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RAMANUJAN GRAPHS, LIFTS and WORD MAPS

NATI LINIAL

and DORON PUDER

- Hebrew University
- Jerusalem, Israel
- http://www.cs.huji.ac.il/∼nati/

Plan of this talk

- A very brief review of expansion and its connection with the spectrum.
- A very brief review of lifts, random lifts and their spectra.
- Statement of the new results.
- A little about the proof - Word maps and the associated cycle structure of permutations.

A very quick review on expansion in graphs

There are three main perspectives of expansion:

- Combinatorial - isoperimetric inequalities
- Linear Algebraic - spectral gap
- Probabilistic - Rapid convergence of the random walk (which we do not discuss today)

For (much) more on this: Our survey article with Hoory and Wigderson

The combinatorial definition

A graph $G = (V, E)$ is said to be ϵ-edge-expanding if for every partition of the vertex set V into X and $X^c = V \setminus X$, where X contains at most a half of the vertices, the number of cross edges

$$e(X, X^c) \geq \epsilon |X|.$$

In words: in every cut in G, the number of cut edges is at least proportionate to the size of the smaller side.

The combinatorial definition (contd.)

The edge expansion ratio of a graph $G = (V, E)$, is

$$h(G) = \min_{S \subseteq V,\, |S| \leq |V|/2} \frac{|E(S, \overline{S})|}{|S|}.$$

The linear-algebraic perspective

The **Adjacency Matrix** of an n-vertex graph G, denoted $A = A(G)$, is an $n \times n$ matrix whose (u, v) entry is the number of edges in G between vertex u and vertex v. Being real and symmetric, the matrix A has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Some simple things the spectrum of $A(G)$ tells about G

- If G is d-regular, then $\lambda_1 = d$. The corresponding eigenvector is $v_1 = \mathbf{1}/\sqrt{n}$
- The graph is connected iff $\lambda_1 > \lambda_2$. We call $\lambda_1 - \lambda_2$ the spectral gap.
- The graph is bipartite iff $\lambda_1 = -\lambda_n$
- $\chi(G) \geq -\frac{\lambda_1}{\lambda_n} + 1$
- A substantial spectral gap implies logarithmic diameter.

Spectrum vs. expansion

Theorem 1. *Let G be a d-regular graph with spectrum $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{(d+\lambda_2)(d-\lambda_2)}.$$

The bounds are tight.

- Left inequality is easy and powerful.
- Right inequality is surprising but, unfortunately, it is weak.

What's a "large" spectral gap?

If expansion is "good" and if a large spectral gap yields large expansion, then it's natural to ask:

Question 1. *How small can* λ_2 *be in a d-regular graph? (i.e., how large can the spectral gap get)?*

Theorem 2 (Alon, Boppana)**.**

$$\lambda_2 \geq 2\sqrt{d-1} - o(1)$$

What is the meaning of the number $2\sqrt{d-1}$?

A good approach to extremal problems is to come up with a candidate for an ideal example, and show that there are no better instances.

What, then, is the ideal expander? A good candidate is the infinite d-regular tree. It is possible to define a spectrum for infinite graphs (we'll see this later). It turns out that the supremum of the spectrum for the d-regular infinite tree is....

$$2\sqrt{d-1}$$

Some questions

How tight is this bound?

Problem 1. *Are there d-regular graphs with second eigenvalue*

$$\lambda_2 \leq 2\sqrt{d-1} \quad ?$$

When such graphs exist, they are called Ramanujan Graphs.

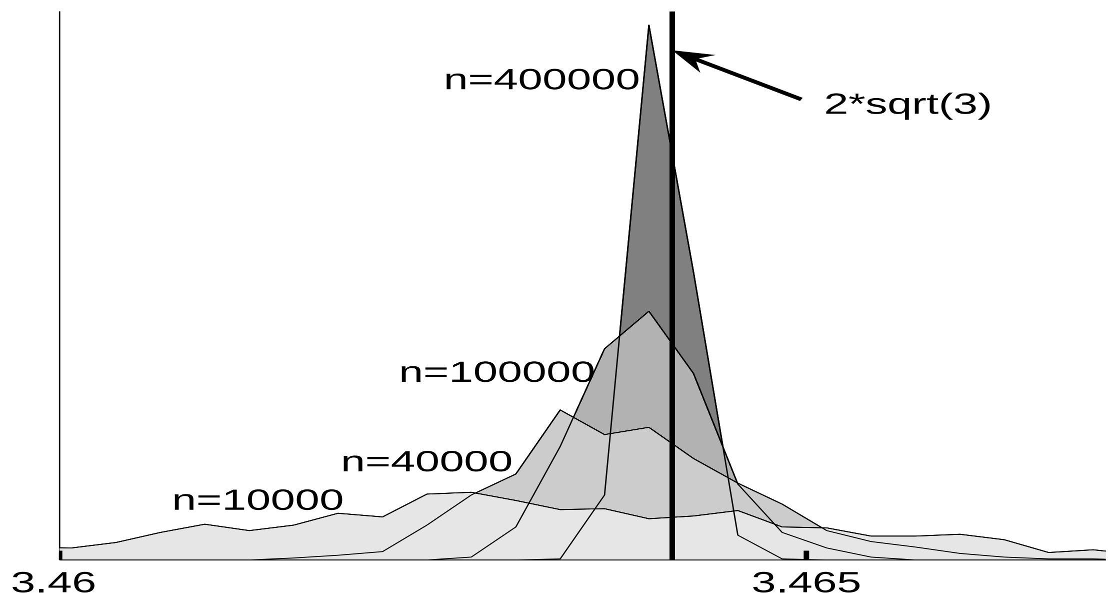
What is the typical behavior?

