Minors in Lifts of Graphs

Yotam Drier *, Nathan Linial †

June 3, 2005

Abstract

We study here lifts and random lifts of graphs, as defined in [1]. We consider the Hadwiger number η and the Hajós number σ of ℓ-lifts of K^n, and analyze their extremal as well as their typical values (that is, for random lifts). When $\ell = 2$, we show that $\frac{n}{2} \leq \eta \leq n$, and random lifts achieve the lower bound (as $n \to \infty$). For bigger values of ℓ, we show $\Omega\left(\frac{n}{\sqrt{\log n}}\right) \leq \eta \leq n\sqrt{\ell}$. We do not know how tight these bounds are, and in fact, the most interesting question that remains open, is whether it is possible for η to be $o(n)$. When $\ell \leq O(\log n)$, almost every ℓ-lift of K^n satisfies $\eta = \Theta(n)$ and for $\Omega(\log n) \leq \ell \leq n^{\frac{1}{3}-\varepsilon}$, almost surely $\eta = \Theta\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}}\right)$. For bigger values of ℓ, $\Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log \ell}}\right) \leq \eta \leq n\sqrt{\ell}$ almost always. The Hajós number satisfies $\Omega(\sqrt{n}) \leq \sigma \leq n$, and random lifts achieve the lower bound for bounded ℓ, and approach the upper bound when ℓ grows.

1 Introduction

In this paper we study the Hadwiger and Hajós numbers of lifts of graphs. We provide both upper and lower bounds on these parameters for lifts, and analyze the typical behavior of random lifts. We restrict ourselves to lifts of

*School of Computer Science and Engineering, Hebrew University, Jerusalem 91904, Israel. Email: yotamd@cs.huji.ac.il

†School of Computer Science and Engineering, Hebrew University, Jerusalem 91904, Israel. Email: nati@cs.huji.ac.il. Research supported in part by a grant from the Israel Science Foundation

the complete graph K^n. This, however, easily yields some bounds on lifts of general graphs, as shown in Section 5.

	$\ell \le \sqrt{\log n}$	$\sqrt{\log n} \le \ell \le \log n$	$\log n \le \ell \le \sqrt[3]{n}$	$\ell \ge \sqrt[3]{n}$
$m(n,\ell)$	$\frac{n}{2\ell-1} \le m < n$	$\Omega\left(\frac{n}{\sqrt{\log n}}\right)$		
$T(n,\ell)$	$\Theta(n)$		$\Theta\left(\sqrt{\frac{\ell}{\log n}} \cdot n\right)$	$\Omega\left(\sqrt{\frac{\ell}{\log \ell}} \cdot n\right)$
$M(n,\ell)$	$\Theta(\sqrt{\ell} \cdot n)$			
ℓ		$\sqrt{\log n}$	$\log n$	$\sqrt[3]{n}$

Table 1: Bounds on Hadwiger numbers as function of ℓ

Table 1 above summarizes the main results proven here. Here $m(n,\ell)$, $M(n,\ell)$ and $T(n,\ell)$ stand for the smallest, largest and typical Hadwiger number of an ℓ-lift of K^n. It can be seen that the bounds are tight for most values of ℓ. The case of $\ell = 2$ is fully characterized : $m(n,2) = \frac{n}{2}$, $M(n,2) = n$, and T achieves the lower bound $(\frac{n}{2})$. All bounds are tight. For general ℓ we have the following:

- $m(n,\ell) \ge \frac{n}{2\ell-1}$ and $m(n,\ell) \ge \Omega\left(\frac{n}{\sqrt{\log n}}\right)$.
- $\frac{n\sqrt{\ell}}{4} \le M(n,\ell) \le n\sqrt{\ell}$
- For random lifts:
  - $T(n,\ell) = \Theta(n)$ for $\ell \le O(\log n)$
  - $T(n,\ell) = \Theta\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}}\right)$ for $\Omega(\log n) \le \ell \le n^{\frac{1}{3}-\varepsilon}$
  - For larger values of ℓ, $\Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log \ell}}\right) \le T(n,\ell) \le n\sqrt{\ell}$

We start with a simple and thought provoking example - the icosahedron. It is a 2-lift of K^6 without K^5 minor. Indeed the most interesting question that remains open, is how small the Hadwiger number can be in any lift of K^n. We return to this in the sequel. Generally speaking, we found it easier to prove lower bounds on both Hadwiger and Hajós numbers (both are questions in NP), and more difficult to establish non-trivial upper bounds on these parameters.

1.1 Lifts

Definition 1.1. An *ℓ-lift* of the labelled undirected graph (V, E) is a graph with vertices $V \times [\ell]$. The edge set is the union of a perfect matching between $\{u\} \times [\ell]$ and $\{v\} \times [\ell]$ for each edge $uv \in E$. In a *random ℓ-lift* of G, these matchings are selected uniformly at random. The set $\{v\} \times [\ell]$ is called the *fiber* over v. Let $L_\ell(G)$ be the set of all ℓ-lifts of G.

Definition 1.2. A *section* of a lift $L \in L_\ell(G)$ is a set of vertices, exactly one from every fiber. A *levelling* is a given partition of L into sections, each section called a *level*. A *flat edge* is an edge between two vertices at the same level. A *star levelling* is a levelling where each level contains a spanning star. If G contains a vertex that is adjacent to all the other vertices, then such levelling exists, by making the edges of the appropriate star flat.

A comprehensive introduction to random lifts can be found in [1, 2, 3].

1.2 Minors and topological minors

Definition 1.3. We recall that a *minor* H of G (denoted $H \preceq G$) is a graph that is obtained from G by a series of edge contractions and deletions, and possibly omitting vertices. Thus, each vertex $v \in V(H)$ corresponds to a connected subset of $V(G)$, called the *branch set* of v. The branch sets are dependent on the way G was contracted. $\eta(G) = max\,\{n \in \mathbb{N} \mid K^n \preceq G\}$ is called the *Hadwiger number* of G.

Definition 1.4. A graph that is obtained by replacing the edges of H with openly disjoint paths is called a *subdivision* of H. If X is isomorphic[1] to a subgraph of G, and X is a subdivision of a graph H, we say that H is a *topological minor* of G. The vertices of G which correspond to the original vertices of H are called *branch vertices*.
$\sigma(G) = max\,\{n \in \mathbb{N} \mid K^n \text{ is a topological minor of } G\}$ is called the *Hajós number* of G.

Definition 1.5.

$$M(n, \ell) = max\,\{\eta(G) \mid G \in L_\ell(K^n)\}$$

$$m(n, \ell) = min\,\{\eta(G) \mid G \in L_\ell(K^n)\}$$

[1] We occasionally identify between graphs that are isomorphic.

$$\tau(n,\ell,\delta) = max\left\{h \mid Prob_{G\in L_\ell(K^n)}[\eta(G) \geq h] > 1-\delta\right\}$$

We also say that $T(n,\ell) = h$ if $\tau(n,\ell,\delta) = h$ for every $\delta > 0$ and n large enough. That is, m is the smallest Hadwiger number of ℓ-lifts of K^n, M is the largest, and T indicates the typical Hadwiger number.

We recall several relevant results from extremal graph theory. The Hadwiger number of a graph with average degree d is $\Omega\left(\frac{d}{\sqrt{\log d}}\right)$. The bound is achieved by random graphs, as Bollobás, Catlin and Erdös first proved in [4].

