leq v_A \leq \widetilde{x_1}) \\
> \;& (F_A(\widetilde{x_1}) - F_A(0)) \cdot (F_B(y_1) - F_B(y_0)) \cdot E_{v_A}(v_A | 0 \leq v_A \leq \widetilde{x_1}) \\
& + (F_A(\widetilde{x_2}) - F_A(\widetilde{x_1})) \cdot (F_B(y_1) - F_B(y_0)) \cdot E_{v_A}(v_A | \widetilde{x_2} \leq v_A \leq \widetilde{x_1}) \\
= \;& (F_A(\widetilde{x_2}) - F_A(0)) \cdot (F_B(y_1) - F_B(0)) \cdot E_{v_A}(v_A | 0 \leq v_A \leq \widetilde{x_2}) \\
= \;& (F_A(\widetilde{x_2}) - F_A(0)) \cdot (F_B(y_1) - F_B(0)) \cdot E_{v_A}(v_A | 0 \leq v_A \leq x_1) \\
= \;& \begin{pmatrix} welfare\, from\, entry \\ (0,0)\ in\ g \end{pmatrix}
\end{aligned}$$

Case 2. We can insert the new line as a last row

In this case we know that B wins the item in g when both players bid maximal bids (i.e. $(b_A, b_B) = (k-2, k-1)$). We will create a new game $\widetilde{g} \in G_{2,(k,k)}$ by adding a $k-1$th row identical to g's $k-2$th row, except that A wins the item when $b_B = k-1$. Let $x_{k-1}' = E(v_B | y_{k-1} \leq v_B \leq 1) + \varepsilon$. When A uses the strategy $\widetilde{x} = (0, x_1, ..., x_{k-2}, x_{k-1}', 1)$, and B still uses y as a threshold-strategy, the total welfare is strictly greater than the optimal welfare in g, achieved by the strategies (x, y). It can be shown similarly as in the previous case: the expected welfare in $\widetilde{g}$ is the same as in g, except in entry $(k-1, k-1)$ when it strictly increases.

Figure B.2: Adding a middle row to $(k-1 \times k)$ game with optimal welfare



Case 3. We can insert a row between 2 existing rows

Now there must be 2 rows i and $i+1$ and 2 columns j and $j+1$ such that we allocate the item to B when the bids are $(i,j),(i,j+1)$ and to A when the bids are $(i+1,j),(i+1,j+1)$ (see figure B.2). We will create a new mechanism $\widetilde{g}$ by adding a row i' identical to row i except for the entry $(i',j+1)$ where B will win the item. The way we construct the new threshold-strategy $\widetilde{x}$ for A depends on whether the expected valuation of B when bidding j is smaller than x_i or not:

When $E(v_B|y_{j-1} \le v_B \le y_j) < x_i$**:** let $x_i^{'} = E(v_B|y_{j-1} \le v_B \le y_j)$, and let $\widetilde{x} = (0, x_1, ..., x_{i-1}, x_i^{'}, x_i, ..., 1)$. As in previous cases, we can see that the expected welfare in all entries hasn't changed, except a strictly positive increase in the (i',j) index (because $E(v_A|x_i^{'} \le v_B \le x_i) > x_i$), so the total welfare has increased. See figure B.2 part (a).

When $E(v_B|y_{j-1} \le v_B \le y_j) \ge x_i$**:** Let $x_i^{'} = E(v_B|y_{j-1} \le v_B \le y_j)$. Because the probabilities are always positive, $E(v_B|y_j \le v_B \le y_{j+1}) > E(v_B|y_j \le v_B \le y_{j+1}) \ge x_i$. Player A wins the item in entry $(i+1,j+1)$ thus $E(v_B|y_j \le v_B \le y_{j+1}) \le E(v_A|x_i \le v_A \le x_{i+1}) < x_{i+1}$ so $x_i < x_i^{'} < x_{i+1}$. Let $\widetilde{x} = (0, x_1, ..., x_i, x_i^{'}, x_{i+1}, ..., 1)$. Again, the expected welfare was changed only in the entry (i',j), where it was strictly increased (because $x_i^{'} > E(v_A|x_i \le v_A \le x_i^{'})$, so the total welfare has increased. See figure B.2 part (b) ☐

B.3 The welfare optimizing mechanism

Proposition B.9. *For any pair of distribution functions on the players' valuations, there exist a unique pair of mutually-centered threshold-strategies.*

Proof. (sketch) We know that $x_0 = y_0 = 0$. If x_1 is known, we can calculate y_1 (that solves $x_1 = E_{v_B}(v_B\,|y_0 \le v_B \le y_1\,)$), which is unique since the function $h(y) = E(v|0 \le v \le y)$ is continuous and strictly monotone (assuming continuous, always-positive distributions). Similarly we can calculate x_2, then y_2, then x_3 and so on. Finally, we calculate y_{k-1} such that it solves $x_{k-1} = E(v_B|y_{k-2} \le$

$v_B \leq y_{k-1})$. It is easy to see that all the variables x_i and y_i can be viewed as a continuous, strictly-monotone functions of x_1. Now, let y' be the solution for the equation $y_{k-1} = E(v_A|x_{k-1} \leq v_A \leq y')$. For the satisfiability of all the $2(k-1)$ equations, $y' = 1$ must hold. Because y' is also a continuous strictly-monotone function of x_1, there is only a single value of x_1 for which all the equations hold. □

Theorem B.10. *For any pair of distribution functions on the players' valuations, the mechanism $PG_k(\overrightarrow{x}, \overrightarrow{y})$ achieves optimal welfare, among all mechanisms (i.e. $w^{opt}_{2,k}$), where $\overrightarrow{x}, \overrightarrow{y}$ are mutually-centered threshold-strategies.*

Proof. According to theorem B.2 and lemma B.1 there is a priority-game $(B \succ A)$ $g \in G_{2,k}$, and threshold strategies $\overrightarrow{x} = (x_0, x_1, ..., x_k), \overrightarrow{y} = (y_0, y_1, ..., y_k)$ such that $w(g, (x, y)) = w^{opt}_{2,k}$. Note that $x_0 = y_0 = 0$ and $x_1 = y_1 = 1$, so we have $2 \cdot (k-1)$ variables to optimize. Also note that player i bids m with probability $F_B(y_m) - F_B(y_{m-1})$ (recall that F_i is the cumulative distribution function for player i), and that her average valuation when she bids m is:

$$E(v_i|s_i(v_i) = m) = E(v_i|y_{m-1} \leq v_i \leq y_m) = \frac{\int_{y_{m-1}}^{y_m} f_i(v_i)v_i dv_i}{F_i(y_m) - F_i(y_{m-1})}$$

We will calculate the total expected welfare by summing first the expected welfare in the entries of the game matrix where B wins the item, then summing the entries where A is the winner. Note that the average valuation of B is fixed per column, and for A is fixed per row.

$$\begin{aligned}
w(g,f) &= \sum_{i=1}^{k} (F_B(y_i) - F_B(y_{i-1})) \cdot (F_A(x_i) - F_A(x_0)) \cdot \frac{\int_{y_{i-1}}^{y_i} f_B(v_B)v_B dv_B}{F_B(y_i) - F_B(y_{i-1})} \\
&\quad + \sum_{i=2}^{k} (F_A(x_i) - F_A(x_{i-1})) \cdot (F_B(y_{i-1}) - F_B(y_0)) \cdot \frac{\int_{x_{i-1}}^{x_i} f_A(v_A)v_A dv_A}{F_A(x_i) - F_A(x_{i-1})} \\
&= \sum_{i=1}^{k} \cdot (F_A(x_i) - F_A(x_0)) \cdot \int_{y_{i-1}}^{y_i} f_B(v_B)v_B dv_B \\
&\quad + \sum_{i=2}^{k} \cdot (F_B(y_{i-1}) - F_B(y_0)) \cdot \int_{x_{i-1}}^{x_i} f_A(v_A)v_A dv_A \\
&= \sum_{i=1}^{k} \cdot F_A(x_i) \cdot \int_{y_{i-1}}^{y_i} f_B(v_B)v_B dv_B + \sum_{i=2}^{k} \cdot F_B(y_{i-1}) \cdot \int_{x_{i-1}}^{x_i} f_A(v_A)v_A dv_A
\end{aligned}$$