Problem 2. *How likely is a (large) random d-regular graph to be Ramanujan?*

What is currently known about Ramanujan Graphs?

Margulis; Lubotzky-Phillips-Sarnak; Morgenstern: d-regular Ramanujan Graphs exist when $d-1$ is a prime power. The construction is easy, but the proof uses a lot of heavy mathematical machinery.

Friedman: If you are willing to settle for $\lambda_2 \leq 2\sqrt{d-1}+\epsilon$, they exist. Moreover, almost every d-regular graph satisfies this condition.


n=400000
2*sqrt(3)
n=100000
n=40000
n=10000
3.46
3.465

Some open problems on Ramanujan Graphs

- Can we construct arbitrarily large d-regular Ramanujan Graphs for every d? Currently no one seems to know. The first unknown case is $d = 7$.

- Can we find combinatorial/probabilistic methods to construct graphs with large spectral gap (or even Ramanujan)? Constructions based on random lifts of graphs (Bilu-L.) yield graphs with

$$\lambda_2 \leq O(\sqrt{d} \log^{3/2} d).$$

The signing conjecture

The following, if true, would prove the existence of arbitrarily large d-regular Ramanujan graphs for every $d \geq 3$.

Conjecture 1. *Every d regular graph G has a signing with spectral radius $\leq 2\sqrt{d-1}$.*

A signing is a symmetric matrix in which some of the entries in the adjacency matrix of G are changed from $+1$ to -1. The spectral radius of a matrix is the largest absolute value of an eigenvalue.

This conjecture, if true, is tight.

Covers and lifts - the abstract approach

Definition 1. *A map* $\varphi : V(H) \to V(G)$ *where* G, H *are graphs is a covering map if for every* $x \in V(H)$*, the neighbor set* $\Gamma_H(x)$ *is mapped* $1:1$ *onto* $\Gamma_G(\varphi(x))$*.*
When such a mapping exists, we say that H *is a lift of* G*.*

This is a special case of fundamental concept from topology. From that perspective a graph is a one-dimensional simplicial complex, so covering maps can be defined and studied for graphs.