Theorem 1.6 (Kostochka 1982 [11]; Thomason 1984 [14]). *There exists a real $c > 0$ such that for every $h \in \mathbb{N}$, every graph of average degree $d \geq ch\sqrt{\log h}$ has a K^h minor. This bound is tight up to the value of c.*

Theorem 1.7 (Thomason 1999 [15]). *Let $c(t)$ be the minimum number such that for every graph G with average degree $d \geq c(t)$, $K^t \preceq G$. Then $c(t) = (\alpha + o(1))\, t\sqrt{\log t}$, where $\alpha = 0.6382...$*

2 Two-Lifts

In this section we consider the special case of 2-lifts. First we exhibit examples for 2-lifts of K^n without a K^n minor. Next, we show tight bounds on the Hadwiger number: $\frac{n}{2} \leq \eta(G) \leq n$. And finally we show that a random lift achieves the lower bound (for $n \to \infty$).

2.1 Examples

Let us first see a 2-lift of K^6 without even a K^5 minor. This is the graph of the icosahedron, which is a 2-lift of K^6, as shown by the labelling in Figure 1. Since the graph of the icosahedron is planar, it has no K^5 minor.

Other such clean and simple examples are not known. However, a computer program was written in order to seek for more examples. It found (among others) the 2-lift of K^8 shown in Figure 2, which has Hadwiger number 7. For simplicity only the first level with its flat edges is shown. This is clearly enough to define the graph (for more details see Section 2.3). We omit the (somewhat tedious) verification that indeed $\eta = 7$.

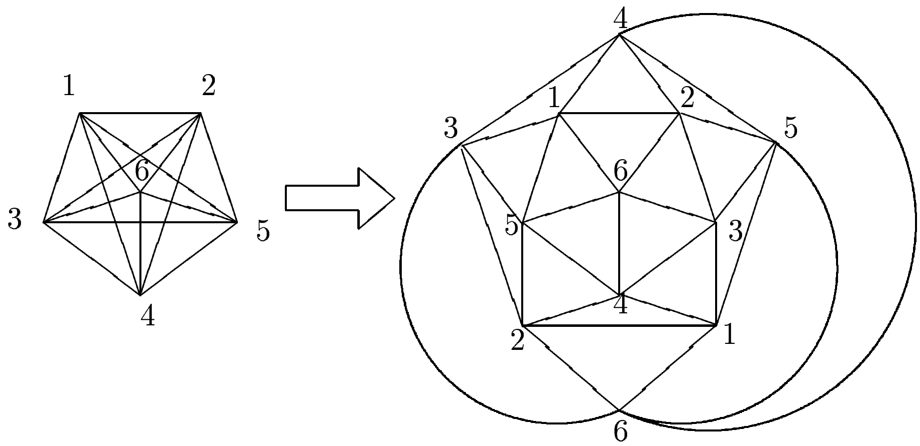


Figure 1: The icosahedron as a 2-lift of K^6 with $\eta = 4$

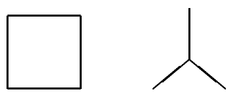


Figure 2: A 2-lift of K^8 with $\eta = 7$

2.2 The largest possible Hadwiger number

The answer and the proof are simple.

Theorem 2.1.

$$M(n, 2) = n$$

Proof. The trivial lift (two disjoint copies of K^n) achieves $\eta(L) = n$, so we only need to show the upper bound on M. Suppose that $K^m \preceq L$. If there is a branch set of size 1, then it is adjacent to $n-1$ branch sets, since L is $(n-1)$-regular. Hence, $m \leq n$ as required. But if all branch sets are of size 2 or more, there are at most n branch sets, since L has only $2n$ vertices. Again, $m \leq n$.

□

2.3 The smallest possible Hadwiger number

Here the answer is simple, but the proof already requires some work.

Theorem 2.2.

$$m(n,2) \geq \frac{n}{2}$$

The proof of Theorem 2.2 requires a detailed analysis of 2-lifts of K^4. To do so, we need some basic facts from the theory of switching classes (also known as two-graphs). For a survey of switching classes, and their many connections to other parts of mathematics, see Seidel [12], Seidel and Taylor [13], and Cameron [7].

Definition 2.3. For a graph $G = (V, E)$ and $S \subset V$, the *switch* of G by S is the graph $G^S = (V, \hat{E})$, where $xy \in \hat{E}$ if and only if :

$$(xy \in E \wedge |\{x,y\} \cap S| \equiv 0 \pmod 2) \vee (xy \notin E \wedge |\{x,y\} \cap S| = 1)$$

The *switching class* of G is $[G] = \{G^S \mid S \subset V\}$.

Every $L \in L_2(K^n)$ can be uniquely encoded by G, the n-vertex graph consisting of the flat edges (in one of the levels). It is easy to observe that G and G^S encode the same lift of K^n for every n-vertex graph G and every set $S \subset V(G)$. Therefore, we freely identify between L and $[G]$.

Remark 2.4. Notice that $(G^S)^T = G^{S \oplus T}$ and that $G^{V(G) \setminus S} = G^S$. Therefore every switching class of (V, E) is of size $2^{|V|-1}$

We turn to explore all the switching classes of graphs with 4 vertices. This will come in handy in the analysis of 2-lifts in general.

Lemma 2.5. *If $|V| \geq 4$ then no switching class contains two distinct graphs with a single edge.*

Proof. For every G and nonempty $S \subsetneq V$, every edge in the cut $(S, V \setminus S)$ appears in either G or G^S. But $|V| \geq 4$ and so $|S|\,|V \setminus S| \geq 3$, and therefore at least one of G and G^S must have more than one edge.

□

By Remark 2.4 each switching class of any 4-vertex graph contains exactly 8 graphs. There are 64 labelled graphs on 4 vertices, and therefore 8 distinct switching classes. By Lemma 2.5 six of the switching classes are the classes defined by a single edge. The other two classes are shown in Figure 3.

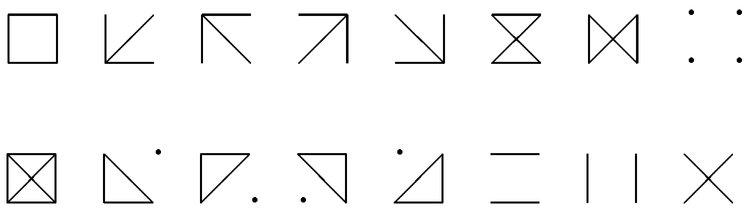
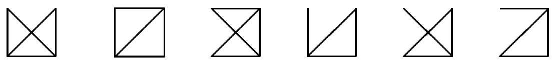


Figure 3: Two of the switching classes on 4 vertices

Remark 2.6. Notice that the lift $[K^4]$ is disconnected. Also, each of the seven other lifts contains two connected sets, each containing a fiber.

Lemma 2.7. *Let $L \in L_2(K^n)$ be such that every four fibers form the lift $[K^4]$. Then L is $[K^n]$.*

Proof. By induction on n. This is clear for $n = 4$. Assume it is true for $n-1$. Let G be a representative of the switching class defined by L. Pick $n-1$ vertices from G. By induction, they can be switched to form K^{n-1}. Assume that the remaining vertex v is now adjacent to some (but not all) of the other vertices. Pick v and three other vertices of which it is adjacent to one or two. These four vertices form one of the following graphs:



But none of them is in $[K^4]$ - a contradiction. Hence, v is either adjacent to all or none of the vertices. Switch on v, if needed, to get K^n.