Because the probability functions are continuous and derivable, we can formulate the partial derivatives for all variables (Note that by definition $\frac{\partial F_i(x)}{\partial x} = (F_i(x))'_x = f_i(x)$ and that $\left(\int_c^x f(v) \cdot v \cdot dv\right)'_x = f(x)x$).

$$(w(g,s))'_{x_i} = \left(\int_{y_{i-1}}^{y_i} f_B(v_B)v_B dv_B\right) \cdot f_A(x_i) + f_A(x_i) \cdot x_i \cdot F_B(y_{i-1}) - f_A(x_i) \cdot x_i \cdot F_B(y_i) = 0$$

$$x_i = \frac{\int_{y_{i-1}}^{y_i} f_B(v_B)v_B dv_B}{F_B(y_i) - F_B(y_{i-1})} = E_{v_B}(v_B\, |y_{i-1} \leq v_B \leq y_i)$$

$$(w(g,s))'_{y_i} = \left(\int_{x_i}^{x_{i+1}} f_A(v_A)v_A dv_A\right) \cdot f_B(y_i) + f_B(y_i) \cdot y_i \cdot F_A(x_i) - f_B(y_i) \cdot y_i \cdot F_A(x_{i+1}) = 0$$

$$y_i = \frac{\int_{x_i}^{x_{i+1}} f_A(v_A) v_A dv_A}{F_A(x_{i+1}) - F_A(x_i)} = E_{v_A}(v_A \,|x_{i+1} \le v_A \le x_i)$$

□

B.4 The uniform distribution case

Theorem B.11. *When players' valuations are uniformly distributed, the mechanism $PG_k(\overrightarrow{x}, \overrightarrow{y})$ achieves optimal welfare, among all mechanisms, where*

$$x = (0, \frac{1}{2k-1}, \frac{3}{2k-1}, ..., \frac{2k-3}{2k-1}, 1)$$

$$y = (0, \frac{2}{2k-1}, \frac{4}{2k-1}, ..., \frac{2k-2}{2k-1}, 1)$$

and the welfare loss it incurs is exactly $\frac{1}{6\cdot(2k-1)^2}$. The optimal welfare is achieved with dominant-strategies equilibrium and ex-post IR.

Proof. First, we define the welfare loss.

Definition 21. Denote w_n^{opt} as the expected welfare from optimal n-players single-item auctions (e.g. Vickrey's 2nd-price auction) with unbounded communications. The *welfare loss* (or efficiency loss) from a mechanism $g \in G_{n,k}$ when the players use strategies s is

$$wl(g, f) = w_n^{opt} - w(g, s)$$

We need to prove that $\min_{g \in G_{2,k}, s \in \varphi_k \times \varphi_k} wl(g, s) = \frac{1}{6\cdot(2k-1)^2}$. According to theorem 4.3 , $w_{2,k}^{opt}$ can be achieved with $PG_k(\overrightarrow{x}, \overrightarrow{y})$, when the thresholds $(\overrightarrow{x}, \overrightarrow{y})$ of players A, B must hold (when valuations are distributed uniformly for both players):

$$\forall_{1 \le i \le k-1} \quad x_i = E_{v_B}(v_B \,|y_{i-1} \le v_B \le y_i) = \frac{y_{i-1} + y_i}{2}$$

$$\forall_{1 \le i \le k-1} \quad y_i = E_{v_A}(v_A \,|x_i \le v_A \le x_{i+1}) = \frac{x_i + x_{i+1}}{2}$$

There is a single solution for this set of equations, when

$$x = (0, \frac{1}{2k-1}, \frac{3}{2k-1}, ..., \frac{2k-3}{2k-1}, 1)$$

$$y = (0, \frac{2}{2k-1}, \frac{4}{2k-1}, ..., \frac{2k-2}{2k-1}, 1)$$

Note that this mechanism can make the wrong allocation (compared to the efficient auction with no communication bounds) only in bids' combinations that are on the diagonal or on the lower secondary diagonal in the matrix representation of the game (i.e. when $b_A = b_B$ or when $b_A = b_B + 1$). It is easy to verify, that for a bids' vector (i, j), the overlapping segment of $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$ is of the exact size of $\frac{1}{2k-1}$. Given the bids' vector (i, j), if one of the valuations is not in this overlapping segment, the allocation is optimal (note that we allocate the item to B on the main diagonal, and to A on the secondary diagonal). The probability that both valuation are in this overlapping range is $\frac{1}{(2k-1)^2}$. The expected valuation in our priority-game (when both valuation are in this overlapping segment) is exactly in the middle of this segment. The expected valuation in the optimal auction will be exactly

	0	1	..	k-2	k-1
0	$B,0$	$B,0$	...	$B,0$	$B,0$
1	$A,\frac{1}{2k-1}$	$B,\frac{2}{2k-1}$	...	$B,\frac{2}{2k-1}$	$B,\frac{2}{2k-1}$
2	$A,\frac{1}{2k-1}$	$A,\frac{3}{2k-1}$	...	$B,\frac{4}{2k-1}$	$B,\frac{4}{2k-1}$
...	...	...	...	...	...
k-2	$A,\frac{1}{2k-1}$	$A,\frac{3}{2k-1}$	...	$B,\frac{2k-4}{2k-1}$	$B,\frac{2k-4}{2k-1}$
k-1	$A,\frac{1}{2k-1}$	$A,\frac{3}{2k-1}$	...	$A,\frac{2k-3}{2k-1}$	$B,\frac{2k-2}{2k-1}$

Figure B.3: Optimal 2-players k-possible-bids game with dominant-strategy equilibrium and ex-post IR

in the $\frac{2}{3}$ point of this range. Thus, the welfare loss, given that both players are in this overlapping segment, is a $\frac{1}{6}$ of the segment, i.e. $\frac{1}{6}\frac{1}{2k-1}$. Thus, for every bids' vector on the main diagonal or on the secondary-diagonal the expected welfare loss is $\frac{1}{6}\frac{1}{(2k-1)^3}$. There are $(2k-1)$ such bids' vector, thus the total welfare loss is $\frac{1}{6}\frac{1}{(2k-1)^2}$.

Figure B.3 describes such mechanism for games with k possible bids. Figure B.3 describes this optimal mechanism. $\square$

B.5 Asymptotic analysis of welfare loss

Theorem B.12. *For any pair of distribution functions on the players' valuations, the mechanism $PG_k(x^w, y^w)$ achieves optimal welfare ($w_{2,k}^{opt}$), among all mechanisms in $G_{2,k}$. The optimal welfare is achieved with dominant-strategies equilibrium and ex-post IR and the welfare loss it incurs, compared with the optimal auction with no communication bounds, is $O(\frac{1}{k^2})$.*

Proof. Theorem B.10 proves that the mechanism $PG_k(\overrightarrow{x}, \overrightarrow{y})$ achieves $w_{2,k}^{opt}$. Following is the proof for the $O(\frac{1}{k^2})$ upper bound. We show a priority game in which both players have the same dominant threshold-strategy, where the probability that a player chooses a certain bid is $O(\frac{1}{k})$ and both players will choose this bid w.p. $O(\frac{1}{k^2})$. Because the thresholds are identical, "bad" allocation of the item might occur only when both bids are equal (i.e. on the diagonal). Simple calculation shows welfare loss of $O(\frac{1}{k^2})$.

