Lecture 7

The Brunn-Minkowski Theorem and Influences of Boolean Variables

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Theorem 7.1 (Brunn-Minkowski). *If $A, B \subseteq \mathbb{R}^n$ satisfy some mild assumptions (in particular, convexity suffices), then*

$$[\mathrm{vol}\,(A+B)]^{\frac{1}{n}} \geq [\mathrm{vol}\,(A)]^{\frac{1}{n}} + [\mathrm{vol}\,(B)]^{\frac{1}{n}}$$

where $A + B = \{a + b : a \in A$ and $b \in B\}$.

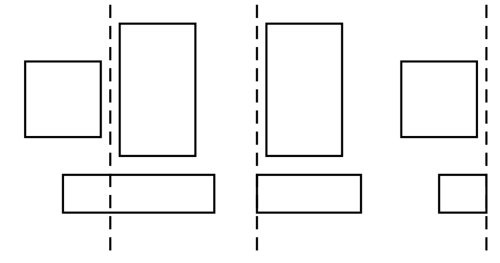
Proof. First, suppose that A and B are axis aligned boxes, say $A = \prod_{j=1}^{n} I_j$ and $B = \prod_{i=1}^{n} J_i$, where each I_j and J_i is an interval with $|I_j| = x_j$ and $|J_i| = y_i$. We may assume WLOG that $I_j = [0, x_j]$ and $J_i = [0, y_i]$ and hence $A + B = \prod_{i=1}^{n}[0, x_i + y_i]$. For this case, the BM inequality asserts that

$$\prod_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}} \geq \prod_{i=1}^{n} x_i^{\frac{1}{n}} \cdot \prod_{i=1}^{n} y_i^{\frac{1}{n}}$$

$$\Leftrightarrow \qquad 1 \geq \left[\prod\left(\frac{x_i}{x_i + y_i}\right)\right]^{\frac{1}{n}} \cdot \left[\prod\left(\frac{y_i}{x_i + y_i}\right)\right]^{\frac{1}{n}}$$

Now, since the arithmetic mean of n numbers is bounded above by their harmonic mean, we have $(\prod \alpha_i)^{\frac{1}{n}} \leq \frac{\sum \alpha_i}{n}$ and $(\prod(1-\alpha_i))^{\frac{1}{n}} \leq \frac{\sum(1-\alpha_i)}{n}$. Taking $\alpha_i = \frac{x_i}{x_i+y_i}$ and hence $1 - \alpha_i = \frac{y_i}{x_i+y_i}$, we see that the above inequality always holds. Hence the BM inequality holds whenever A and B are axis aligned boxes.

Now, suppose that A and B are the disjoint union of axis aligned boxes. Suppose that $A = \bigcup_{\alpha \in \mathcal{A}} A_\alpha$ and $B = \bigcup_{\beta \in \mathcal{B}} B_\beta$. We proceed by induction on $|\mathcal{A}| + |\mathcal{B}|$. We may assume WLOG that $|\mathcal{A}| > 1$. Since the boxes are disjoint, there is a hyperplane separating two boxes in $\mathcal{A}$. We may assume WLOG that this hyperplane is $x_1 = 0$.



Let $A^+ = \{x \in A : x_1 \geq 0\}$ and $A^- = \{x \in A : x_1 \leq 0\}$ as shown in the figure above. It is clear that both A^+ and A^- are the disjoint union of axis aligned boxes. In fact, we may let $A^+ = \bigcup_{\alpha \in \mathcal{A}^+} A_\alpha$ and $A^- = \bigcup_{\alpha \in \mathcal{A}^-} A_\alpha$ where $|\mathcal{A}^+| < |\mathcal{A}|$ and $|\mathcal{A}^-| < |\mathcal{A}|$. Suppose that $\frac{\mathrm{vol}(A^+)}{\mathrm{vol}(A)} = \alpha$. Pick a λ so that

$$\frac{\mathrm{vol}\left(\{x \in B : x_1 \geq \lambda\}\right)}{\mathrm{vol}\left(B\right)} = \alpha$$

We can always do this by the mean value theorem because the function $f(\lambda) = \frac{\mathrm{vol}(\{x \in B : x_1 \geq \lambda\})}{\mathrm{vol}(B)}$ is continuous, and $f(\lambda) \to 0$ as $\lambda \to \infty$ and and $f(\lambda) \to 1$ as $\lambda \to -\infty$.