□

Proof of Theorem 2.2

Let $L \in L_2(K^n)$. Construct a set F of disjoint quadruples of fibers, none of which form $[K^4]$. This is done by picking such disjoint quadruples as long as possible. Let $f = |F|$. By Lemma 2.7, the subgraph induced on the remaining vertices is $[K^{n-4f}]$. As mentioned in Remark 2.6, in every quadruple of F we can find two connected sets, each containing a fiber. Choose each of them to be a branch set, and so we have $2f$ branch sets so far. Choose the section K^{n-4f} out of the remaining vertices, and let every vertex of it also be a branch set. We have selected $n-2f$ connected sets, and it remains to show they are pairwise adjacent. Naturally the $n-4f$ singleton branch sets form a clique. Each of the other branch sets contains a fiber and is therefore adjacent to every vertex. Therefore, $\eta(L) \geq n - 2f \geq \frac{n}{2}$.

□

2.4 The typical case

In this section we show that the Hadwiger number of a random 2-lift of K^n is almost always very close to the lower bound in Theorem 2.2.

Theorem 2.8. *$T(n,2) < \frac{n}{2-\varepsilon}$ for every $\varepsilon > 0$ and n large enough.*

In order to show that, we even relax the condition that branch sets span connected subgraphs. This leads to the notion of pseudoachromatic number (see [6, 9]) :

Definition 2.9. A (typically improper) coloring $C : V \mapsto [k]$ of a graph H is called *pseudocomplete* if for every $1 \leq i < j \leq k$ there is an edge xy of H with $C(x) = i$ and $C(y) = j$. We think of each color class as a possible branch set in a complete minor of the graph H. The maximal order of a pseudocomplete coloring of V is called the *pseudoachromatic number.*

Proof of Theorem 2.8

The proof is probabilistic. We estimate the probability that any of the (only n^{2n}) colorings is pseudocomplete in the space of all the 2-lifts of K^n (there are $2^{\binom{n}{2}}$ such 2-lifts). Let $t = \frac{n}{2-\varepsilon}$ and fix a coloring $C : [2n] \mapsto [t]$. The average size of a color class is $\frac{2n}{t} < 4$, so we consider only those $\Omega(n)$ colors that appear 3 times or less. We may assume that no such color class contain a fiber (or else it is disconnected). For each color class A, eliminate all (at most three) color classes that meet any fiber that contain a vertex in A. We still have a collection S of $\Omega(n)$ color classes, with at most three vertices each, and they reside in distinct fibers. It follows that :

$$Prob_L \left[\bigwedge_{i \neq j \in S} \text{There is an edge } xy \text{ in } L \text{ with } C(x) = i, C(y) = j \right] =$$

$$= \prod_{i \neq j \in S} Prob_L [\text{ There is an edge } xy \text{ in } L \text{ with } C(x) = i, C(y) = j] \leq$$

$$\leq \prod_{i \neq j \in S} \left(1 - (\frac{1}{2})^9\right) \leq 2^{-\Omega(n^2)}$$

The first equality follows since the events (for distinct pairs $i, j \in S$) are independent. This follows from the fact that $\bigcup_{i \in S} C^{-1}(i)$ meets every fiber at most once. For the second inequality, note that we select independently with probability $\frac{1}{2}$ from the edges of $K_{a,b}$ with $a, b \leq 3$. And so, the probability that no edge is chosen is at most $1 - (\frac{1}{2})^9$.

There are $t^{2n} < n^{2n}$ possible colorings, and so the union bound implies that a random lift L has no K^t minor almost always, and so $T(n, 2) < t$, if n is large enough.

□

3 ℓ-Lifts

We now shift our attention from 2-lifts to ℓ-lifts. Here our results are more fragmentary. The bounds on the Hadwiger number of random lifts are quite tight:

- $T(n, \ell) = \Theta(n)$ for $\ell \leq O(\log n)$
- $T(n, \ell) = \Theta\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}}\right)$ for $\Omega(\log n) \leq \ell \leq n^{\frac{1}{3}-\varepsilon}$
- For larger values of ℓ, $\Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log \ell}}\right) \leq T(n, \ell) \leq n\sqrt{\ell}$

We determine $M(n, \ell)$ up to a multiplicative constant, i.e. $M(n, \ell) = \Theta(n\sqrt{\ell})$. As for the lower bound, Theorem 1.6 implies $m(n, \ell) \geq \Omega\left(\frac{n}{\sqrt{\log n}}\right)$. This is at most $O(\sqrt{\log n})$ away from the truth since $m(n, \ell) \leq n$ follows from the trivial lift. We also show $\eta(L) \geq \frac{n}{2\ell-1}$ which is useful for $\ell \leq O(\sqrt{\log n})$. In particular for $\ell = O(1)$ both $m(n, \ell)$ and $M(n, \ell)$ (and certainly $T(n, \ell)$) are $\Theta(n)$.

3.1 The smallest possible Hadwiger number

Theorem 3.1.

$$\textit{For every } n, \ell, \quad m(n, \ell) \geq \left\lceil \frac{n}{2\ell - 1} \right\rceil$$

Lemma 3.2. *Every connected ℓ-lift of K^n contains a connected ℓ-lift of K^m for some $m \leq 2\ell - 1$.*

Proof. Let $L \in L_\ell(K^n)$ be a connected ℓ-lift. Consider a star levelling of L, and let F be the fiber defined by the centers of these stars. Let G be the graph that result by contracting each level to a vertex. G is connected and so it contains a spanning tree T, with $\ell - 1$ edges. Let E be edges in L that correspond to the edges of T (if there are more than one edge corresponding to the same edge of T, pick one arbitrarily). Construct the desired lift from the fibers intersecting with E (at most $2\ell - 2$), and F itself.

□

Remark 3.3. Suppose that L is a disconnected ℓ-lift of a connected graph G. Find a spanning tree of G and keep its edges flat, so each level is connected. It follows easily that each connected component is the disjoint union of several levels. In other words, each connected component of L is an $\tilde{\ell}$-lift of G, for some $\tilde{\ell} < \ell$.

Proof of Theorem 3.1

Let $L \in L_\ell(K^n)$. If L is disconnected, the previous remark shows that it contains an $\tilde{\ell}$-lift of K^n for some $\tilde{\ell} < \ell$, and so we finish by induction on ℓ. If L is connected, then by Lemma 3.2 it contains a connected subgraph $H \in L_\ell(K^m)$ for some $m \leq 2\ell - 1$. By induction on n, the graph $L \setminus H \in L_\ell(K^{n-m})$ has a complete minor on at least $\lceil \frac{n-m}{2\ell-1} \rceil$ vertices. Now H is connected and since it contains a whole fiber it can be added as a branch set to yield a complete minor of order $\geq \lceil \frac{n-m}{2\ell-1} \rceil + 1 \geq \lceil \frac{n}{2\ell-1} \rceil$.

□

Remark 3.4. Notice that for $\ell < 0.319\sqrt{\log n}$, this bound is better than the bound that follows from Theorem 1.7. For constant ℓ-s we get $\eta(L) \geq \Omega(n)$. For large ℓ Theorem 1.7 yields the best bound known to us.