Consider a priority game $g \in G_{2,k}$. Let a, b be integers such that $k = a + b$ and $a, b \geq \lfloor \frac{k}{2} \rfloor$ (clearly such numbers exist). Assume w.l.o.g that $a \geq b$. Let $X = \{x_1, ... x_{a-1}\} \in [0, 1]^{a-1}$ be a set of $a - 1$ points that divide the distribution function f_A to a segments with the same weight (when $x_0 = 0, x_a = 1$), i.e.:

$$\forall_{1 \leq i \leq a-1} \qquad F_A(x_i) - F_A(x_{i-1}) = \frac{1}{a}$$

Similarly, let $Y = \{y_1, ..., y_b\} \in [0, 1]^b$ be a set of b points that divide f_B to $b + 1$ equal segments (when $y_0 = 0, y_{b+1} = 1$), i.e.:

$$\forall_{1 \leq i \leq b+1} \qquad F_B(y_i) - F_B(y_{i-1}) = \frac{1}{b+1}$$

Let $T = X \bigcup Y$, $|T| = (a-1)+b = k-1$, and let $\overrightarrow{t} = (0, t_1, t_2, ..., t_{k-1}, 1)$ be the threshold-strategy created by ordering the elements in T from smallest to largest. For every $1 \leq i \leq k$,

$$F_A(t_i) - F_A(t_{i-1}) \leq \frac{1}{a} \leq \frac{1}{\lfloor \frac{k}{2} \rfloor} \leq \frac{1}{\frac{k}{2}-1} = \frac{2}{k-2}$$

$$F_B(t_i) - F_B(t_{i-1}) \leq \frac{1}{b+1} \leq \frac{1}{\lfloor \frac{k}{2} \rfloor + 1} \leq \frac{1}{\frac{k}{2}} = \frac{2}{k}$$

g is a priority-game (w.l.o.g $B \succ A$), therefore, when $b_A > b_B$ (i.e. player A bids higher bid), player A wins the item. Because both player use the same threshold strategy, clearly $v_A > v_B$. Thus, the allocation is as in the Vickrey's 2nd-price auction. In the same way, when $b_A < b_B$ there will also be no welfare loss. Welfare loss is only possible in entries on the diagonal, i.e. when $b_A = b_B$. When $b_A = b_B = m$, the players' valuation are in the range $[t_{m+1}, t_m]$. The maximal welfare loss in this case can occur when $v_A = t_{m+1}$,$v_B = t_m$, but B wins. Thus, we can have an upper bound for the total welfare loss, by summing the welfare losses only on the entries of the diagonal (note that the optimal allocation always allocates the item to the player with the maximal valuation):

$$\begin{aligned}
&wl(g, f) = \\
&= \sum_{i=1}^{k} (F_A(t_i) - F_A(t_{i-1}))\,(F_B(t_i) - F_B(t_{i-1}))\, E\left(max(v_A, v_B) - v_B \;\middle|\; \begin{array}{l} v_A \in [t_{i-1}, t_i] \\ v_B \in [t_{i-1}, t_i] \end{array} \right) \\
&\leq \sum_{i=1}^{k} (F_A(t_i) - F_A(t_{i-1}))\,(F_B(t_i) - F_B(t_{i-1}))\,(t_i - t_{i-1}) \\
&\leq \sum_{i=1}^{k} \frac{2}{k-2} \cdot \frac{2}{k} \cdot (t_i - t_{i-1}) \leq \frac{4}{k^2 - 2k} \sum_{i=1}^{k} (t_i - t_{i-1}) \\
&= \frac{4}{k^2 - 2k}(t_k - t_0) = \frac{4}{k^2 - 2k}
\end{aligned}$$

We know that we can set the payments in g (priority game based on t and t) such that the threshold strategy t will be dominant for A and B, with ex-post IR. ☐

Theorem B.13. *The mechanism $PG_k(x, y)$, where $x = y = (0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1)$, incurs an expected welfare loss of $O(\frac{1}{k})$ for any pair of distribution functions of the players' valuations. Moreover, for any mechanism there exist a pair of distribution functions for which the expected welfare loss is $\Omega(\frac{1}{k})$.*

Proof. When both players use the same threshold strategy $x = (0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1)$ in a priority-game, non-optimal allocation is possible only on the diagonal. Then, when both player bids the same bid, a maximal welfare loss of $\frac{1}{k}$ can be incurred (maximal difference between valuations). Thus the expected welfare loss will be smaller than

$$\begin{aligned}
& \sum_{i=0}^{k-1} (F_B(x_{i+1}) - F_B(x_i))\,(F_A(x_{i+1}) - F_A(x_i))\,\frac{1}{k} \\
= \;& \frac{1}{k} \sum_{i=0}^{k-1} (F_B(x_{i+1}) - F_B(x_i))\,(F_A(x_{i+1}) - F_A(x_i)) \\
\leq \;& \frac{1}{k} \sum_{i=0}^{k-1} (F_B(x_{i+1}) - F_B(x_i)) = \frac{1}{k}
\end{aligned}$$

Now, consider a mechanism $g \in G_{2,k}$ with dominant strategies s_A, s_B. We can prove, similarly to the proof of lemma B.1, every mechanism that has a dominant strategy equilibrium, has an equilibrium with dominant threshold-strategies. Thus, we can assume that s_A, s_B are threshold strategies with the threshold-values vectors x, y. For any bids-combination b_A, b_B with overlapping valuations, i.e. $[x_{b_A+1}, x_{b_A}] \bigcap [y_{b_B+1}, b_B] \neq \phi$, let z_{b_A,b_B} be the size of the overlapping segment. clearly, the sum of all such z_{b_A,b_B} is 1, and there are no more than $2k-1$ such bids combination. Thus, there must be a bids' combination b_A, b_B with an overlapping segment with size of at least $\frac{1}{2k-1}$. Denote this segment with $[m_1, m_2]$ when $m_2 > m_1$. Assume w.l.o.g that for the bids vector b_A, b_B player B wins the item with probabilty not greater than $\frac{1}{2}$. Now assume that players' valuations are distributed such that B always have the valuation m_2 and A's valuation is always m_1. Then, the allocation will not be optimal (i.e. A wins or no player wins) with probability of at least $\frac{1}{2}$, and the welfare loss is at least $m_2 - m_1 \geq \frac{1}{2k-1}$. Thus, the expected welfare loss is at least $\frac{1}{2} \cdot \frac{1}{2k-1}$, i.e. $\Omega(\frac{1}{k})$. ☐

C Revenue optimization in 2-players mechanisms

C.1 Threshold-strategies' optimality

Lemma C.1. *Given a mechanism $g \in G_{n,(k_1,\ldots,k_n)}$ with ex-post Individual-Rationality, there is a strategies-vector $\widetilde{s} \in \times_{i=1}^{n} \varphi_k$ such that for every player i, $\widetilde{s}_i$ is a threshold-strategy and*

$$r(g, \widetilde{s}) = \max_{s \in \times_{i=1}^{n} \varphi_k} r(g, s)$$

Proof. Given a strategies vector s^* that achieves optimal revenue in g (i.e. $r(g, s^*) = \max_{s \in \times_{i=1}^{n} \varphi_k} r(g, s)$), we will show that for every player i we can modify s_i^* to be a threshold-strategy, and the welfare will not decrease.

Assume s_i^* is not a threshold-strategy. Therefore, there must be $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$, $\alpha_1 < \alpha_2 < \alpha_3$ such that $s_i^*(\alpha_1) = s_i^*(\alpha_3) = m$ but $s_i^*(\alpha_2) = t$ ($t \neq m$). We will show that a strategies vector $\widetilde{s}$ identical to s^*, except $\widetilde{s}_i(\alpha_2) = m$, holds $r(g, \widetilde{s}) \geq r(g, s^*)$.

Denote $Feasible_{i,v} = \left\{ j \,\middle|\, v \geq max_{b_{-i}} \left(a_i(j, s_{-i}) \right) \right\}$, i.e. the set of all feasible bids for a player with ex-post IR when her valuation is v. From the definition of $Feasible_{i,v}$ we derive that if $m \in Feasible_{i,\alpha_1}$ and $\alpha_2 > \alpha_1$ then $m \in Feasible_{i,\alpha_2}$ (i.e. a bid will remain feasible when the valuation increases).

