Lecture 5

Isoperimetric Problems

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Lecturer: Nati Linial
Notes: Yuhan Cai & Ioannis Giotis

Codes: densest sphere packing in $\{0,1\}^n$.

$A(n,d) = max\{|\varphi|, \varphi \subseteq \{0,1\}^n, dist(\varphi) \geq d\}$

$R(\delta) = limsup\{\frac{1}{n}\log_2(\varphi)|\varphi \subseteq \{0,1\}^n, dist(\varphi) \geq \delta_n\}$

'Majority is the stablest' -

- Gaussian: $\frac{1}{(2\pi)^(n/2)} e^{-\|x\|^2/2}$
- Borell: isoperimetric problem is solved by a half-space

Isoperimetric Questions on the cube (Harper): Vertex and Edge isoperimetric questions.

The edge problem is defined as follows: Given that $S \subseteq \{0,1\}^n, |S| = R$, how small $e(S, \bar{S})$ be?

Answer: $\forall S \subseteq \{0,1\}^n, e(S) \leq 1/2|S|\log_2|S|, |S| = 2^k, S = \{(* \ldots * 0 \ldots 0\}$ with k *s.

Proof (induction on dim):

$e(S) \leq e(S_0) + e(S_1) + |S_0|, |S| = x, |S_0| \geq \alpha x, \alpha < 1/2.$

$1/2x\log_2 x \geq 1/2(\alpha x)\log_2 \alpha x) + 1/2(1-\alpha)x\log[(1-\alpha)x] + \alpha x$

$0 \geq \alpha\log\alpha + (1-\alpha)\log(1-\alpha) + 2\alpha$

$H(\alpha) \geq 2\alpha$ at $\alpha = 0, 1/2$.

The vertex isoperimetric problem is defined as $min\sharp\{y \mid y \notin S, \exists x \in S\}$ such that $xy \in E(\langle 0,1\rangle^n)$, $S \subseteq \{0,1\}^n$, $|S| \leq k$. The answer is an optimal S-ball. Specifically, if $k = |S| = \sum_{j=0}^{t}\binom{n}{j}$, then $|S| \geq \binom{n}{t+1}$.

We will use the Kraskal-Katona theorem. If $f \subseteq \left(\begin{array}{c}\text{[n]}\\ \text{k}\end{array}\right)$, then the *shadow of f* is

$$\sigma(f) = \left\{ y \in \left(\begin{array}{c}\text{[n]}\\ \text{k}\end{array}\right) \mid \exists x \in f, x \supseteq y \right\}$$

We wish to minimize $|\sigma(f)|$.

To do this, take f as an initial segment in the reverse lexicographic order. The lexicographic order is defined as

$$A < B, \text{ if } min(A\backslash B) < min(B\backslash A)$$

while the reverse lexicographic order is

$$A <_{RL} B, \text{ if } max(A\backslash B) <_{RL} \max(B\backslash A)$$

For example:

$$\begin{aligned} Lex &: \langle 1,2\rangle\langle 1,3\rangle\langle 1,4\rangle, \ldots \\ RLex &: \langle 1,2\rangle\langle 1,3\rangle\langle 2,3\rangle, \ldots \end{aligned}$$

Margulis and Talagrand gave the following definition for $S \subseteq \langle 0,1\rangle^n$

$$h(x) = \{y \notin S \mid xy \in E\}, x \in S$$

We now have the 2 problems

- Vertex Isoperimetric, $min_{|S|=k} \sum_{x\in S}(h(x))^{0\to\rho=0}$
- Edge Isoperimetric, $min_{|S|=k} \sum_{x\in S}(h(x))^{\to\rho=1}$

We have $|S| \geq 2^{n-1} \Rightarrow \Sigma\sqrt{h(x)} \geq \Omega(2^n)$, for $p = 1/2$.

Kleitman: $|S| = \Sigma_{j=0}^{t}\binom{n}{j}$, $S \subseteq \{0,1\}^n$, $t < n/2 \Rightarrow diam(S) \geq 2t$. Can you show that S necessarily contains a large code?

Question: (answered by Friedgut) suppose that $|S| \simeq 2^{n-1}$ and $\varphi(S, S^C) \sim 2^{n-1}$, then is S roughly a dictatorship?

Answer: yes. subcube $x_1 = 0 \Leftrightarrow f(x_1, \ldots, x_n) = x_1$. $R(\delta) = limsup_{n\to\infty}\{\frac{1}{n}\log(\varphi)|\varphi \subseteq \{0,1\}^n, dist(\varphi) \geq \delta n\}$.

5.1 Delsarte's LP

Having $g = 1_C$, $f = 2^n g * g/|C|$, Delsarte's LP is

$$\begin{aligned} A(n,d) \leq max\Sigma_{x\in\{0,1\}^n} f(x) \\ f \geq 0 \\ f(\mathbf{0}) = 1 \\ \hat{f} \geq 0 \\ f|_{1,\ldots,d-1} = 0 \end{aligned}$$

Some useful equations

$$g * g(0) = \frac{1}{2^n}\sum g(y)g(y) = \frac{|C|}{2^n}$$

$$g * g(S) = \frac{1}{2^n}\sharp\{x, y \in C \mid x \oplus y = S\}$$

We start with an observation. Without loss of generality, f is symmetric or in other words $f(x)$ depends only on $|x| = \alpha_{|x|}$. We look for $\alpha_0 = 1, \alpha_1 = \ldots = \alpha_{d-1} = 0$, $\alpha_d, \ldots, \alpha_n \geq 0$.