Let $B^+ = \{x \in B : x_1 \geq \lambda\}$ and $B^- = \{x \in B : x_1 \leq \lambda\}$. By induction, we may apply BM to both (A^+, B^+) and (A^-, B^-), obtaining

$$\left[\mathrm{vol}\left(A^+ + B^+\right)\right]^{\frac{1}{n}} \geq \left[\mathrm{vol}\left(A^+\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B^+\right)\right]^{\frac{1}{n}}$$
$$\left[\mathrm{vol}\left(A^- + B^-\right)\right]^{\frac{1}{n}} \geq \left[\mathrm{vol}\left(A^-\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B^-\right)\right]^{\frac{1}{n}}$$

Now,

$$\left[\mathrm{vol}\left(A^+\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B^+\right)\right]^{\frac{1}{n}} = \alpha^{\frac{1}{n}} \left[\left[\mathrm{vol}\left(A\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B\right)\right]^{\frac{1}{n}}\right]$$
$$\left[\mathrm{vol}\left(A^-\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B^-\right)\right]^{\frac{1}{n}} = (1-\alpha)^{\frac{1}{n}} \left[\left[\mathrm{vol}\left(A\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B\right)\right]^{\frac{1}{n}}\right]$$

Hence

$$\left[\mathrm{vol}\left(A^+ + B^+\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(A^- + B^-\right)\right]^{\frac{1}{n}} \geq \left[\left[\mathrm{vol}\left(A\right)\right]^{\frac{1}{n}} + \left[\mathrm{vol}\left(B\right)\right]^{\frac{1}{n}}\right]$$

The general case follows by a limiting argument (without the analysis for the case where equality holds). □

Suppose that $f : \mathbb{S}^1 \to \mathbb{R}$ is a mapping having a Lipshitz constant 1. Hence

$$\|f(x) - f(y)\| \leq \|x - y\|_2$$

Let μ be the median of f, so

$$\mu = \mathrm{prob}\left[\{x \in \mathbb{S}^n : f(x) < \mu\}\right] = \frac{1}{2}$$

We assume that the probability distribution always admits such a μ (at least approximately). The following inequality holds for every $\epsilon > 0$ as a simple consequence of the isoperimetric inequality on the sphere.

$$\{\mathrm{x} \in \mathbb{S}^{\mathrm{n}} : |\mathrm{f} - \mu| > \epsilon\} < 2e^{-\epsilon n/2}$$

For $A \subseteq \mathbb{S}^n$ and for $\epsilon > 0$, let

$$A_\epsilon = \{x \in \mathbb{S}^n : \operatorname{dist} x, A < \epsilon\}$$

Question 7.1. Find a set $A \subseteq \mathbb{S}^n$ with $\mathrm{A} = a$ for which A_ϵ is the smallest.

The probability used here is the (normalized) Haar measure. The answer is always a spherical cap, and in particular if $a = \frac{1}{2}$, then the best A is the hemisphere (and so $A_\epsilon = \{x \in \mathbb{S}^n : x_1 < \epsilon\}$). We will show that for $A \subseteq \mathbb{S}^n$ with $\mathrm{A} = \frac{1}{2}$, $\mathrm{A}_\epsilon \geq 1 - 2e^{-\epsilon^2 n/4}$. If A is the hemisphere, then $\mathrm{A}_\epsilon = 1 - \Theta(e^{-\epsilon^2 n/2})$, and so the hemisphere is the best possible set.