We know that the strategies' vector s^* achieve optimal revenue in g, and that $s_i^*(\alpha_3) = m$. Thus, no bid achieves greater expected revenue than m when i's valuation is α_3, i.e.

$$m \in \arg \max_{j \in Feasible_{i,\alpha_3}} \left(\sum_{b_{-j}} Pr(b_{-j}) a_i(j, b_{-j}) p_i(j, b_{-j}) \right)$$

where $Pr(b_{-j})$ is the probability that the other players bid b_{-i}. We saw that $m \in Feasible_{i,\alpha_2}$ and clearly $Feasible_{i,\alpha_2} \subseteq Feasible_{i,\alpha_3}$, therefore

$$m \in \arg \max_{j \in Feasible_{i,\alpha_2}} \left(\sum_{b_{-j}} Pr(b_{-j}) a_i(j, b_{-j}) p_i(j, b_{-j}) \right)$$

Thus, if we modify s_i^* such that $s_i^*(\alpha_2) = m$, the expected revenue will not decrease. This way we can modify s_i^* to be a threshold strategy without decreasing the revenue, and we can do it for each player i. ☐

C.2 Lower bound for revenue loss

Theorem C.2. *([19]) Consider a model with unbounded communication, in which non-winning players pay zero. Let h be a direct-revelation mechanism, which is incentive-compatible (i.e. truth telling by all players forms Nash equilibrium) and interim individually-rational. Then in h, the expected revenue is equal to the expected virtual utility.*

Proposition C.3. *Consider a mechanism $g \in G_{2,k}$ with dominant-strategies equilibrium. Then, in the unbounded communications model, there is an incentive-compatible, individually-rational mechanism that achieves exactly the same expected revenue and expected virtual-utility as g does.*

Proof. Let $g \in G_{2,k}$ be a mechanism with dominant strategies s_A,s_B. Consider the following direct-revelation mechanism: each player i bids her true valuation v_i. The mechanism calculates $s_i(v_i)$ for every i, and determines the allocation and payments according to g, as if the players bids were $s_i(v_i)$. The new mechanism is incentive-compatible: Let w_i be player i's bid. If $s_i(v_i) = s_i(w_i)$, the allocation and payments are identical, weather she bids v_i or w_i. If $s_i(v_i) \neq s_i(w_i)$, and player i gains positive utility by bidding i, then bidding $s_i(w_i)$ gains more utility for her in g than bidding $s_i(v_i)$. Contradiction to s_i being a dominant-strategy.

The new mechanism is clearly ex-post individually-rational, since the allocation and payments are exactly as in g, when the players use the strategies s_A, s_B, and g is ex-post individually-rational. Since the allocation and payments are exactly as in g, the expected revenue, and expected virtual-utility are equal in both mechanisms □

Proposition C.4. *Consider a mechanism $g \in G_{2,k}$, with dominant-strategies equilibrium. Then, over all the 2-players mechanisms with k-possible bids, which are ex-post individually-rational and have dominant-strategy equilibrium, g achieves maximal revenue if and only if g achieves maximal virtual utility*

Proof. An immediate result of proposition C.3 and C.2 is that the expected virtual utility in a mechanism with k-possible bids and dominant strategies equilibrium is equal to the expected revenue. Thus maximizing the revenue is equivalent to maximizing the virtual utility □

Theorem C.5. *When both players' valuations are distributed with the same regular distribution function, the mechanism $MPG_k(x^r, y^r)$ achieves optimal expected revenue among all the individually-rational mechanisms. It incurs a revenue loss, compared with the optimal auction with no communication limitations, of $O(\frac{1}{k^2})$.*

Proof. First, we prove the optimality of the given mechanism $MPG_k(x, y)$.

Consider the model where players' consider their virtual-utilities as their valuations, (then, the valuations are in $[\frac{1}{-p(0)}, 1]$), and that there is a third player with a constant valuation of zero (player Z which is actually the seller). Consider a mechanism h with k-possible bids in this model, and a pair of threshold-strategies $\widetilde{x}, \widetilde{y}$ that achieve optimal **welfare** among all mechanisms and strategies (due to lemma B.1, h achieves optimal welfare with threshold strategies, player Z is not relevant for this lemma). We will make some modifications in h, which cannot decrease the expected welfare it achieves with $\widetilde{x}, \widetilde{y}$. In the interim stages $\widetilde{x}, \widetilde{y}$ won't necessarily stay dominant, but in the final mechanism they will. Let a be the first index such that $E(v_A|\widetilde{x_a} \leq v_A \leq \widetilde{x_{a+1}}) \geq 0$. Let b be the first index such that $E(v_B|\widetilde{y_b} \leq v_B \leq \widetilde{y_{b+1}}) \geq 0$. Consider a bids' vector $(i.j)$. When $i < a$ and $j < b$, the expected valuations of both A and B are negative. Thus, allocating the item to Z will not decrease the expected welfare. When $i < a$ and $j \geq b$, the expected welfare of player B is positive, and A's expected welfare is negative, thus we can allocated the item to B and the welfare will not decrease. Similarly, we can allocate the item to A when $i \geq a$ and $j < b$. When $i < a$, the allocation is done regardless to i, thus we can assume that x_a is the first threshold (i.e. $a = 1$), and similarly we can assume that $b = 1$.

Next, we show the optimal allocation for bids' combinations (i, j) such that $i \geq a$ and $j \geq b$. We can ignore the player with the zero valuation, and we actually perform an auction with $k - 1$ possible bids for each player, when the players' valuation are in the range $[\widetilde{x_a}, 1]$, $[\widetilde{y_b}, 1]$. The proof for theorem B.2 is true when players' valuations are in these new ranges, and we know (theorem B.10) that the optimal welfare in this case can be achieved in a priority game, with mutually-centered strategies $\widetilde{x} = (\widetilde{x_1}, \widetilde{x_2}, ..., \widetilde{x_{k-1}}, 1)$ and $\widetilde{y} = (\widetilde{y_1}, \widetilde{y_2}, ..., \widetilde{y_{k-1}}, 1)$. The only degrees of freedom we have, are the values for $\widetilde{x_1}$ and $\widetilde{y_1}$. Thus, the new welfare-optimal mechanism is actually a modified priority-game , based on $\widetilde{x} = (\frac{1}{-p(0)}, \widetilde{x_1}, \widetilde{x_2}, ..., \widetilde{x_{k-1}}, 1)$ and $\widetilde{y} = (\frac{1}{-p(0)}, \widetilde{y_1}, \widetilde{y_2}, ..., \widetilde{y_{k-1}}, 1)$. Consider the same mechanism, except each payment c in the mechanism is replaced with $\widetilde{v}^{-1}(c)$. Clearly, we get the given mechanisms $MPG_k(x, y)$, were x, y are $\widetilde{v}^{-1}$-transformation of $\widetilde{x}, \widetilde{y}$. The distribution function p is regular, thus $\widetilde{v}^{-1}$ is also strictly increasing, and thus $MPG_k(x, y)$, when players use their original valuations, will have exactly the same allocation as $MPG_k(\widetilde{x}, \widetilde{y})$, when the players consider their virtual-utilities as their valuations. Thus, $MPG_k(x, y)$ achieves optimal expected virtual-utility. Due to proposition C.4, $MPG_k(x, y)$ achieves optimal revenue.

Next, we prove the second part of the theorem, i.e. that revenue-optimal mechanisms incurs expected revenue loss of $O(\frac{1}{k^2})$. We show that the mechanism created when $\widetilde{x_1} = \widetilde{y_1} = 0$ (which is not necessarily optimal) incurs a revenue loss of $O(\frac{1}{k^2})$. The optimal mechanism cannot do worse. Consider the model when every player replaces his valuation with his virtual-utility, and consider the mechanism $MPG_k(\widetilde{x}, \widetilde{y})$ described above (when $\widetilde{x_1} = \widetilde{y_1} = 0$, and when $(\widetilde{x_1}, ..., \widetilde{x_{k-1}}, 1)$ and $(\widetilde{y_1}, ..., \widetilde{y_{k-1}}, 1)$ are mutually-centered). Let (i, j) be a bids' vector for A, B. Clearly, no welfare loss occurs when either $i = 0$ or when $j = 0$. When $i > 0$ and $j > 0$, we opt to maximize the expected welfare (where players' valuations are in range $[0, 1]$). Due to theorem 4.3, mutually-centered strategies do just that (the only difference is that the cumulative distribution function in $[0, 1]$ is smaller than 1, but this fact does not effect the proof of theorem 4.3). Due to [9], $RP(p) = \widetilde{v}^{-1}(0)$ (recall that in our model, the seller valuation for the item is zero). Thus, the expected welfare loss from this mechanism is $(1 - F(RP(p)))^2 \cdot O(\frac{1}{k^2}) = O(1) \cdot O(\frac{1}{k^2}) = O(\frac{1}{k^2})$. We know that the allocation in both mechanisms is the same, whether the players consider their original valuation in $MPG_k(x, y)$ or if they consider their virtual-utility as their valuation in $MPG_k(\widetilde{x}, \widetilde{y})$. Therefore, $MPG_k(x, y)$ incurs the same welfare loss as $MPG_k(\widetilde{x}, \widetilde{y})$, i.e. it incurs welfare loss of $O(\frac{1}{k^2})$. $MPG_k(x, y)$ has a dominant strategies equilibrium, and using proposition C.3 and theorem C.2 it incurs revenue loss of $O(\frac{1}{k^2})$. $\square$