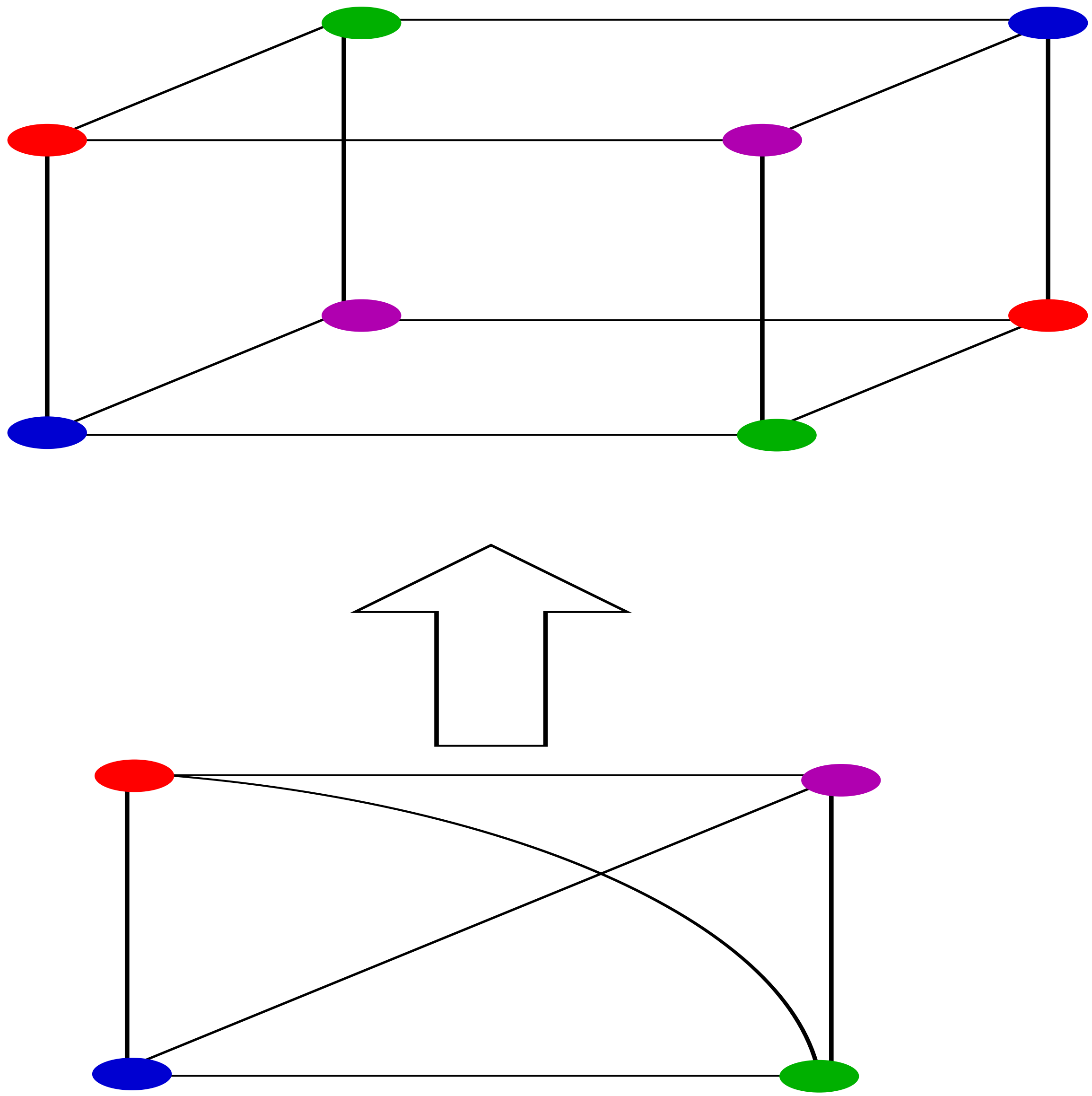


Figure 1: The 3-dimensional cube is a 2-lift of K_4

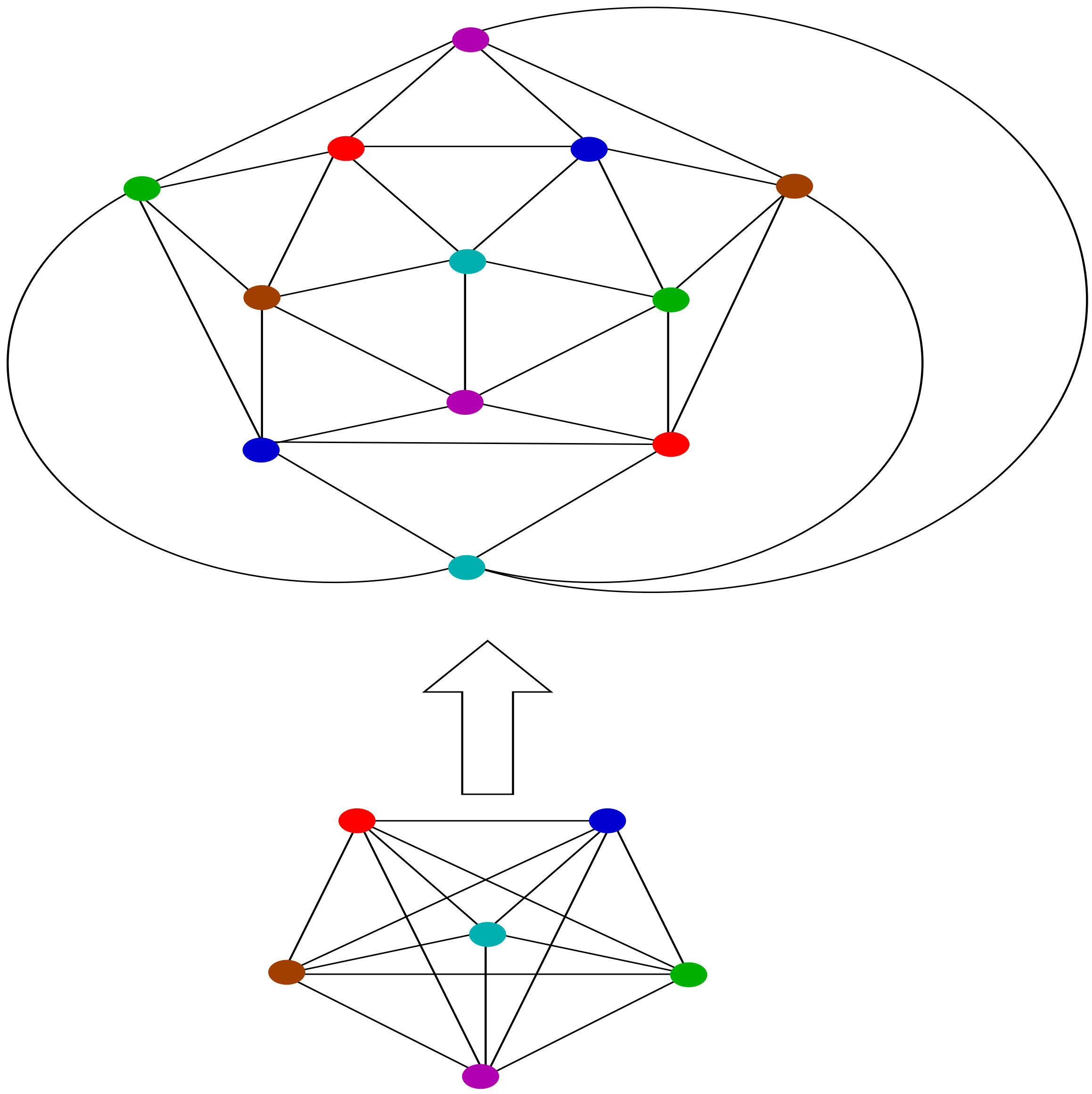


Figure 2: The icosahedron graph is a 2-lift of K_6

Making this definition more concrete

We see in the previous examples that the covering map φ is $2:1$.