But first, a small variation on BM :

$$\operatorname{vol}\left(\frac{A+B}{2}\right) \geq \sqrt{\operatorname{vol}(A) \cdot \operatorname{vol}(B)}$$

This follows from BM because

$$\begin{aligned}\operatorname{vol}\left(\frac{A+B}{2}\right)^{\frac{1}{n}} &\geq \operatorname{vol}\left(\frac{A}{2}\right)^{\frac{1}{n}} + \operatorname{vol}\left(\frac{B}{2}\right)^{\frac{1}{n}} \\ &= \frac{1}{2}\left[\operatorname{vol}(A)^{\frac{1}{n}} + \operatorname{vol}(B)^{\frac{1}{n}}\right] \\ &\geq \sqrt{\operatorname{vol}(A)^{\frac{1}{n}} + \operatorname{vol}(B)^{\frac{1}{n}}}\end{aligned}$$

For $A \subseteq \mathbb{S}^n$, let $\tilde{A} = \{\lambda a : a \in A, 1 \geq \lambda \geq 0\}$. Then $\mathrm{A} = \mu_{n+1}(\tilde{A})$. Let $B = \mathbb{S}^n \setminus A_\epsilon$.

Lemma 7.2. *If $\tilde{x} \in \tilde{A}$ and $\tilde{y} \in \tilde{B}$, then*

$$\left|\frac{\tilde{x}+\tilde{y}}{2}\right| \leq 1 - \frac{\epsilon^2}{8}$$

It follows that $\frac{\tilde{A}+\tilde{B}}{2}$ is contained in a ball of radius at most $1 - \frac{\epsilon^2}{8}$. Hence

$$\begin{aligned}\left(1 - \frac{\epsilon^2}{8}\right)^{n+1} &\geq \operatorname{vol}\left(\frac{\tilde{A}+\tilde{B}}{2}\right) \\ &\geq \sqrt{\operatorname{vol}\left(\tilde{A}\right) \cdot \operatorname{vol}\left(\tilde{B}\right)} \\ &\geq \sqrt{\frac{\operatorname{vol}\left(\tilde{B}\right)}{2}}\end{aligned}$$

Therefore, $2e^{-\epsilon^2 n/4} \geq \operatorname{vol}\left(\tilde{B}\right)$.

7.1 Boolean Influences

Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function. For a set $S \subseteq [n]$, the influence of S on f, $I_f(S)$ is defined as follows. When we pick $\{x_i\}_{i \notin S}$ uniformly at random, three things can happen.

1. $f = 0$ regardless of $\{x_i\}_{i \in S}$ (suppose that this happens with probability q_0).
2. $f = 1$ regardless of $\{x_i\}_{i \in S}$ (suppose that this happens with probability q_1).
3. With probability $\mathrm{Inf}_f(S) := 1 - q_0 - q_1$, f is still undetermined.

Some examples:

- (Dictatorship) $f(x_1, x_2, \ldots, x_n) = x_1$. In this case

$$\mathrm{Inf}_{\textsf{dictatorship}}(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

- (Majority) For $n = 2k+1$, $f(x_1, x_2, \ldots, x_n)$ is 1 if and only if a majority of the x_i are 1. For example, if $S = \{1\}$,

$$\mathrm{Inf}_{\textsf{majority}}(\{1\}) = \mathrm{prob}\left(x_1 \text{ is the tie breaker }\right) = \frac{\binom{2k}{k}}{2^{2k}} = \Theta\left(\frac{1}{\sqrt{k}}\right)$$

  For fairly small sets S,

$$\mathrm{Inf}_{\textsf{majority}}(S) = \Theta\left(\frac{|S|}{\sqrt{n}}\right)$$