Lemma C.6. *Denote* $t = \frac{-(1+\alpha)+\sqrt{1+3\alpha}}{1-\alpha}$ *where* $\alpha = \frac{1}{(2k-3)^2}$ *and let*

$$\widetilde{x} = (-1, 0, t + \frac{1(1-t)}{2k-3}, t + \frac{3(1-t)}{2k-3}, ..., t + \frac{(2k-5)(1-t)}{2k-3}, 1)$$

$$\widetilde{y} = (-1, 0, t, t + \frac{2(1-t)}{2k-3}, t + \frac{4(1-t)}{2k-3}, ..., t + \frac{(2k-4)(1-t)}{2k-3}, 1)$$

be threshold-strategies for A, B respectively. Consider a mechanism $MPG_k(\widetilde{x}, \widetilde{y})$, in a model of a single item auction among 3 players, $\{A, B, Z\}$ where the valuations for the players A, B are distributed uniformly on $[-1, 1]$, and player Z has a constant valuation of zero. Players A, B have k possible bids. Then, h achieves optimal welfare in this model (i.e. $w(h, x, y) = max_{h', x', y'} w(h', x', y')$), but with welfare loss greater than $\frac{1}{96 \cdot (2k-3)^2}$.

Proof. Due to theorem C.5, we know that the optimal welfare in this case can be achieved with a priority game $MPG_{k-1}(x, y)$ where x, y are mutually-centered. Assuming w.l.o.g. that $y_b \geq x_a$, these equations have a unique solution which is $x = (0, x_a, y_b + \frac{1-y_b}{2k-3}, y_b + \frac{3(1-y_b)}{2k-3}, ..., y_b + \frac{(2k-5)(1-y_b)}{2k-3}, 1)$, $y = (0, y_b, y_b + \frac{2(1-y_b)}{2k-3}, y_b + \frac{4(1-y_b)}{2k-3}, ..., y_b + \frac{(2k-4)(1-y_b)}{2k-3}, 1)$. We should find the values for x_a and y_b for which the mechanism achieves minimal welfare loss. For that, we present an expression for the welfare loss, and find its minimum value.

First, we calculate the welfare loss given that $v_A, v_B \geq y_B$. As in the proof for theorem B.11, the mechanism's allocation can be different from the allocation of 2nd-price auction only in $2 \cdot (k-1) - 1$ bids combinations (on the diagonal, and on the secondary diagonal). For any such bids' combination (l, m) the segments $[x_l, x_{l+1}]$ and $[y_m, y_{m+1}]$ have an overlapping segment of size $\frac{(1-y_b)}{2(k-1)-1}$. In priority games, the allocation can be different form the optimal allocation only only when both valuations are within such segment, and this happens with probability $\left(\frac{(1-y_b)}{2(k-1)-1}\frac{1}{2}\right)^2$. Given that the both valuations are in such range, the expected welfare in optimal auction is $\frac{2}{3}$ of this range, and the expected welfare in our mechanism is in the middle of this range, so the expected welfare loss is $\left(\frac{(1-y_b)}{2(k-1)-1}\frac{1}{6}\right)$. Thus, the total welfare loss added when $v_A, v_B \geq y_B$ is

$$\left(\frac{(1-y_b)}{2(k-1)-1}\frac{1}{2}\right)^2 \left(\frac{(1-y_b)}{2(k-1)-1}\frac{1}{6}\right)(2 \cdot (k-1) - 1)$$

Similarly, we can calculate the welfare loss in all possible cases and get (for example, the 2nd expression is the expected welfare loss when $0 \leq v_B \leq y_B$ and $-1 \leq v_A \leq 0$, and so on):

$$\left(\frac{(1-y_b)}{(2(k-1)-1)}\frac{1}{2}\right)^2 \left(\frac{(1-y_b)}{(2(k-1)-1)}\frac{1}{6}\right)(2 \cdot (k-1) - 1) +$$
$$\frac{y_B}{2}\frac{1}{2}\frac{y_B}{2} + \frac{x_a}{2}\frac{1}{2}\frac{x_a}{2} + \frac{(y_b - x_a)}{2}\frac{x_a}{2}\frac{(y_b + x_a)}{2} +$$
$$\frac{x_a}{2}\frac{x_a}{2}\frac{2x_a}{3} + \frac{(y_b - x_a)}{2}\frac{(y_b - x_a)}{2}\frac{(y_b - x_a)}{6}$$

For convenience, denote $\alpha = \frac{1}{(2k-3)^2}$. This welfare-loss function has a unique minimum value when $x_a = 0$ and $y_B = \frac{-(1+\alpha)+\sqrt{1+3\alpha}}{1-\alpha}$. Thus we have the exact formula for the minimal welfare loss. However, this formula is a bit complicated. First, we notice the existence of the following inequalities:

$$y_B = \frac{-(1+\alpha) + \sqrt{1+2\alpha+\alpha}}{1-\alpha} \geq \frac{-(1+\alpha) + \sqrt{1+2\alpha+\alpha^2}}{1-\alpha} = \frac{-(1+\alpha)+1+\alpha}{1-\alpha} = 0 \tag{C.1}$$

$$y_B = \frac{-(1+\alpha) + \sqrt{1+3\alpha}}{1-\alpha} \leq \frac{-(1+\alpha)+1+3\alpha}{1-\alpha} = \frac{2\alpha}{1-\alpha} \tag{C.2}$$

Rearranging the welfare loss expression results (when $x_a = 0$):

$$3\frac{y_B^2}{24}(1+\alpha) + \frac{y_B^3}{24}(1-\alpha) + \frac{\alpha}{24} - \frac{3\alpha y_B}{24}$$

Which is greater than $\frac{\alpha}{24} - \frac{3\alpha y_B}{24}$ (using equation C.1). Using equation C.2, and assuming $k \geq 3$:

$$\frac{\alpha}{24} - \frac{3\alpha y_B}{24} \geq \frac{\alpha}{24} - \frac{3\alpha\frac{2\alpha}{1-\alpha}}{24} = \frac{\alpha}{24}\left(\frac{1-7\alpha}{1-\alpha}\right) \geq \frac{\alpha}{24}\frac{1}{4} = \frac{1}{96 \cdot (2k-3)^2}$$

Thus, the welfare loss is greater than $\frac{1}{96 \cdot (2k-3)^2}$ for $k > 2$. □

Theorem C.7. *When players' valuations are distributed uniformly, the modified priority-game $MPG_k(x,y)$ achieves optimal expected revenue among all the individually-rational mechanisms, where*

$$x = (0, \frac{1}{2}, t + \frac{1\cdot(1-t)}{2k-3}, t + \frac{3\cdot(1-t)}{2k-3}, ..., t + \frac{(2k-5)\cdot(1-t)}{2k-3}, 1)$$

$$y = (0, t, t + \frac{2\cdot(1-t)}{2k-3}, t + \frac{4\cdot(1-t)}{2k-3}, ..., t + \frac{(2k-4)\cdot(1-t)}{2k-3}, 1)$$

and $t = \frac{-2\alpha+\sqrt{1+3\alpha}}{2(1-\alpha)}$ *for* $\alpha = \frac{1}{(2k-3)^2}$. *This mechanism incurs revenue loss of* $\Omega(\frac{1}{k^2})$.