Remark 3.5. To prove upper bounds on $m(n, \ell)$ we need to find lifts with no complete minors. In the case of the icosahedron (as a 2-lift of K^6) this followed from the planarity of the lifted graph. Could a similar argument be applied for higher n as well (with embeddability into higher-genus surfaces)? This will unfortunately not work when $n \geq 8$. Euler's formula implies that the genus of a graph G is at least $\geq \left\lceil \frac{|E(G)|-3|V(G)|+6}{6} \right\rceil$ (and $\tilde{g} \geq \left\lceil \frac{|E(G)|-3|V(G)|+6}{3} \right\rceil$ for the nonorientable case). It follows that for a fixed base graph with $|E(G)| > 3|V(G)| + 6$, the genus of an ℓ-lift grows like $\Omega(\ell)$.

3.2 The largest possible Hadwiger number

Theorem 3.6.

$$\text{For every } n, \ell, \quad \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \sqrt{\ell} \right\rfloor \leq M(n,\ell) < \left\lceil n\sqrt{\ell} \right\rceil$$

Proof. We prove the lower bound by constructing a lift L with Hadwiger number $\lfloor \frac{n}{4} \rfloor \lfloor \sqrt{\ell} \rfloor$, as shown in Figure 4. The numbers in the sketch stand for the relevant steps described below.

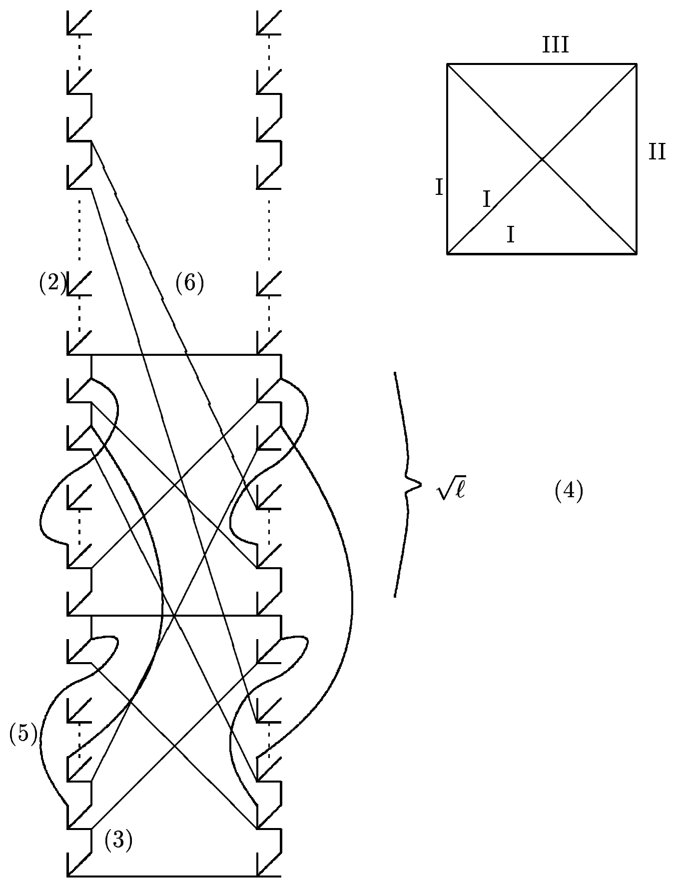


Figure 4: Sketch of an ℓ-lift with $\eta \geq \lfloor \frac{n}{4} \rfloor \lfloor \sqrt{\ell} \rfloor$

1. Partition the fibers of L into quadruples, with up to three remaining fibers.

2. In every level three edges (the "I" edges in the sketch) remain flat to make the 4-vertex graph connected.

3. Another edge (edge "II" in the sketch) is defined by the permutation $i \mapsto i+1 \pmod{\ell}$, so that it connects every two consecutive levels.

4. Partition the levels of each quadruple into $\lfloor\sqrt{\ell}\rfloor$ blocks of size $\lfloor\sqrt{\ell}\rfloor$.

5. In each quadruple, connect the i^{th} level of the j^{th} block to the j^{th} level of the i^{th} block, by swapping the fifth edge (edge "III" in the sketch) between them.

6. Between two quadruples we lift the edges so that every block in one is adjacent to all the blocks of the other.

The last two steps can be done thanks to the fact the the number of levels in each block is not smaller than the number of blocks in each quadruple. Now every block is connected (by steps 2, 3), and we choose it as a branch set. We have found a $K^{\lfloor\frac{n}{4}\rfloor\lfloor\sqrt{\ell}\rfloor}$ minor, because every two branch sets are adjacent (by steps 5, 6).

To prove the opposite inequality, note that L has $\ell\binom{n}{2} < \binom{n\sqrt{\ell}}{2}$ edges, so it cannot contain a $K^{n\sqrt{\ell}}$ minor.

□

3.3 The typical case

We saw before that the Hadwiger number of a random 2-lift roughly equals the lowest possible value (The bounds in Theorems 2.2 and 2.8 are nearly equal). On the other hand for $\ell > \Omega(\log n)$, the Hadwiger number of a random lift is closer to the largest possible value, as we show next. It is interesting to compare η of a random lift with the trivial lift (for which $\eta = n$). This change in behavior occurs around $\ell \approx \log n$. For smaller values of ℓ, the Hadwiger number is smaller than n, and for larger values of ℓ, it is bigger.

Theorem 3.7. *For every $\varepsilon > 0$ and large enough n :*

1. $\ell \leq O(\log n) \Rightarrow T(n,\ell) = \Theta(n)$

2. $\Omega(\log n) \leq \ell \leq n^{\frac{1}{3}-\varepsilon} \Rightarrow T(n,\ell) = \Theta\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}}\right)$

3. $\ell > n^{\frac{1}{3}-\varepsilon} \Rightarrow \Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log \ell}}\right) \leq T(n,\ell) \leq n\sqrt{\ell}$

In the proof of this Theorem, we deal with the different ranges for ℓ in the Lemmas below. We begin with upper bounds on T :

Lemma 3.8. *If* $\ell \leq 2\log n$*, then* $T(n,\ell) < n$

Lemma 3.9. *For every* $\varepsilon > 0$ *and large enough* n*, if* $\log n \leq \ell \leq n^{\frac{1}{3}-\varepsilon}$ *then* $T(n,\ell) \leq O\left(n\sqrt{\frac{\ell}{\log n}}\right)$.

The upper bound in case 3 of Theorem 3.7 is contained in Theorem 3.6. Next we turn to the lower bounds :

Lemma 3.10. *For every* n *and* ℓ*,* $T(n,\ell) \geq \Omega(n)$*.*

Lemma 3.11. *If* $\ell \geq 2.5\log n$*, then :*

$$T(n,\ell) > \Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}+\sqrt{\log \ell}}\right)$$

Remark 3.12. It is of interest to determine the critical value for the equality $T(n,\ell) = n$. If $\ell > 289\log n$ then $T(n,\ell) > n$, as implied by Lemma 3.11 (with a more careful analysis of the constants). On the other hand for $\ell < 2\log n$, $T(n,\ell) < n$ by Lemma 3.8. We do not know whether $T(n,\ell)$ can be significantly lower than n. Specifically, is there some $\varepsilon > 0$ such that $T(n,\ell) < (1-\varepsilon)n$ for infinitely many n,ℓ.

The following Lemmas will be useful in proving the above Lemmas :

Lemma 3.13. *Let* $A, B \subseteq [\ell]$*,* $|A| = a$*,* $|B| = b$*. If* $a+b \leq \ell$*, then :*

$$Prob_{\pi \in S_\ell}[\pi(A) \cap B = \emptyset] = \frac{(\ell-a)!(\ell-b)!}{\ell!(\ell-a-b)!}$$

And Furthermore,

$$Prob_{\pi \in S_\ell}[\pi(A) \cap B = \emptyset] < e^{-\frac{ab}{\ell}}$$

The proof is by simple calculation.