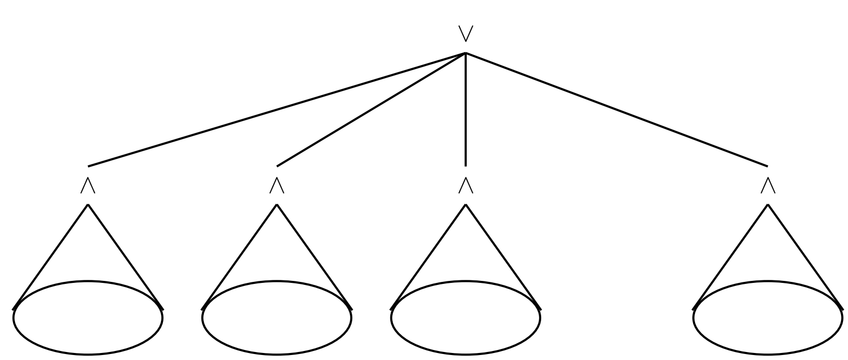
- (Parity) $f(x_1, x_2, \ldots, x_n) = 1$ if and only if an even number of the x_i's are 1. In this case

$$\mathrm{Inf}_{\textsf{parity}}(\{x_i\}) = 1$$

  for every $1 \leq i \leq n$.

Question 7.2. What is the smallest $\delta = \delta(n)$ such that there exists a function $f : \{0,1\}^n \to \{0,1\}$ which is balanced (i.e., $Ef = \frac{1}{2}$) for which $\mathrm{Inf}_f(\{x_i\}) < \delta$ for all x_i ?

Consider the following example, called tribes. The set of inputs $\{x_1, x_2, \ldots, x_n\}$ is partitioned into tribes of size b each. Here, $f(x_1, x_2, \ldots, x_n) = 1$ if and only if there is a tribe that unanimously 1.



Since we want $Ef = \frac{1}{2}$, we must have $\text{prob}(f = 0) = \left(1 - \frac{1}{2^b}\right)^{\frac{n}{b}} = \frac{1}{2}$. Therefore, $\frac{n}{b} \ln\left(1 - \frac{1}{2^b}\right) = -\ln 2$. We use the Taylor series expansion for $\ln(1-\epsilon) = -\epsilon - \epsilon^2/2 - \cdots = -\epsilon - O(\epsilon^2)$ to get $\frac{n}{b}\left(\frac{1}{2^b} + O\left(\frac{1}{4^b}\right)\right) = -\ln 2$. This yields $n = b \cdot 2^b \ln 2\,(1 + O(1))$. Hence $b = \log_2 n - \log_2 \ln n + \Theta(1)$.

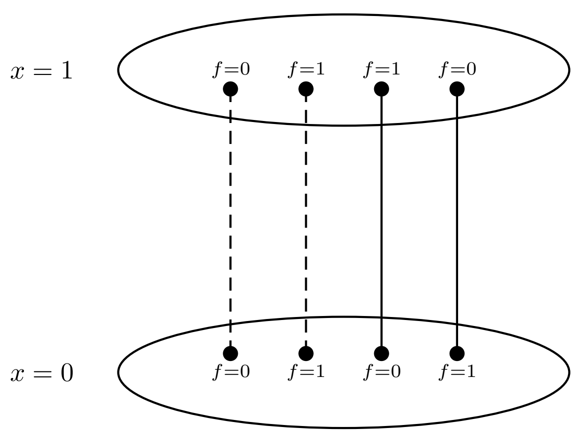
Hence,

$$\begin{aligned}
\text{Inf}_{\mathsf{tribes}}(x) &= \left(1 - \frac{1}{2^b}\right)^{\frac{n/b}{-}1} \cdot \left(\frac{1}{2}\right)^{b-1} \\
&= \frac{\left(1 - \frac{1}{2^b}\right)^{\frac{n}{b}}}{1 - \frac{1}{2^b}} \cdot \frac{1}{2^{b-1}} \\
&= \frac{1}{1 - \frac{1}{2^b}} \cdot \frac{1}{2^b} \\
&= \frac{1}{2^{b-1}} = \Theta\left(\frac{\log b}{n}\right)
\end{aligned}$$

In this example, each individual variable has influence $\Theta(\log n/n)$. It was later shown that this is lowest possible influence.