- The 3-cube is a 2-lift of K_4.
- The graph of the icosahedron is a 2-lift of K_6.

In general, if G is a connected graph, then every covering map $\varphi : V(H) \to V(G)$ is $n : 1$ for some integer n (easy).

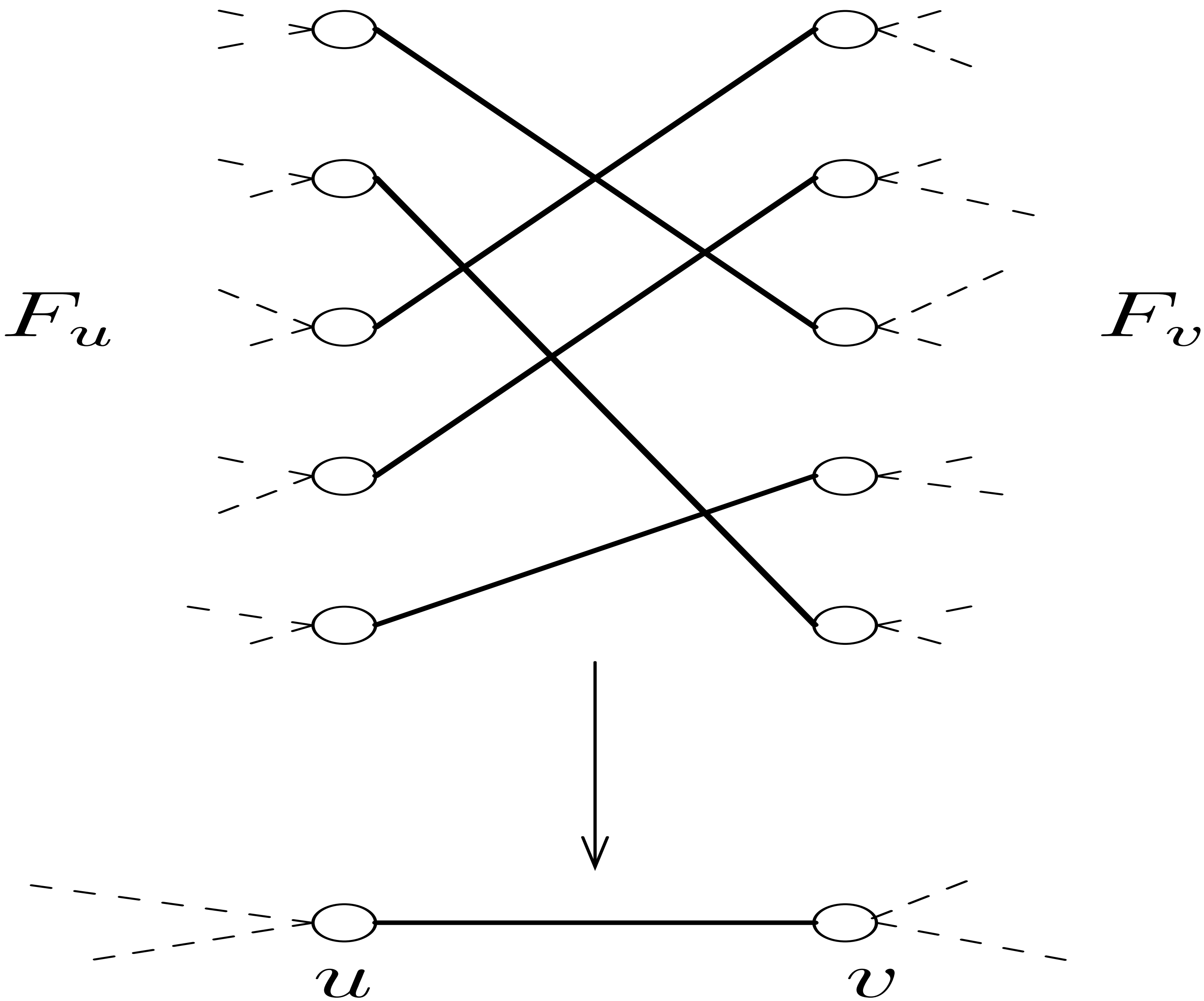
Fold numbers etc.