Lemma 3.14.

$$Prob_{L \in L_\ell(K^4)}[L \text{ is disconnected }] \leq \frac{2}{\ell^2}$$

Proof. Fix a star levelling of L. By Remark 3.3, L is disconnected if and only if its levels can be partitioned to two sets, say of k and $\ell - k$ levels, with no edge between them. Every level has three free (non star) edges. So, if we fix such a partition of the levels, there are $(k!(\ell-k)!)^3$ lifts that satisfy this condition. Therefore:

$$Prob_{L \in L_\ell(K^4)}[L \text{ is disconnected }] \leq \frac{1}{\ell!^3} \sum_{k=1}^{\ell/2} \binom{\ell}{k} (k!(\ell-k)!)^3 =$$

$$= \sum_{k=1}^{\ell/2} \binom{\ell}{k}^{-2} < \frac{1}{\ell^2} + \left(\frac{\ell}{2} - 1\right) \binom{\ell}{2}^{-2} < \frac{2}{\ell^2}$$

□

Proof of Lemma 3.9

We use an argument similar to the proof of the upper bound for random 2-lifts (Theorem 2.8). Let $k = \sqrt{\frac{4\ell}{\log n - 3\log \ell}}$. Notice that $2 \leq k \leq \ell$, since $\log n \leq \ell \leq o(\sqrt[3]{n})$. Let $t = 2kn$. We will show that $T(n, \ell) < t$, and conclude the Lemma. Pick a coloring $C : [n\ell] \mapsto [t]$, and let us consider the probability that C is pseudocomplete relative to a random ℓ-lift. Clearly C must be onto. Also, we distinguish between color classes of more than $\frac{\ell}{k}$ vertices and smaller color classes. For the former we pessimistically assume the color classes to be connected and adjacent to all other color classes. For a color class that is no bigger than $\frac{\ell}{k}$ to be connected, it must contain fewer than $\frac{\ell}{2k}$ vertices in each fiber it meets, so we assume this about C.

Now let $1 \leq a_1 \leq \cdots \leq a_t$ be the sizes of the color classes. Suppose that $a_m \leq \frac{\ell}{k} < a_{m+1}$. Clearly,

$$n\ell = \sum a_i \geq m + (t-m)(\frac{\ell}{k} + 1) \Rightarrow m \geq \left(1 + \frac{k}{\ell}\right) t - kn > kn$$

We now select a sub-collection of the m smaller color classes. Pick such a color class A, and eliminate all (at most $\frac{\ell^2}{k}$) color classes that reside the same fiber with a vertex in A. Select one of the remaining small classes and do the same. Repeat until exhaustion. This process yields a collection S of at least $\frac{k^2 n}{\ell^2}$ color classes, with at most $\frac{\ell}{k}$ vertices each, that reside in distinct fibers. Denote $A_{f,c}$ the set of all the vertices from fiber F_f that are colored with color c, and $a_{f,c} = |A_{f,c}|$. Lemma 3.13 implies that :

$$Prob_L[\text{ There is no edge between color classes } c \text{ and } d\ ] \geq$$

$$\geq \prod_{f,g} \frac{(\ell - a_{f,c})!\,(\ell - a_{g,d})!}{\ell!\,(\ell - a_{f,c} - a_{g,d})!} > \prod_{f,g} e^{-\frac{2a_{f,c}a_{g,d}}{\ell - a_{f,c}+1}} > e^{-2\sum \frac{a_{f,c}a_{g,d}}{\ell - a_{f,c}}} \geq e^{-2\frac{(\ell/k)^2}{\ell-\ell/2k}} =$$

$$= e^{-\frac{4\ell}{2k^2-k}}$$

The second inequality follows by simple calculation as $a_{f,c}, a_{g,d} \leq \frac{\ell}{4}$. And so,

$$Prob_L\left[\bigwedge_{i\neq j\in S} \text{There is an edge } xy \text{ in } L \text{ with } C(x)=i, C(y)=j\right] =$$

$$= \prod_{i\neq j\in S} Prob_L\,[\text{ There is an edge } xy \text{ in } L \text{ with } C(x)=i, C(y)=j] \leq$$

$$\leq \prod_{i\neq j\in S}\left(1 - e^{-\frac{4\ell}{2k^2-k}}\right) \leq \left(1 - e^{-\frac{4\ell}{2k^2-k}}\right)^{O\left(\frac{k^4n^2}{\ell^4}\right)}$$

There are less than $n^{n\ell}$ possible colorings and therefore the union bound implies:

$$Prob_L[L \text{ has pseudoachromatic number } \leq t] \leq n^{n\ell}\left(1 - e^{-\frac{4\ell}{2k^2-k}}\right)^{O\left(\frac{k^4n^2}{\ell^4}\right)} <$$

$$< e^{n^{4/3-\varepsilon}\log n - O\left(\left(\frac{k^2n}{\ell^2}\right)^2\left(\frac{\ell^3}{n}\right)^{\frac{2}{3}}\right)} = e^{n^{4/3-\varepsilon}\log n - O\left(\frac{n^{4/3}}{\log^2 n}\right)}$$

And this is arbitrarily small when n is large enough.

□

Proof of Lemma 3.8

The proof is a small variation of the former one. Let $t = \left(\frac{\ell}{\ell+1} + \frac{1}{\ell^2}\right)n$. We show that $T(n,\ell) < t$. As before, pick a coloring C and observe its color classes of size ℓ or less. In this case $m \geq \left(1+\frac{1}{\ell}\right)t - n > \frac{n}{\ell^2}$ and we can find a collection S of at least $\frac{n}{\ell^4}$ color classes, with at most ℓ vertices each, that reside in distinct fibers.

$$Prob_L\left[\bigwedge_{i\neq j\in S} \text{There is an edge } xy \text{ in } L \text{ with } C(x)=i, C(y)=j\right] \leq$$

$$\leq \prod \frac{\left(\frac{\ell}{2}\right)!\,\left(\frac{\ell}{2}\right)!}{\ell!} \leq \left(1 - \frac{1}{O\left(\frac{2^\ell}{\sqrt{\ell}}\right)}\right)^{O\left(\frac{n^2}{\ell^8}\right)}$$

And so,

$$Prob_L[L \text{ has a pseudoachromatic number } \leq t] \leq n^{n\ell} e^{-O\left(\frac{n^2}{\ell^{7.5} 2^\ell}\right)}$$

And this is arbitrarily small when $\ell \leq 2 \log n$.

□

Proof of Lemma 3.10

We prove that $T(n, \ell) > \frac{\ell^2-2}{\ell^2} \frac{n}{4}$. Fix a partition of the fibers into sets of size four (quadruples). Let X_q be the following indicator random variable :

$$X_q = \begin{cases} 1 & \text{if quadruple } q \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = \sum X_i$ be the number of connected quadruples in L. Lemma 3.14 implies that $\mu = \mathbb{E}[X] = \sum \mathbb{E}[X_i] \geq \left\lfloor \frac{n}{4} \right\rfloor \frac{\ell^2-2}{\ell^2}$. Chernoff inequality implies that :

$$Prob\left[X < \left(\frac{1}{4} - \varepsilon\right) \frac{\ell^2-2}{\ell^2} n\right] = Prob\left[X < (1-4\varepsilon)\mu\right] < e^{-8\varepsilon^2 \mu} \to 0$$

We have thus found at least $\frac{\ell^2-2}{\ell^2} \left\lfloor \left(\frac{1}{4} - \varepsilon\right) n \right\rfloor$ disjoint connected quadruples, for every $\varepsilon > 0$. Let each be a branch set. Each of them contains a whole fiber and they are therefore adjacent to each another.