Proof. Let $\widetilde{w}$ be the optimal expected welfare achieved by the optimal mechanism described in lemma C.6. Assume there is a mechanism $g' \in G_{2,k}$ that achieves expected virtual utility greater than $\widetilde{w}$ in dominant strategy equilibrium. Let h' be a mechanism which is identical to g', except each payment p in g' is replaced with $2p-1$. It is easy to see that the same transformation on x, y results dominant strategy equilibrium in h'. Since the virtual utility $(2v-1)$ is strictly increasing with v, each player's bid in g' when her valuation is v will be identical to her bid in h' when she considers the virtual utility as her valuation. Thus, the allocation in these two cases will be the same and the expected welfare in h' is equal the expected virtual utility in g'. Therefore, h' is a mechanism in the model described in lemma C.6, which achieves strictly greater welfare than $\widetilde{w}$. Contradiction to lemma C.6. Thus, no mechanism in $G_{2,k}$ achieves expected virtual utility which is greater than $\widetilde{w}$.

Clearly, in the given mechanism g, if we replace each payment p to be $2p-1$, we get exactly the mechanism described in lemma C.6. For similar reasons, the expected virtual utility in g will be $\widetilde{w}$. Due to proposition C.4, g maximize expected revenue if and only if g maximize expected utility. Thus, g is revenue-optimal, and the revenue loss is equal to the virtual-utility loss, which is $\Omega(\frac{1}{k^2})$. □

C.3 Thresholds values for revenue-optimal mechanisms

Let $\widetilde{x} = (\widetilde{x_0}, ..., \widetilde{x_{k-2}}, 1)$ and $\widetilde{y} = (\widetilde{y_0}, ..., \widetilde{y_{k-2}}, 1)$ be threshold-values. Let $x^r = (x^r_0, ..., x^r_k)$ and $y^r = (y^r_0, ..., y^r_k)$ be threshold-values that satisfy $x^r_0 = y^r_0 = 0$, $x^r_k = y^r_k = 1$ and for every $1 \le i \le k-1$ $\widetilde{x_{i-1}} = \widetilde{v}(x^r_i)$.

Proposition C.8. *When the valuations of both players are distributed with the same regular function, $MPG_k(x^r, y^r)$ achieves optimal revenue if $\widetilde{x}$ and $\widetilde{y}$ are mutually-centered and the following constraints hold:*

$$-\frac{N-1+y_1}{N} f(\widetilde{x_1})\widetilde{x_1} + \int_{y_1}^{1} v_B f(v_B) dv_B \left(\frac{1}{N} - f(\widetilde{x_1})\right) = 0 \quad \text{(C.3)}$$

$$-\frac{N-1+x_1}{N} f(\widetilde{y_1})\widetilde{y_1} + \int_{x_1}^{1} v_A f(v_A) dv_A \left(\frac{1}{N} - f(\widetilde{y_1})\right) - \left(F(\widetilde{x_2}) - F(\widetilde{x_1})\right) f(\widetilde{y_1})\widetilde{y_1} = 0 \quad \text{(C.4)}$$

Proof. Consider the model where the players consider their virtual-utilities as their valuations. We show in the proof for theorem 4.6 that the optimal welfare in this model can be achieves in a modified prioirty-game. The expected welfare in a $MPG_k(x,y)$ where $x = (-\frac{1}{f(0)}, x_1, ..., x_{k-1}, 1)$, $y = (-\frac{1}{f(0)}, y_1, ..., y_{k-1}, 1)$ is (denote $N = \frac{1}{f(0)} + 1$ to be the normalization factor):

$$\begin{aligned} E(w) \quad = \quad & \frac{N-1+y_1}{N} \int_{x_1}^{1} v_A f(v_A) dv_A + \frac{N-1+x_1}{N} \int_{y_1}^{1} v_B f(v_B) dv_B \\ & + \sum_{i=2}^{k} \left(F(x_i) - F(x_1)\right) \int_{y_{i-1}}^{y_i} v_B f(v_B) dv_B + \sum_{i=3}^{k} \left(F(y_{i-1}) - F(y_1)\right) \int_{x_{i-1}}^{x_i} v_A f(v_A) dv_A \end{aligned}$$

Partial derivatives (first order conditions) on $x_1, ..., x_{k-1}, y_1, ..., y_{k-1}$ show that $(x_1, ..., x_{k-1}, 1)$ and $(y_1, ..., y_{k-1}, 1)$ are mutually centered, and x_1 and y_1 holds the following constraints:

$$-\frac{N-1+y_1}{N}f(x_1)x_1 + \int_{y_1}^{1} v_B f(v_B) dv_B \left(\frac{1}{N} - f(x_1)\right) = 0$$

$$-\frac{N-1+x_1}{N}f(y_1)y_1 + \int_{x_1}^{1} v_A f(v_A) dv_A \left(\frac{1}{N} - f(y_1)\right) - (F(x_2) - F(x_1)) f(y_1)y_1 = 0$$

As proved in theorem 4.6, the mechanism in which for every $1 \le i \le k-1$ $\widetilde{x_{i-1}} = \widetilde{v}(x_i^r)$ has the same allocation (in the original model), and thus achieves optimal expected virtual utility. By proposition C.4 this mechanism achieves optimal revenue. □

D n-players mechanisms

We consider games among n players, where each player has 2 possible bids (i.e. she can send 1 bit to the mechanism). In this section, we assume all players are symmetric, i.e. all players have the same distribution of their valuations.

D.1 Optimality of priority games

Lemma D.1. *Let $g \in G_{n,2}$ be a priority game for n symmetric players. Then, there exists a vector of strategies s^* for which g achieves maximal welfare, i.e.*

$$w(g, s^*) = w_{n,2}^{opt}$$

Proof. From lemma B.1 we can assume s^* consists of threshold-strategies. Let $x_1, ..., x_n$ be the thresholds for the players (w.l.o.g $x_1 \le ... \le x_n$). The following claim derives immediately from the symmetric-players assumption (i.e. all the valuations' distributions are the same).

Claim. If $x_i \ge x_j$ then

$$E\left(v_i \,|x_i \le v_i \le 1\right) \ge E\left(v_j \,|x_j \le v_j \le 1\right)$$

and

$$E\left(v_i \,|0 \le v_i \le x_i\right) \ge E\left(v_j \,|0 \le v_j \le x_j\right)$$

Mechanisms with optimal allocation must allocate the item for the player with the maximal expected valuations. Using the claim above, it is easy to see that priority-games' allocation (where player n has the highest priority, player 1 has the lowest priority, etc.) holds this rule. □

Lemma D.2. *There is a mechanism $g \in G_{n,2}$ that achieves maximal revenue with some strategies' vector s^* (i.e. $r(g, s^*) = r_{n,2}^{opt}$), such that g is a modified priority-game where each player i pays a fixed payment x_i whenever he wins. Moreover, this game has dominant-strategies equilibrium.*

Proof. We will find an ex-post IR mechanism with optimal evenue, then we will show that it has a dominant-strategies equilibrium. From lemma B.1 we can assume s^* consists of threshold-strategies. Let $x_1, ..., x_n$ be the thresholds for the players, and w.l.o.g $x_1 \le ... \le x_n$. If all players have ex-post IR a player cannot pay more than his valuation, i.e. $x_i \ge p_i(1, s_{-i})$ for every b_{-i}. If there is b_{-i} for which $x_i > p_i(1, b_{-i})$ then the mechanism in which $x_i = p_i(1, b_{-i})$ achieves strictly greater revenue than g. If we modify the mechanism such that every winning player pays his threshold value, we thus get a mechanism with higher expected revenue than g. Thus, we can assume player i will always pay its threshold value upon winning.

The ex-post IR assumption also derives that every player must have a bid that ensures him a zero payment in case her valuation is zero. W.l.o.g this bid will be "0" for each player. Given a revenue-optimal game g, we can modify it to give the item to the player with the highest payment over all players who bids 1, and the revenue will not decrease. Therefore, we allocate the item to the player whose threshold value is the highest among all players that bid 1 (take the player with the higher index in case of equal thresholds). This is exactly the allocation of a modified priority-game where player n is with highest priority, player 1 with the lowest priority etc.