- We call n the fold number of φ.
- We say that H is an n-lift of G.
- Sometime we say that H is an n-cover of G.

A direct, constructive perspective

The set of those graphs that are n-lifts of G is called $L_n(G)$.

- Every $H \in L_n(G)$, has vertex set $V(H) = V(G) \times [n]$.
- We call the set $F_x = \{x\} \times [n]$ the fiber over x.
- For every edge $e = xy \in E(G)$ we have to select some perfect matching between the fibers F_x and F_y, i.e., a permutation $\pi = \pi_e \in S_n$ and connect (x, i) with $(y, \pi(i))$ for $i = 1, \ldots, n$.
- This set of edges is denoted by F_e, the fiber over e.
- We refer to G as our base graph.


F_u
F_v
u
v

Random lifts - A new class of random graphs

- When the permutations π_e are selected at random, we call the resulting graph a random n-lift of G.

- They can be used in essentially every way that traditional random graphs are employed:
  - To construct graphs with certain desirable properties. In our case, to achieve large spectral gaps.
  - To model various phenomena.
  - To study their typical properties.

A bit more on lifts

- Vertex degrees are maintained. If x has degree d, the so do all the vertices in the fiber of x. In particular, a lift of a d-regular graph is d-regular.

- The cycle C_n is a lift of C_m iff $m|n$.

- The d-regular tree covers every d-regular graph. This is the *universal cover* of a d-regular graph. Every connected base graph has a universal cover which is an infinite tree.

- An important special case: Every $2r$-regular graph is a lift of the *r-bouquet*: The graph with a single vertex and r loops.

A few words on the spectrum of the universal cover

Let T be the universal cover of some finite graph, and let A be the adjacency matrix of T.

We think of the (infinite) matrix A as a linear operator on $l_2(V(T))$. As such, A need not have any eigenvalues at all, and a modified definition is in place:

- The spectrum of T is defined as the set of all real numbers t for which the operator $A - tI$ is not invertible.
- In the finite-dimensional case this reduces to the usual definition, but here things are different. Two major differences are that

- T has a continuous spectrum. In particular, for $T = T_d$, the infinite d-regular tree, the spectrum is the whole interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

- Moreover, note that d is not in the spectrum of T_d since the constant function 1 is not in $l_2(V(T))$.

The largest absolute value of points in T's spectrum is called the spectral radius of T and is usually denoted by $\rho(T)$.

Thus in particular $\rho(T_d) = 2\sqrt{d-1}$.

Irregular Ramanujan Graphs?

- The second eigenvalue of every d-regular graph G is at least

$$2\sqrt{d-1} - o(1)$$

.

- The spectral radius of the universal cover of G (i.e. the infinite d-regular tree) is

$$2\sqrt{d-1}$$

.

- We say that a d-regular G is Ramanujan if every eigenvalue λ of G, except for the highest eigenvalue d satisfies $|\lambda| \leq 2\sqrt{d-1}$.

Irregular Ramanujan Graphs? (contd.)

This suggests the following definition:

Definition 2. *A (not necessarily regular) graph G is said to be Ramanujan, if every eigenvalue λ of G, except for the highest eigenvalue (the Perron eigenvalue) satisfies*

$$|\lambda| \leq \rho$$

where ρ is the spectral radius of G's universal cover.

Work by Lubotzky-Greenberg ('95) shows that for large graphs ($|V(G)| \to \infty$) this inequality is best possible.