□

Proof of Lemma 3.11

The proof is based on the following two properties of random lifts :

- Almost every four fibers form a connected subgraph (Lemma 3.14).
- Two big enough portions of two distinct fibers are almost surely adjacent (Lemma 3.13).

Since every connected quadruple of fibers contains a spanning tree, we can find there many large enough subtrees that will serve as branch sets. Each of the branch sets is limited to four fibers and therefore contains big portions of some of the fibers. Consequently, the branch sets are very likely to be adjacent to each other, as needed.

Let $L \in L_\ell(K^n)$ with fibers $\{F_i\}_{i=1}^n$. Fix $k = \sqrt{32\ell \log n} + \sqrt{16\ell \log \ell}$. Consider a partition of the fibers into quadruples. As we saw in the proof of Lemma 3.10, we can almost always find $\left(\frac{1}{4} - \varepsilon\right) \frac{\ell^2-2}{\ell^2} n$ connected quadruples.

Fix a spanning tree in such a quadruple. We show how to find subtrees of size s where $k \le s \le 3k-2$, thanks to the fact that the maximal degree is 3. Pick an arbitrary root, and select the largest child subtree. Proceed this way and stop just before you need to pick a subtree smaller than $k-1$. The tree you are left with is of size at least k (with the current root), and at most $3k-2$, since there are at most 3 subtrees. Hence, we may assume (pessimistically) that we have found $\frac{4\ell}{3k}$ subtrees of size k that way. Let each subtree be a branch set, and denote them $\{B_j\}_{j=1}^{m}$, where $m = \Theta\left(\frac{n\ell}{k}\right) = \Theta\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}+\sqrt{\log \ell}}\right)$. Let $A_{ij} = F_i \cap B_j$. For every j, $|\{i \mid A_{ij} \ne \emptyset\}| \le 4$, and so we can denote the four subsets into which B_j is split, by α_j, β_j, γ_j and δ_j, such that $|\alpha_j| \ge |\beta_j| \ge |\gamma_j| \ge |\delta_j|$. Notice that it is always true that $\frac{k}{4} \le |\alpha_j| \le \frac{k}{2}$. Next we show that :

$$Prob\left[B_j \text{ and } B_{\tilde{j}} \text{ are not connected}\right] < e^{-\frac{k^2}{16\ell}}$$

Let $i, \tilde{i}$ be the fibers of $\alpha_j, \alpha_{\tilde{j}}$, i.e. $\alpha_j = A_{ij}, \alpha_{\tilde{j}} = A_{\tilde{i}\tilde{j}}$. If $i \ne \tilde{i}$ we can use Lemma 3.13 to get :

$$Prob[N(B_j) \cap B_{\tilde{j}} = \emptyset] < Prob[N(\alpha_j) \cap \alpha_{\tilde{j}} = \emptyset] < e^{-\frac{|\alpha_j||\alpha_{\tilde{j}}|}{\ell}} \le e^{-\frac{k^2}{16\ell}}$$

Suppose $i = \tilde{i}$. Assume *w.l.o.g.* that $|\alpha_j| \le |\alpha_{\tilde{j}}|$. Hence, $|\beta_j| \ge \frac{k-|\alpha_j|}{3} \ge \frac{k-|\alpha_{\tilde{j}}|}{3}$. $\forall x \in [\frac{k}{4}, \frac{k}{2}] \quad \frac{k-x}{3}x \ge \frac{k^2}{16}$ and therefore $|\beta_j||\alpha_{\tilde{j}}| \ge \frac{k^2}{16}$. Surely β_j is included in a fiber other than i and so we can use Lemma 3.13 once again to get :

$$Prob[N(B_j) \cap B_{\tilde{j}} = \emptyset] < e^{-\frac{|\beta_j||\alpha_{\tilde{j}}|}{\ell}} \le e^{-\frac{k^2}{16\ell}}$$

Hence, it is very likely that the branch sets form a complete minor.

$$Prob\left[\exists\ j, \tilde{j} \text{ s.t. } N(B_j) \cap B_{\tilde{j}} = \emptyset\right] < \binom{m}{2} e^{-\frac{k^2}{16\ell}} \le O\left(\frac{n\ell}{k} e^{-\frac{k^2}{32\ell}}\right)^2$$

This upper bound is $o(1)$ by our choice of k, and so

$$Prob\left[\eta(L) \ge \Omega\left(\frac{n\sqrt{\ell}}{\sqrt{\log n}+\sqrt{\log \ell}}\right)\right] \to 1$$

□

4 Topological Minors of Lifts

We show tight bounds on the Hajós number of an ℓ-lift L of K^n :

$$\Omega\left(\sqrt{n}\right) \leq \sigma(L) \leq n \tag{1}$$

Furthermore, we show that random lifts span this whole range for different values of $\ell = \ell(n)$. For $\ell = O(1)$ there holds $\sigma(L) = \Theta(\sqrt{n})$, and for $\ell = \Theta(n)$ we get $\sigma(L) = \Theta(n)$.

4.1 Bounds

The upper bound is very easy :

Theorem 4.1. *For every $L \in L_\ell(K^n)$, $\sigma(L) \leq n$. This bound is tight.*

Proof. Every vertex in L has only $n-1$ neighbors, and thus only $n-1$ different paths leaving it. The bound is attained by the trivial lift.

□

The lower bound follows immediately from a known Theorem :

Theorem 4.2 (Komlós Szemerédi 1996 [10]; Bollobás Thomason 1996 [5]). *Every graph of average degree d, contains a subdivision of $K^{\Omega\left(\sqrt{d}\right)}$.*

Corollary 4.3. *For every $L \in L_\ell(K^n)$, $\sigma(L) \geq \Omega\left(\sqrt{n}\right)$.*

The lower bound is tight up to a constant. It is attained for random 2-lifts (or the union of random 2-lifts, for higher order lifts), as we show below.

4.2 Random lifts

Theorem 4.4. *For almost every $L \in L_\ell(K^n)$ we have $\sigma(L) \leq O\left(\sqrt{n\ell}\right)$.*

Proof. Let $t = (2+\varepsilon)\sqrt{n\ell}$ and suppose L contains a subdivision of K^t. Let V be the set of its branch vertices, and X be the random variable that counts the number of missing edges in the induced subgraph $L[V]$. Every missing edge must be replaced by a path with at least one additional vertex. Since there are only $n\ell$ vertices in L, the number of missing edges cannot exceed this bound.