Proposition 7.3. *If $Ef = \frac{1}{2}$, then $\sum_x \text{Inf}_f(x) \geq 1$.*

This is a special case of the edge isoperimetric inequality for the cube, and the inequality is tight if f is dictatorship.



The variable x is influential in the cases indicated by the solid lines, and hence

$$\text{Inf}_f(x) = \frac{\#\text{ of mixed edges}}{2^{n-1}}$$

Let $S = f^{-1}(0)$. Then $\sum \text{Inf}_f(x) = \frac{1}{2^{n-1}} e(S, S^c)$.

One can use $\hat{f}$ to compute influences. For example, if f is monotone (so $x \prec y \Rightarrow f(x) \leq f(y)$), then

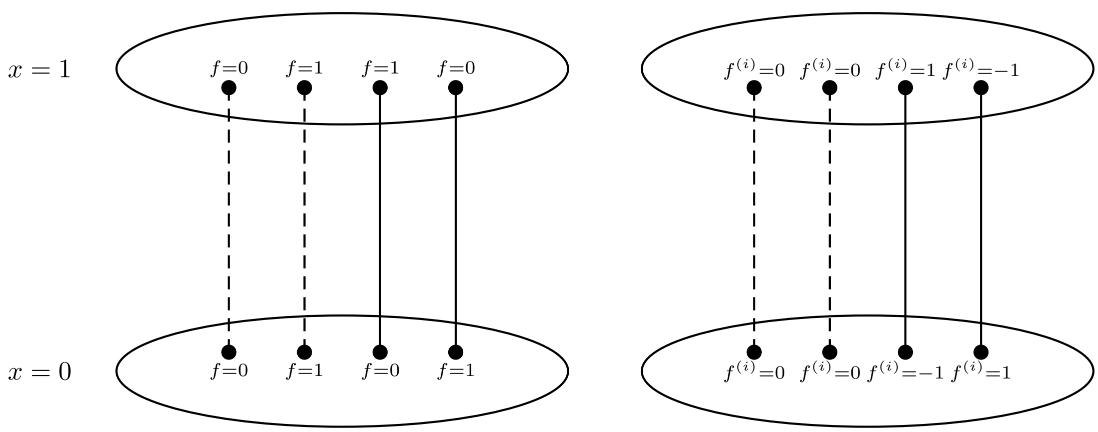
$$\hat{f}(S) = \sum_T \frac{(-1)^{|S \cap T|}}{2^n}$$

Therefore,

$$\begin{aligned}
\hat{f}(\{i\}) &= \frac{1}{2^n} \sum_{i \notin T} f(T) - \frac{1}{2^n} \sum_{i \in T} f(T) \\
&= \frac{1}{2^n} \sum_{i \notin T} (f(T) - f(T \cup \{i\})) \\
&= \frac{-1}{2^n} \cdot \#\text{ mixed edges in the direction of } i \\
&= -\frac{1}{2} \mathrm{Inf}_f\,(x_i)
\end{aligned}$$

Hence $\mathrm{Inf}_f\,(x_i) = -2\hat{f}(\{i\})$. What can be done to express $\mathrm{Inf}_f\,(x)$ for a general f? Define

$$f^{(i)}(z) = f(z) - f(z \oplus e_i)$$



Then

$$\mathrm{Inf}_f\,(x_i) = \left|\text{support} f^{(i)}\right| = \sum_w \left(f^{(i)}(w)\right)^2$$

The last term will be evaluated using Parseval. For this, we need to compute the Fourier expression of $f^{(i)}$ (expressed in terms of $\hat{f}$).