This further suggests

Problem 3. *Do there exist arbitrarily large irregular Ramanujan graphs?*

Recall Friedman's Theorem: For every $\epsilon > 0$, almost every d-regular graph satisfies $\lambda_2 \leq 2\sqrt{d-1} + \epsilon$.

If we view a $2r$-regular graph as a random lift of the r-bouquets, this raises the question:

Problem 4. *What can be said about the eigenvalues in random lifts of a given graph G?*

Old vs. New Eigenvalues

Here is an easy observation:

The lifted graph inherits every eigenvalue of the base graph.

Namely, if H is a lift of G, then every eigenvalue of G is also an eigenvalue of H (Pf: Pullback, i.e., take any eigenfunction f of G, and assign the value $f(x)$ to every vertex in the fiber of x. It is easily verified that this is an eigenfunction of H with the same eigenvalue as f in G).

These are called the old eigenvalues of H. If G is given, the old eigenvalues appear in every lift, and we can only hope to control the values of the new eigenvalues.

The exact formulation of the problem

We now understand what we should ask:

Problem 5. *What can be said about the new eigenvalues in random lifts of a given graph G?*

(The old eigenvalues, namely the eigenvalues of G will be present in every lift).

Friedman strikes again - twice

Theorem 3 (Friedman '03)**.** *Let G be a finite connected graph and let D be its largest (Perron) eigenvalue. Let T be the universal cover of G and let ρ be the spectral radius of T. Then for almost every lift H of G it holds that every new eigenvalues of H satisfies*

$$\mu \leq D^{1/2}\rho^{1/2} + o(1).$$

Conjecture 2 (Friedman, ibid.)**.** *With the same notations, for almost every lift H of G it holds that every new eigenvalues of H satisfies*

$$\mu \leq \rho + o(1).$$

So, what's new?

Theorem 4 (L + Doron Puder, '08). *With the same notations, for almost every lift H of G it holds that every new eigenvalues of H satisfies*

$$\mu \leq O(D^{1/3}\rho^{2/3}).$$

We also have several conjectures that suggest an approach to proving the same statement with $O(\rho)$.
A bit about this - below.

A little about the proof - The trace method

This is an adaptation of a very old and powerful idea in the study of spectra which goes back to Wigner in the early 50's. Here is how it works:

Let H be an n-lift of G and let A_G, A_H be the adjacency matrices of G, H resp. New eigenvalues of H are denoted by μ. The trace of ${A_H}^t$ equals the number of closed paths of length t in H. Therefore:

$$ {\mu_{max}}^t \leq \sum_{\mu} \mu^t = \left(\sum_{\mu} \mu^t + \sum_{i=1}^{|V(G)|} \lambda_i^t \right) - \sum_{i=1}^{|V(G)|} \lambda_i^t = tr({A_H}^t) - tr({A_G}^t) $$

The trace method (contd.)

Each closed path in H is a lift of a closed path in G. If G's edges are labelled $g_1, \ldots, g_k$, then every closed path in G corresponds to some formal word w in $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$. Closed lifts of this path correspond to fixed points of $w(\sigma_1, \ldots, \sigma_k)$ - The permutation that's obtained when you plug σ_i for g_i in w, i.e. when you select the permutations that define the random lift.

– We denote by $\mathcal{CP}_t(G)$ is the set of all closed paths of length t in G (in particular, $|\mathcal{CP}_t(G)| = tr(A_G^t)$).

– We denote by $X_w^{(n)}(\sigma_1, \ldots, \sigma_k)$ is the random variable that counts the number of fixed points in the permutation $w(\sigma_1, \ldots, \sigma_k)$ when the σ_i are sampled at random from S_n.

The trace method (contd.)

$$\mu_{max}{}^{t} \leq tr(A_H{}^{t}) - tr(A_G{}^{t}) = \sum_{w \in \mathcal{CP}_t(G)} \left[X_w^{(n)}(\sigma_1, \ldots, \sigma_k) - 1\right]$$

Taking expectations and using Jensen's Inequality, we obtain:

$$\mathbb{E}(\mu_{max}) = \mathbb{E}\left[\left(\mu_{max}{}^{t}\right)^{1/t}\right] \leq \left[\mathbb{E}\left(\mu_{max}{}^{t}\right)\right]^{1/t} \leq \left[\sum_{w \in \mathcal{CP}_t(G)} \left[\mathbb{E}(X_w^{(n)}) - 1\right]\right]^{1/t}$$

Word maps

We were led to study fixed points in random permutations of the form $w(\sigma_1, \dots, \sigma_k)$. These questions are related to a subject that goes back 100 years or so to Frobenius and Schur. Let

$$w = g_{i_1}^{\alpha_1} \cdots g_{i_m}^{\alpha_m}$$

be a formal word in formal letters $g_1^{\pm 1}, \dots, g_k^{\pm 1}$. We consider w also as a map from $w : S_n^k \to S_n$ as follows: Select for each j, uniformly and independently a permutation $\sigma_j \in S_n$ and define

$$w(\sigma_1, \dots, \sigma_k) = \sigma_{i_1}^{\alpha_1} \cdots \sigma_{i_m}^{\alpha_m}$$

Word maps (contd.)