By simple calculation, for every $\ell \geq 2$, $\mathbb{E}[X] \geq \left(1 - \frac{1}{\ell}\right)\binom{t}{2}$. Let us calculate the variance - When does a pair of edges have a bigger probability to be missing together? Only when they are between the same two fibers and are disjoint. There are at most $O\left(t^2\right)$ pairs of fibers, and for each such only $O\left(\ell^4\right)$ choices of the two pairs of vertices, and so the total number of such pairs is less then $O(\ell^5 n)$. For such edges e and f we have :

$$Pr(\text{both } e,\, f \text{ are missing}) - Pr(e \text{ is missing})Pr(f \text{ is missing}) \leq$$

$$\leq \frac{1}{\ell} + \left(1 - \frac{2}{\ell}\right)\left(1 - \frac{1}{\ell - 1}\right) - \left(1 - \frac{1}{\ell}\right)^2 = \frac{1}{\ell^2(\ell - 1)}$$

And Therefore :

$$Var(X) = E(X^2) - E^2(X) =$$

$$= \sum_{e,f \in V^2} Pr(e,\, f \text{ are missing}) - Pr(e \text{ is missing})Pr(f \text{ is missing}) \leq O(\ell^2 n)$$

And so, Chebyshev bound implies

$$Pr(X \leq n\ell) \leq Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X] - n\ell) \leq \frac{Var(X)}{(\mathbb{E}[X] - n\ell)^2}$$

But as we saw, $\mathbb{E}[X] \geq \left(1 - \frac{1}{\ell}\right)\binom{t}{2}$, so $\mathbb{E}[X] - n\ell \geq \Omega(n\ell)$ and it follows that $Pr(X \leq n\ell) \leq O\left(\frac{1}{n}\right)$. Therefore :

$$Pr(K^t \text{ is a topological minor of } L) \leq Pr(X \leq n\ell) \to 0$$

□

We have just seen that for random lifts of constant height, σ typically takes the lower bound in Equation 1. However for higher values of ℓ, it reaches the upper bound.

Theorem 4.5. *If $n \geq \Omega(\ell)$ then $\sigma \geq \Omega(\ell)$ almost always.*

Proof. Let $0 < \varepsilon < 1$ and $k = (1 - \varepsilon)\frac{\ell}{3}$. Pick $n \geq \log\left(\frac{4}{\varepsilon(1+\varepsilon)}\right) \cdot \ell + \ell + 1$. We construct a K^k topological minor, by selecting all the branch vertices from a single fiber f, and connecting them by paths of length five. To do so, we choose a star levelling by f and partition the other fibers into two classes: the class R, regarded "horizontally" as a collection of levels (rows), and C

which is considered "vertically" as a collection of fibers (columns). Allocate ℓ fibers for C, and $\log\left(\frac{4}{\varepsilon(1+\varepsilon)}\right)\cdot\ell$ fibers for R (ignore the rest of the fibers, if there are any left). Enumerate arbitrarily the levels in R as $\{R^i\}_{i=1}^{\ell}$ and the fibers in C as $\{C_i\}_{i=1}^{\ell}$. The path between the branch vertices f_i and f_j will be (see Figure 5 below):

$$f_i \to R^i \to C_i \to C_j \to R^j \to f_j$$

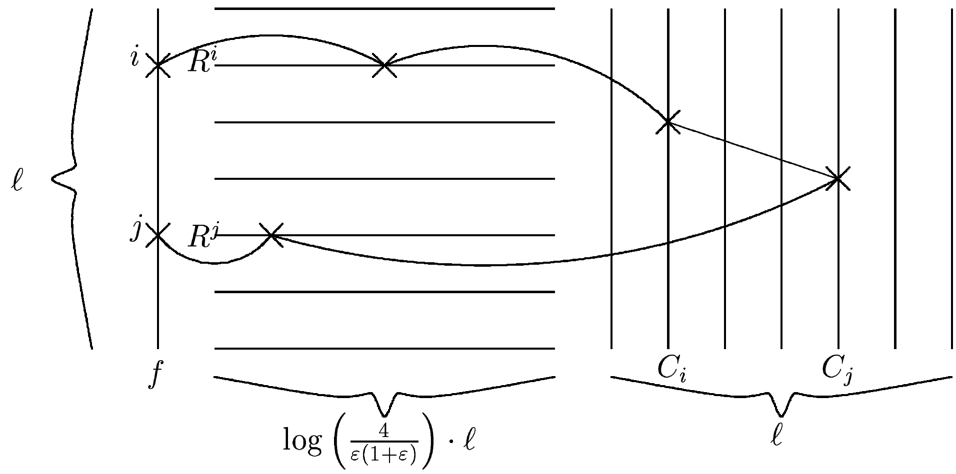


Figure 5: The path found between f_i and f_j

Next we show that these paths exist and are disjoint :

Let ${R^i}_v$ and ${C_i}^v$ be the v-th vertex in R^i and C_i, respectively. For every $1 \le i \le \ell$ do the following : Omit all the vertices in C_i with no neighbor in R^i. Every vertex ${C_i}^c$ that is not omitted is adjacent to some ${R^i}_r$, and so we define $\varphi_i(c) = r$. Notice that φ_i is injective due to the structure of lifts. ($\varphi_i(c) = r = \varphi_i(c')$ would mean that ${R^i}_r$ has two neighbors in the fiber C_i). Lemma 4.6 below implies that there are at least $3k$ edges between the first k fibers (we assume they are first *w.l.o.g.*). Hence, for each $1 \le i, j \le k$ we can pick an edge between C_i and C_j, say ${C_i}^c - {C_j}^d$, and omit its vertices. The $f_i - {R^i}_{\varphi_i(c)}$, ${R^j}_{\varphi_j(d)} - f_j$ edges exist because of the star levelling, and the ${R^i}_{\varphi_i(c)} - {C_i}^c$, ${C_j}^d - {R^j}_{\varphi_j(d)}$ exist by the definition of φ. We have just shown the paths exist, and it remains to show that they are disjoint, but this follows directly from our choice of distinct C_i-C_j edges, and the fact that φ is bijective. Thus we have found a subdivision of K^k as promised. $\square$

Lemma 4.6. *After omitting vertices in C_i with no neighbor in R^i, there remain almost always at least k fibers with at least $\left(1-\frac{\varepsilon}{2}\right)\ell$ vertices each.*

Proof. Let X be the random variable that counts the number of vertices omitted from C_i. Markov inequality implies :

$$Pr(X > \frac{1}{2}\varepsilon\ell) \leq \frac{2\mathbb{E}[X]}{\varepsilon\ell} = \frac{2}{\varepsilon}\left(\frac{\ell-1}{\ell}\right)^{\log\left(\frac{4}{\varepsilon(1+\varepsilon)}\right)\cdot\ell} < \frac{1}{2}(1+\varepsilon)$$

Let Y be the random variable that counts the number of fibers from C for which $X \leq \frac{1}{2}\varepsilon\ell$.

$$\mathbb{E}[Y] \;=\; \left(1 - Pr(X > \frac{1}{2}\varepsilon\ell)\right)\ell \;>\; \frac{1}{2}(1-\varepsilon)\ell = \frac{3k}{2}$$

Chernoff bound implies (Notice that Y is a sum of independent variables, since they count the different fibers) :

$$Pr(Y < k) < e^{-\frac{1}{2}\mathbb{E}[Y]\left(1-\frac{k}{\mathbb{E}[Y]}\right)^2} < e^{-\frac{1}{36}(1-\varepsilon)\ell} \xrightarrow[\ell\to\infty]{} 0$$

$\square$

What can be said about σ for larger ℓ? We expect that if $\ell \geq \Omega(n)$ then almost surely $\sigma = \Theta(n)$.