In this game, the threshold strategies with the threshold values $x_1, ..., x_n$ are clearly dominant strategies. $\square$

D.2 Welfare optimizing mechanisms

D.2.1 Optimal welfare in n-players 2-possible-bids games

Theorem D.3. *Let $(x_1, ..., x_n) \in [0,1]^n$ be threshold-strategies for the n players, such that the following constraints hold:*

$$\forall_{m \le n-2}\, x_{m+1} = (1 - F(x_m)) \cdot E\left(v_m \,|v_m \in [x_m, 1]\right) + F(x_m) \cdot x_m \tag{D.1}$$

$$x_1 = E\left(v_n \,|0 \le v_n \le n\right) \tag{D.2}$$

$$x_n = \frac{\sum_{i=1}^{n-1} (\prod_{j=i+1}^{n-1} F(x_j))\,(1 - F(x_i))\, E\left(v_i \,|v_i \in [x_i, 1]\right)}{1 - \prod_{i=1}^{n-1} F(x_i)} \tag{D.3}$$

Then, a priority game g for which $x_1, ..., x_n$ are dominant threshold-strategies, achieves optimal welfare.

Proof. Due to lemma D.1 priority games can achieve optimal welfare. This optimal welfare can be achieved with threshold strategies due to lemma B.1. Our goal in this subsection is to find the threshold strategies that achieve maximal welfare.

Consider a priority game among n players, indexed by their priorities (i.e. $1 \prec 2 ... \prec n$). In priority games, every player wins the item if he bids 1 and the players with higher priority bid 0. When all players bid 0, player n wins. Thus, the probability that player i wins is $\left(\prod_{j=i+1}^{n} F(x_j)\right) \cdot (1 - F(x_i))$ (where $F(x)$ is the cumulative distribution function of all players). The expected welfare from this game, where the players use threshold strategies $x_1, ..., x_n$ is:

$$\begin{aligned} w(g,s) &= \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} F(x_j)\right)(1 - F(x_i))\,\frac{\int_{x_i}^{1} f(v_i) v_i dv_i}{(1 - F(x_i))} \\ &\quad + \left(\prod_{i=1}^{n} F(x_i)\right) \frac{\int_0^{x_n} f(v_n) v_n dv_n}{F(x_n)} \end{aligned}$$

In an internal maximum solution, the derivatives by $x_1, ..., x_n$ should be zero. By rearranging these equations we get a formalization of the optimal solution.

For players $1 \le m \le n-1$:

$$x_m = \sum_{i=1}^{m-1}\left(\prod_{j=i+1}^{m-1} F(x_j)\right)(1-F(x_i))\,E\left(v_i\,|x_i \le v_i \le 1\right) + \left(\prod_{i=1,i\neq m}^{n-1} F(x_i)\right)E\left(v_n\,|0 \le v_n \le x_n\right)$$

For player n:

$$x_n = \frac{\sum_{i=1}^{n-1}\left(\prod_{j=i+1}^{n-1} F(x_j)\right)(1-F(x_i))\,E\left(v_i\,|x_i \le v_i \le 1\right)}{1-\prod_{i=1}^{n-1} F(x_i)}$$

And now we can easily reach a recursive formula by calculating $x_{m+1} - x_m$:

$$x_{m+1} - x_m = (1-F(x_m))\cdot\left(E\left(v_m\,|x_m \le v_m \le 1\right) - x_m\right)$$

Thus, x_{m+1} is a convex combination (with $F(x_m)$) of x_m and the expected valuation of player m when she bid 1:

$$x_{m+1} = (1-F(x_m))\cdot E\left(v_m\,|x_m \le v_m \le 1\right) + F(x_m)\cdot x_m \tag{D.4}$$

When $m = 1$, we have $x_1 = E\left(v_n\,|x_n \le v_n \le 1\right)$.

□

Corollary D.4. *Let $(x_1, ..., x_n) \in [0,1]^n$ be threshold-strategies for the n players, such that the following constraints hold:*

$$\forall_{1\le m\le n-2} \quad x_{m+1} = \frac{1}{2} + \frac{x_m^2}{2} \tag{D.5}$$

$$x_1 = \frac{x_n}{2} \tag{D.6}$$

$$x_n = \frac{\sum_{i=1}^{n-1}\left(\prod_{j=i+1}^{n-1} x_j\right)\left(1-x_i^2\right)}{2\left(1-\prod_{i=1}^{n-1} x_i\right)} \tag{D.7}$$

Then, when players' valuations are distributed uniformly, a priority game g for which $x_1, ..., x_n$ are dominant strategies achieves optimal welfare.

Proof. Trivial result of theorem D.3 □

D.3 Revenue optimizing mechanisms

Classic results in auction theory (see [19]) show that 2nd-price auction with a reservation price achieves optimal revenue for the seller, independent of the number of the (symmetric) buyers. They provide a simple formula for the optimal reservation price, and in case of uniform distribution on the valuations the reservation price is $\frac{1}{2}$. Thus, in the case of uniform distribution, our benchmark auction is 2nd-price auction with reservation price of $\frac{1}{2}$.

Theorem D.5. *Let $(x_1, ..., x_n) \in [0,1]^n$ be threshold-values, such that the following constraints hold:*

$$x_1 = \frac{1}{2} \tag{D.8}$$

$$\forall 1 \leq m \leq n-1 \;\; x_{m+1} = \frac{1}{2} + \frac{x_m^2}{2} \tag{D.9}$$

Then, when the players valuations are distributed uniformly, a priority game $g \in G_{n,2}$ for which the threshold-strategies with $x_1, ..., x_n$ are dominant strategies, achieves optimal revenue.

Proof. Due to lemma D.2 optimal revenue can be achieved in priority games (where no allocation is made when all players bid 0), and the optimal revenue can be achieved with dominant threshold strategies. Let s be the vector of threshold-strategies with $x_1, ..., x_n$ as thresholds.

Again, consider a priority game among n players, indexed by their priorities (i.e. $1 \prec 2 ... \prec n$) with thresholds $x_1, ..., x_n$. The revenue from this game is given by:

$$r(g,s) = \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} F(x_j) \right) \cdot (1 - F(x_i)) \cdot x_i$$

The derivation of this formula is tedious and is out of the scope of this paper. We show here the analysis for the case of uniform distribution of the players' valuations, which is quite elegant.

When the valuations are distributed uniformly, the revenue is given by:

$$r(g,s) = \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} x_j \right) \cdot (1 - x_i) \cdot x_i = \sum_{i=1}^{n} \left(\prod_{j=i}^{n} x_j \right) \cdot (1 - x_i)$$

For internal maximum, the derivations should be zero:

$$\left(\prod_{j=m+1}^{n} x_m \right) (1 - 2x_m) + \sum_{i=1}^{m-1} \left(\prod_{j=i, j \neq i}^{n} x_j \right) (1 - x_i) = 0$$

Or:

$$1 - 2x_m = \sum_{i=1}^{m-1} \left(\prod_{j=i}^{m-1} x_j \right) (1 - x_i)$$

And for $m+1$:

$$1 - 2x_{m+1} = \sum_{i=1}^{m} \left(\prod_{j=i}^{m} x_j \right) (1 - x_i) = x_m(1 - x_m) + x_m(2x_m - 1)$$

So we reach a recursive formula similar to D.5:

$$\forall_{1 \leq m \leq k-2} \qquad x_{m+1} = \frac{1}{2} + \frac{x_m^2}{2}$$

But in this case, $x_1 = \frac{1}{2}$. □

D.4 Analysis of the uniform distribution case

First, we see an easy upper bound for the optimal revenue-loss and welfare-loss when players' valuations are distributed uniformly.

Proposition D.6. *The modified priority-game, in which the threshold-strategy* $1 - \frac{ln(n)}{n}$ *is dominant for all players, incurs both revenue-loss and welfare-loss of* $O(\frac{ln(n)}{n})$.