Question 2. *Given a formal word w, a fixed integer L and $n \to \infty$ consider the random variable that counts the number of L-cycles in a random word in the image of w. How is this random variable distributed?*

We say that a formal word w is imprimitive if it can be expressed as $w = v^r$ for some integer $r \geq 2$.
Every word can be uniquely expressed as $w = u^d$ with u primitive. The above question was answered in

Word maps (contd.)

Theorem 5 (A. Nica '94). *Let $w = u^d$ with u primitive. Let $X_{w,L}^{(n)}$ be the random variable that counts the number of L-cycles in a random permutation of the form $w(\sigma_1, \ldots, \sigma_k)$, where the permutations σ_i are selected uniformly at random from the symmetric group S_n. Then the limit distribution of $X_{w,L}^{(n)}$ (as $n \to \infty$) depends only on the integer d. In particular it is the same as when $w = x^d$ (here x is a single letter).*

Counting fixed points

To calculate $\mathbb{E}(X_w^{(n)})$, the expected number of fixed points in $w(\sigma_1, \ldots, \sigma_k)$, we count fixed points in $w(\sigma_1, \ldots, \sigma_k)$ in *all* choices of $(\sigma_1, \ldots, \sigma_k) \in S_n^{\,k}$.

Let $w = g_{i_1}^{\alpha_1} g_{i_2}^{\alpha_2} \ldots g_{i_m}^{\alpha_m}$ with $\alpha_i \in \{-1, 1\}$. If $s_0 \in \{1, \ldots, n\}$ is a fixed point of $w(\sigma_1, \ldots, \sigma_k)$ we draw the following closed trail:

$$s_0 \xrightarrow{\sigma_{i_1}^{\alpha_1}} s_1 \xrightarrow{\sigma_{i_2}^{\alpha_2}} s_2 \xrightarrow{\sigma_{i_3}^{\alpha_3}} \ldots \xrightarrow{\sigma_{i_{m-1}}^{\alpha_{m-1}}} s_{m-1} \xrightarrow{\sigma_{i_m}^{\alpha_m}} s_0$$