5 Lifts of general graphs

So far we have restricted our discussion to lifts of complete graphs. Note, however, that all the upper bounds we have proved apply for any base graph G on n vertices. We can offer a little less immediate, but more informative observation regarding the lower bounds :

Theorem 5.1. *Let $H \preceq G$ be graphs and $\ell \geq 2$ an integer then :*

1. *For every $L \in L_\ell(G)$ there is a graph $L' \in L_\ell(H)$ with $L' \preceq L$.*
2. *For every $L' \in L_\ell(H)$ there is a graph $L \in L_\ell(G)$ with $L' \preceq L$.*

Proof.

1. In every branch set of $H \preceq G$ pick a spanning tree T_i. It is well known (see [1]) and easy that we may assume (by an appropriate relabelling of the vertices) that each level in L contains flat copies of all the T_i. Contract every T_i in every level to a vertex (i.e. take the T_i's as branch sets). Next, delete arbitrarily edges (if necessary) so that every vertex will have exactly one edge for each edge of H. The resulting graph is an ℓ-lift of H as claimed.

2. Given $L' \in L_\ell(H)$, replace every vertex in H by the corresponding branch set, and connect all the edges of the branch set together. Namely, if $uv \in E(H)$ replace u, v with branch sets V_u, V_v and add the edges $\{\bar{u}\bar{v} \mid \bar{u} \in V_u \times \{i\},\ \bar{v} \in V_v \times \{\pi(i)\}\}$ where π is the matching that belongs to uv in L'.

□

The basic definitions for complete base graphs extend to the general case:

Definition 5.2.

$$M_G(\ell) = max\,\{\eta(H) \mid H \in L_\ell(G)\}$$

$$m_G(\ell) = min\,\{\eta(H) \mid H \in L_\ell(G)\}$$

$$\tau_G(\ell,\delta) = max\,\left\{h \mid Prob_{H \in L_\ell(G)}[\eta(G) \geq h] > 1-\delta\right\}$$

Corollary 5.3.

$$M_G(\ell) \geq M(\eta(G),\ell)$$

$$m_G(\ell) \geq m(\eta(G),\ell)$$

Clearly this bounds hold with equality when the graph G is complete, but for general base graph, the bound on M is not tight. For example take $\ell > 16$, $G = W_{\frac{\ell+1}{2}}$ the $\frac{\ell+1}{2}$ spoked wheel. In this case $M_G(\ell) \geq \ell$ whereas $M(\eta(G),\ell) \leq 4\sqrt{\ell}$, since G is planar and therefore $\eta(G) = 4$.

We may use the same principle of Theorem 5.1 to prove a bound on a random lift of any fixed graph :

Theorem 5.4. *For every $\varepsilon > 0$, $\delta > 0$ and G with large enough Hadwiger number, if $\ell \geq 2.5\log(\eta(G))$ then*

$$\tau_G(\ell,\delta) > \Omega\left(\frac{\eta(G)\sqrt{\ell}}{\sqrt{\log(\eta(G))}+\sqrt{\log \ell}}\right)$$

Proof. Simply repeat the proof the Lemma 3.11, but instead of vertices and their fibers, refer to branch sets of $K^{\eta(G)}$ in G, and their fibers. The probability will only grow since each branch set may be adjacent to a few branch sets in another fiber (see Theorem 5.1).

□

Notice that here too, there is a gap between $\tau_G(\ell, \delta)$ and $\tau(\eta(G), \ell, \delta)$. E.g., let G be a disjoint union of $2^{\binom{n}{2}}$ K^n's. Then $\tau(\eta(G), \ell, \delta) \approx \frac{n}{2}$, while in a 2-lift of G one copy K^n will probably remain flat, and so $\tau_G(\ell, \delta) \geq n$.

We can repeat the same arguments for topological minors, only that instead of looking on spanning trees, we look at the paths. Therefore :

- $\sigma(L) \geq \Omega\left(\sqrt{\sigma(G)}\right)$.
- $\sigma(L) \geq (1-\varepsilon)\frac{\ell}{3}$ almost always for $\sigma(G) \geq \left(\log\left(\frac{4}{\varepsilon(1+\varepsilon)}\right) + 1\right)\ell + 1$.

6 Open Questions

In the last section, we have only scratched the surface on general base graphs. Essentially all questions in this direction are still open. In fact, the question that got us started on this project remains untouched. We were hoping to understand, for a given graph, which minors are "essential" and which are not. Namely, which minors of a given G persist and are to be found in all lifts of G and which are not. We think that this is an interesting concept that is worth studying.

In the more restricted context of lifts of cliques, the most intriguing question that remains open is whether there are lifts of K^n with $\eta(L) = o(n)$. If the answer is positive then the explicit construction of such graphs would be a very interesting challenge.

Furthermore, we were not able to determine the typical behavior of σ in ℓ-lifts of K^n for large ℓ. We suspect that σ is almost surely equal to $\Theta(n)$ for $\ell \geq \Omega(n)$, but this remains open.

References

[1] A. Amit and N. Linial, *Random graph coverings I: General theory and graph connectivity*, Combinatorica **22** (2002), 1–18.

[2] A. Amit, N. Linial, and J. Matoušek, *Random lifts of graphs: Independence and chromatic number*, Random Structures and Algorithms **20** (2002), 1–22.

[3] A. Amit, N. Linial, J. Matoušek, and E. Rozenman, *Random lifts of graphs*, SODA '01: Proceedings of the twelfth annual ACM-SIAM symposium on Discrete algorithms (2001), 883–894.

[4] B. Bollobás, P.A. Catlin, and P. Erdös, *Hadwiger's Conjecture is True for Almost Every Graph*, Europ. J. Combinatorics **1** (1980), 195–199.

[5] B. Bollobás and A. Thomason, *Proof of a Conjecture of Mader, Erdös and Hajnal on Topological Complete Subgraphs*, Europ. J. Combinatorics **19** (1998), 883–887.

[6] J. Bosák, *Decompositions of Graphs*, Kluwer Academic, 1990.

[7] P.J. Cameron, *Cohomological Aspects of Two-graphs*, Math. Z. **157** (1977), 101–119.

[8] D.G. Corneil and R.A. Mathon (eds.), *Geometry and Combinatorics: Selected Works of J.J. Seidel*, Academic Press, Boston, 1991.

[9] R. P. Gupta, *Bounds on the Chromatic and Achromatic Numbers of Complementary Graphs*, Recent Progress in Combinatorics (ed. W. T. Tutte) (1969), 229–235.

[10] J. Komlós and E. Szemerédi, *Topological Cliques in Graphs II*, Combinatorics, Probability and Computing **5** (1996), 79–90.

[11] A. V. Kostochka, *The minimum Hadwiger number for graphs with a given mean degree of vertices*, Metody Diskret. Analiz **38** (1982), 37–58, in Russian.

[12] J.J. Seidel, *A Survey of Two-graphs*, Colloquio Internazionale sulle Teorie Combinatorie **I** (Rome, 1973), 481–511, Rome, 1976. Acc. Naz. Lincei. Reprinted in [8].

[13] J.J. Seidel and D.E. Taylor, *Two-graphs, a second survey*, Algebraic Methods in Graph Theory **II** (Proc. Internat. Colloq., Szeged, 1978), 689–711, Amsterdam, 1981. North-Holland. Reprinted in [8].

[14] A. Thomason, *An Extremal Function for Contractions of Graphs*, Math. Proc. Cambridge Philos. Soc. **95** (1984), 261–265.

[15] ———, *The Extremal Function for Complete Minors*, Journal of Combinatorial Theory, Series B **81** (2001), 318–338.