$$\begin{aligned}
\widehat{f^{(i)}}(S) &= \frac{1}{2^n}\sum_T f^{(i)}(T)(-1)^{|S\cap T|} \\
&= \frac{1}{2^n}\sum_T \Big[f(T) - f(T\oplus\{i\})\Big](-1)^{|S\cap T|} \\
&= \frac{1}{2^n}\sum_{i\notin T}\left(\Big[f(T) - f(T\cup\{i\}\Big](-1)^{|S\cap T|} + \Big[f(T\cup\{i\}) - f(T)\Big](-1)^{|S\cap(T\cup\{i\})|}\right) \\
&= \frac{1}{2^n}\sum_{i\notin T}\Big[f(T) - f(T\cup\{i\}\Big]\left((-1)^{|S\cap T|} - (-1)^{|S\cap(T\cup\{i\})|}\right) \\
&= \begin{cases} 0 & \text{if } i\notin S \\ 2\hat{f}(S) & \text{if } i\in S \end{cases}
\end{aligned}$$

Using Parseval on $\widehat{f^{(i)}}$ along with the fact that $\widehat{f^{(i)}}$ takes on only values $\{0,\pm 1\}$, we conclude that

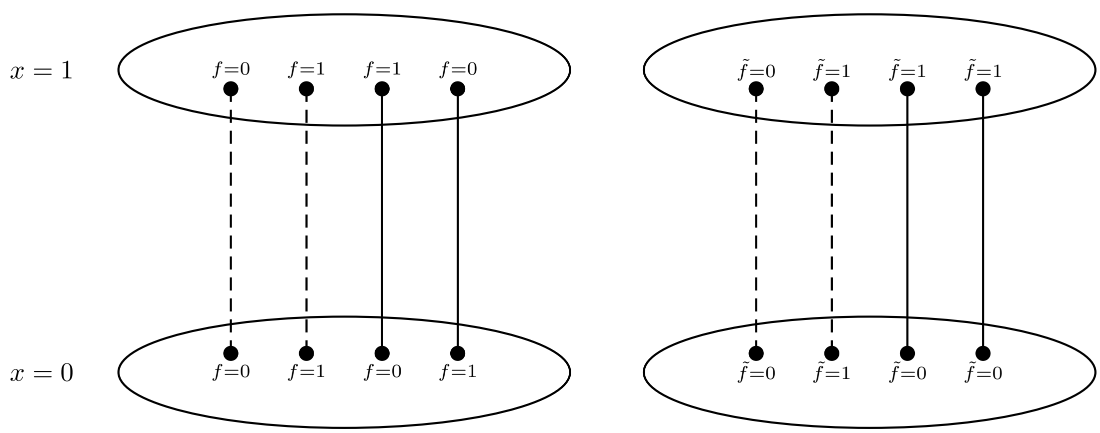
$$\mathrm{Inf}_f\,(x_i) = 4\sum_{i\in S}|hatf(S)|^2$$

Next time, we will show that if $Ef = \frac{1}{2}$, then there exists a i such that $\sum_{i\in S}\left(\hat{f}(S)\right)^2 > \Omega(\ln n/n)$.

Lemma 7.4. *For every* $f : \{0,1\}^n \to \{0,1\}$*, there is a monotone* $g : \{0,1\}^n \to \{0,1\}$ *such that*

- $Eg = Ef$.
- *For every* $s \subseteq [n]$, $\mathrm{Inf}_g\,(S) \le \mathrm{Inf}_f\,(S)$.

Proof. We use a shifting argument.



Clearly $E\tilde{f} = Ef$. We will show that for all S, $\mathrm{Inf}_{\tilde{f}}\,(S) \le \mathrm{Inf}_f\,(S)$. We may keep repeating the shifting step until we obtain a monotone function g. It is clear that the process will terminate by considering the progress measure $\sum f(x)\,|x|$ which is strictly increasing. Therefore, we only need show that $\mathrm{Inf}_{\tilde{f}}\,(()\,S) \le \mathrm{Inf}_f\,(S)$.

$\square$