Counting fixed points - categorizing trails

Let

$$s_0 \rightarrow \ldots \rightarrow s_{m-1} \rightarrow s_0$$

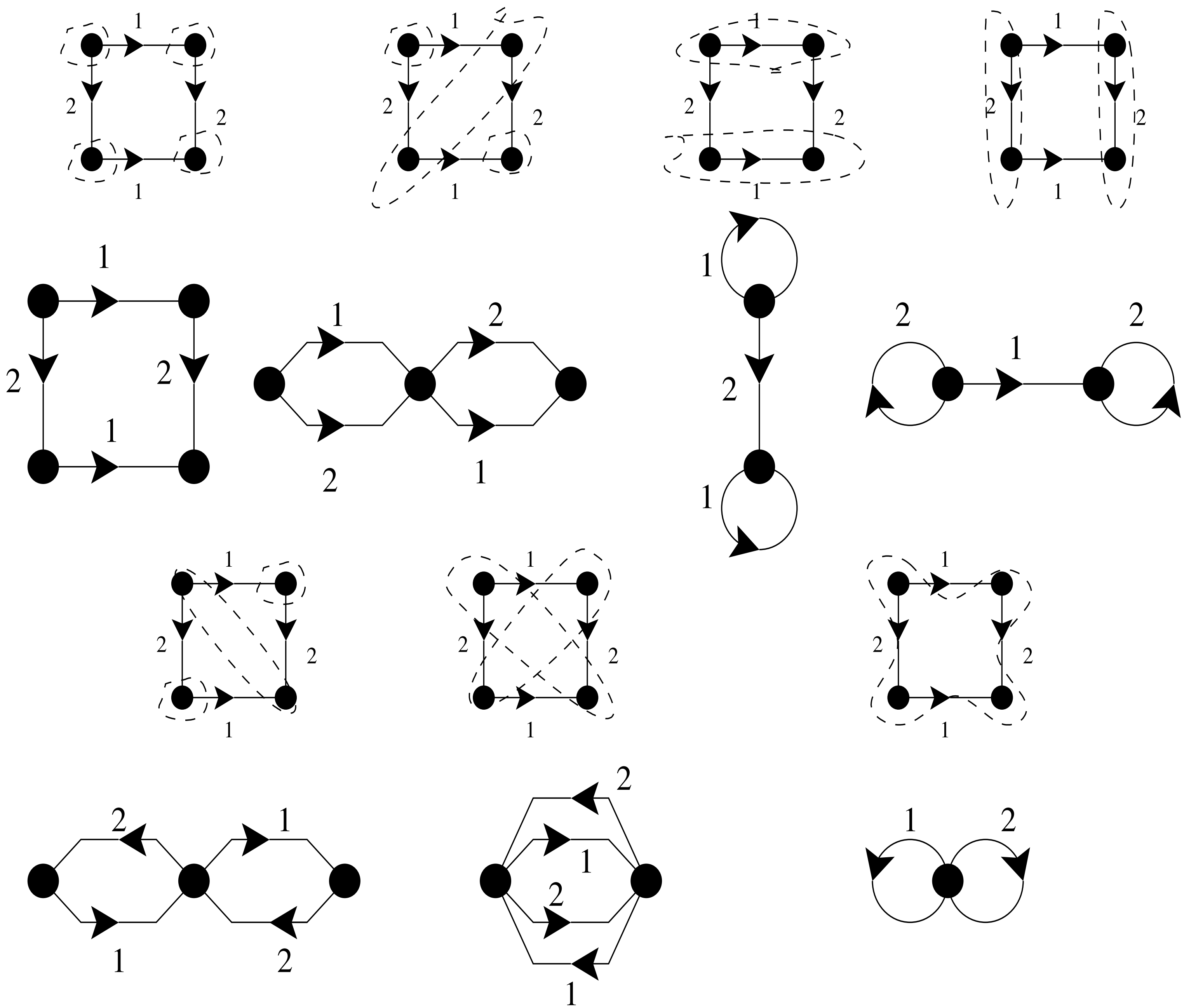
and

$$s'_0 \rightarrow \ldots \rightarrow s'_{m-1} \rightarrow s'_0$$

be the trails of the fixed points s_0 and s'_0 in $w(\sigma_1, \ldots \sigma_k)$ and $w(\sigma'_1, \ldots, \sigma'_k)$ respectively.

We put them in the same category, if they have the same coincidence pattern, i.e., if $\forall i, j \in \{0, \ldots, m-1\}$,

$$s_i = s_j \Leftrightarrow s'_i = s'_j$$



A counting formula for fixed points

It is now easy to count the number of realizations for every Γ that is a consistent quotient of the word w.

$$N_\Gamma(n) = n(n-1)\ldots(n-v_\Gamma+1)\prod_{j=1}^{k}\left[n-e_\Gamma^j\right]!$$

Counting fixed points (contd.)

Therefore

$$\mathbb{E}(X_w^{(n)}) = \frac{1}{(n!)^k} \sum_{\sigma_1,\ldots,\sigma_k \in S_n} X_w^{(n)}(\sigma_1,\ldots,\sigma_k) = \frac{1}{(n!)^k} \sum_{\Gamma \in \mathcal{Q}_w} N_\Gamma(n) =$$

$$\sum_{\Gamma \in \mathcal{Q}_w} \left(\frac{1}{n}\right)^{e_\Gamma - v_\Gamma} \frac{\prod_{l=1}^{v_\Gamma - 1}(1 - \frac{l}{n})}{\prod_{j=1}^{k} \prod_{l=1}^{e_\Gamma^j - 1}(1 - \frac{l}{n})}$$

The beginning of the end...

- Study the Taylor expansion of $\sum_{\Gamma \in \mathcal{Q}_w} \left(\frac{1}{n}\right)^{e_\Gamma - v_\Gamma} \frac{\prod_{l=1}^{v_\Gamma - 1}(1-\frac{l}{n})}{\prod_{j=1}^{k} \prod_{l=1}^{e_\Gamma^j - 1}(1-\frac{l}{n})}$

- Concentrate on the leading term: $\left(\frac{1}{n}\right)^{e_\Gamma - v_\Gamma}(1 + o(1))$

- Note that the exponent $e_\Gamma - v_\Gamma + 1$, that determines the highest order term is the Euler Characteristic of Γ.

- For Nica's Theorem - A quotient Γ of a cycle satisfies $e_\Gamma - v_\Gamma = 0$ iff Γ is a cycle as well.

... and beyond ...

The route to a proof of Friedman's conjecture should probably start with questions such as:

Conjecture 3. *For every formal word w and for every $n \geq 1$,*

$$\mathbb{E}(X_w^{(n)}) \geq 1$$