Proof. Assume each bidder pays $1 - \frac{ln(n)}{n}$ upon winning. Then, the expected revenue is

$$\begin{aligned} E(R) &= (1 - \left(1 - \frac{ln(n)}{n}\right)^n) \cdot (1 - \frac{ln(n)}{n}) \\ &\geq (1 - e^{-\frac{ln(n)}{n} n}) \cdot (1 - \frac{ln(n)}{n}) = (1 - \frac{1}{n}) \cdot (1 - \frac{ln(n)}{n}) \\ &= 1 - O(\frac{ln(n)}{n}) \geq r_n^{opt} - O(\frac{ln(n)}{n}) \end{aligned}$$

·where r_n^{opt} is the optimal revenue in unbounded-commumication mechanisms between n players. Similarly, we can show that the welfare-loss incurred by the given mechanism is $O(\frac{ln(n)}{n})$. ☐

Next, we show that the loss from limiting the communications in n-players games to 1-bit is $O(\frac{1}{n})$ for both welfare and revenue, when the valuations are distributed uniformly. According to computer simulations we made, these upper bounds are tight. However, we haven't proved it, and it is an open question.

D.4.1 Upper bound for revenue loss

Due to section D.3, the threshold strategies given by the following recursion

$$x_{m+1} = \frac{1}{2} + \frac{x_m^2}{2}, \quad x_1 = \frac{1}{2} \tag{D.10}$$

achieve optimal revenue in priority games, when these strategies are dominant. We will show that the revenue loss achieved by this mechanism and strategies is $O(\frac{1}{n})$.

Theorem D.7. *Consider* n*-players mechanisms with* 2 *possible bids, where the players' valuations are distributed uniformly. The efficient mechanism* $g \in G_{n,2}$ *described in theorem D.5 achieves revenue loss of* $O(\frac{1}{n})$

Proof. We know ([19, 9]) that Vickrey's 2nd-price auction achieves optimal revenue. In the priority game described in lemma D.2, if player i wins the item, he pays x_i. In 2nd-price auctions, the valuations of both players can be 1, so the winner pays 1. Thus, the maximal revenue loss when player i wins is $1 - x_i$. Therefore, we can bound the expected revenue loss by

$$rl(g, s) \leq \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} x_j \right) \cdot (1 - x_i) \cdot (1 - x_i)$$

We will first prove 2 inductive claims, about the properties of the recursion elements:

Claim D.8. $\forall_k \quad 1 - x_k \leq \frac{2}{k}$

Proof. By induction on k. For $k=1$, we know $x_k = \frac{1}{2}$, and $1 - x_k = 1 - \frac{1}{2} < 2 = \frac{2}{k}$. We assume $1 - x_k \leq \frac{2}{k}$, and we prove that $1 - x_{k+1} \leq \frac{2}{k+1}$ (we use the fact that $\frac{k^2}{(k-1)(k+1)} \geq 1$ for every $k > 1$):

$$\begin{aligned} 1 - x_{k+1} &= 1 - (\frac{1}{2} + \frac{x_k^2}{2}) \leq \frac{1}{2} - \frac{\left(1-\frac{2}{k}\right)^2}{2} = \frac{2(k-1)}{k^2} \\ &\leq \frac{2(k-1)}{k^2} \cdot \frac{k^2}{(k-1)(k+1)} = \frac{2}{k+1} \end{aligned}$$

□

Claim D.9. $\forall_{k\geq 15} \quad x_k \leq \frac{2k-3}{2k}$

Proof. Again, by induction on k. For $k = 15$, $x_{15} = 0.899 \leq 0.9 = \frac{2\cdot 15-3}{2\cdot 15}$. We assume $x_k \leq \frac{2k-3}{2k}$, and we prove that $x_{k+1} \leq \frac{2(k+1)-3}{2(k+1)}$.

$$x_{k+1} = \frac{1}{2} + \frac{x_k^2}{2} \leq \frac{1}{2} + \frac{\left(\frac{2k-3}{2k}\right)^2}{2} = \frac{8k^2 - 12k + 9}{8k^2}$$

It suffices to prove that $\frac{2(k+1)-3}{2(k+1)} - \frac{8k^2-12k+9}{8k^2} \geq 0$, and indeed when $k \geq 15$:

$$\frac{2(k+1)-3}{2(k+1)} - \frac{8k^2 - 12k + 9}{8k^2} = \frac{6k - 18}{16k^2(k+1)} \geq 0$$

□

Denote $\overline{rl_n} = \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} x_j\right) \cdot (1 - x_i)^2$. We will show that for every $n \geq 15$ we have $\overline{rl_n} \leq \frac{8}{n}$. The proof is by induction on n. We assume $\overline{rl_n} \leq \frac{8}{n}$, and try to prove that $\overline{rl_{n+1}} \leq \frac{8}{n+1}$.

We first notice that $\overline{rl_{n+1}} = (1 - x_{n+1})^2 + x_{n+1}\overline{rl_n}$. Next, using the two claims above and the induction hypothesis:

$$\overline{rl_{n+1}} = (1 - x_{n+1})^2 + x_{n+1}\overline{rl_n} \leq \left(\frac{2}{n+1}\right)^2 + \frac{2(n+1)-3}{2(n+1)} \cdot \frac{8}{n} = \frac{4}{(n+1)^2} + \frac{2n-1}{2n+2} \cdot \frac{8}{n}$$

Now, it suffices to show that $\frac{8}{n+1} - \left(\frac{4}{(n+1)^2} + \frac{2n-1}{2n+2} \cdot \frac{8}{n}\right) \geq 0$:

$$\frac{8}{n+1} - \frac{4}{(n+1)^2} - \frac{2n-1}{2n+2} \cdot \frac{8}{n} = \frac{8n+8-4}{(n+1)^2} - \frac{16n-8}{2n(n+1)} = \frac{n(8-8)+8}{2n(n+1)^2} \geq 0$$

□

D.4.2 Upper bound for welfare loss

The thresholds in equation D.5, which optimize the welfare in a priority game, are similar to the thresholds from equation D.10 that optimize revenue. We will show that using the latter thresholds we get an upper bound of $O(\frac{1}{n})$ for the welfare loss in the priority game. Simulations shows that the optimal welfare loss is proportional to $\frac{1}{n}$, but we have managed only to prove the upper-bound.

Theorem D.10. *Consider n-players mechanisms with 2 possible bids, where the players' valuations are distributed uniformly. The efficient mechanism $g \in G_{n,2}$ described in corollary D.4 achieves welfare loss of $O(\frac{1}{n})$*

Proof. We assume w.l.o.g. that the players have priorities in g according to their index (i.e. $1 \prec 2 ... \prec n$). We also assume that the players use threshold-strategies with the thresholds $x_1, ..., x_n$ described in equation D.10. As shown in section D.2.1 for the general case, the following expression stands for the expected welfare in n-players priority-games, with uniform distribution:

$$w(g,s) \;=\; \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n} x_j\right)(1-x_i)\,\frac{1+x_i}{2} + \left(\prod_{i=1}^{n} x_i\right)\frac{x_n}{2}$$

We compare this expression with the welfare achieved in 2nd-price auction which is known to be optimal. When a player wins after bidding 1, the minimal valuation he might have is x_i but there may be a player with valuation 1 that did not win. 2nd-price auction allocates the item to the player with the highest valuation, thus the maximal welfare loss in this case is $1 - x_i$. When all players bid 0, we will use the trivial upper bound of 1 for the welfare loss . Therefore, we can bound the welfare loss by:

$$wl(g,s) \;\leq\; \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n} x_j\right)(1-x_i)(1-x_i) + \prod_{i=1}^{n} x_i$$

The first expression in the sum is proved to be $\leq \frac{8}{n}$ (for every $n > 14$) in the previous section (D.4.1) . Here we prove that the second expression is also $O(\frac{1}{n})$.

Claim D.11. $\forall_n \quad \prod_{i=1}^{n} x_i \leq \frac{1}{n}$

Proof. By induction on n. For $n = 1$ the claims hold. We assume $\prod_{i=1}^{n} x_i \leq \frac{1}{n}$ and prove that $\prod_{i=1}^{n+1} x_i \leq \frac{1}{n+1}$. We use claim D.9 :

$$\prod_{i=1}^{n+1} x_i = x_{n+1}\prod_{i=1}^{n} x_i \leq x_{n+1}\frac{1}{n} \leq \frac{2n-1}{2n+2}\frac{1}{n} < \frac{2n-1}{2n+2}\frac{1}{n} + \frac{1}{2n(n+1)} = \frac{1}{n+1}$$

□

Thus, for every $n > 14$, $wl(g,s) \leq \frac{8}{n} + \frac{1}{n} = \frac{9}{n}$ □