We've expressed $f \geq 0, f(\hat{0}) = 1$ and we are trying to maximize $\sum \binom{n}{j} \alpha_j$.

$$\begin{aligned} L_j &= \{x \in \{0,1\}^n, |x| = j\} \\ f &= \sum_{j=0}^{n} \alpha_j 1_{L_j} \\ \hat{f} &= \sum_j \alpha_j \hat{1}_{L_j} \end{aligned}$$

Note that L_j is symmetric. It also follows that $\hat{1}_{L_j}$ is symmetric. We need to know $\hat{1}_{L_j}$ if $|y| = t$.

$$\begin{aligned} \hat{\phi}(T) &= \sum \phi(S)(-1)^{|S \cap T|} \\ \hat{1}_{L_j}(T) &= \sum_{|S|=j} (-1)^{|S \cap T|} \\ K_j^{(n)}(x) &= \sum_i (-1)^i \binom{t}{i} \binom{n-t}{j-i} \end{aligned}$$

This is the *Krawtchouk* polynomial presented in the next section.

5.2 Orthogonal Polynomials on $\mathbb{R}$

Interesting books for this section are "Interpolation and Approximation" by Davis and "Orthogonal polynomials" by Szegö.

The weights of orthogonal polynomials on $\mathbb{R}$ are defined by

$$w : \mathbb{R} \to \mathbb{R}^+ \ , \ \int_{\mathbb{R}} w(x) < \infty$$

The inner product on $f : \mathbb{R} \to \mathbb{R}$ is

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) w(x) \, dx$$

and with weights $w_1, w_2, \ldots$, and points $x_1, x_2, \ldots$

$$\langle f, g \rangle = \sum w_i f(x_i) g(x_i)$$

Let's now talk about orthogonality. Start from the functions $1, x, x^2, \ldots$ and carry out a Gram-Schmidt orthogonalization process. You'll end up with a sequence of polynomials $P_0(x), P_1(x), \ldots$ s.t. P_i has degree i and $\langle P_i, P_j \rangle = \delta_{ij}$.

One case of orthogonal polynomials are the *Krawtchouk* polynomials, on discrete points $x_0 = 0, x_1 = 1, \ldots, x_n = n$ with $w_j = \binom{n}{j}/2^n$. The j-th Krawtchouk polynomial $K_j(x)$ is a degree j polynomial in x. It is also the value of $\hat{1}_{L_j}(T)$ whenever $|T| = x$.

$$K_j^{(n)}(x) = \sum_{i=0}^{n}(-1)^i \binom{x}{i}\binom{n-x}{j-i}$$

Let's see why are they orthogonal or in other words

$$\frac{1}{2^n}\sum_{i=0} K_p(i)K_q(i)\binom{n}{i} = \delta_{pq}\binom{n}{p}$$

Starting from

$$\langle 1_p, 1_q\rangle = \frac{1}{2^n}\binom{n}{p}\delta_{pq}$$

and using Parseval's identity we get

$$\langle \hat{1}_{L_p}, \hat{1}_{L_q}\rangle = \frac{1}{2^n}\sum K_p(|S|)K_q(|S|) = \frac{1}{2^n}\sum_{i=0} K_p(i)K_q(i)\binom{n}{i}$$

The first K_j's are

$$K_0(x) = 1, K_1(x) = n-2x, K_2(x) = \binom{x}{2} - (n-x) + \binom{n-x}{2} = \frac{(n-2x)^2 - n}{2}$$

We also have the following identity

$$K_j(n-x) = (-1)^j K_j(x)$$

Lemma 5.1. *Every system of orthogonal polynomials satisfies a 3-term recurrence*

$$xP_j = \alpha_j P_{j+1} + \beta_j P_j + \gamma_j P_{j-1}$$

Proof.

$$\begin{aligned} 1_{L_i} * 1_{L_j}(S) &= \frac{1}{2^n}\sum_i 1_{L_j}(S \oplus i) = \\ &= \frac{1}{2^n}((j+1)1_{L_{j+1}} + (n-j+1)1_{L_{j-1}}) = \\ &= \frac{1}{2^n}((j+1)1_{L_{j+1}} + (n-j+1)1_{L_{j-1}}) \end{aligned}$$

For the Krawtchouk polynomials

$$\begin{aligned} K_i K_j &= (j+1)K_{j+1} + (n-j+1)K_{j-1} \\ (n-2x)K_j &= (j+1)K_{j+1} + (n-j+1)K_{j-1} \end{aligned}$$

□

Theorem 5.2. *For every family of orthogonal polynomials there is*

1. *a 3-term recurrence relation*

$$x \cdot P_j = \alpha_j P_{j+1} + \beta_j P_j + \gamma_j P_{j-1}$$

2. P_j *has* j *real roots all in* $conv[supp\ w]$.

Proof. Observe that $P_0, P_1, \dots, P_t$ form a basis for the space of all polynomials of degree $\leq t$, which means that $\langle P, Q \rangle = 0$, $\forall Q$ polynomials of degree j

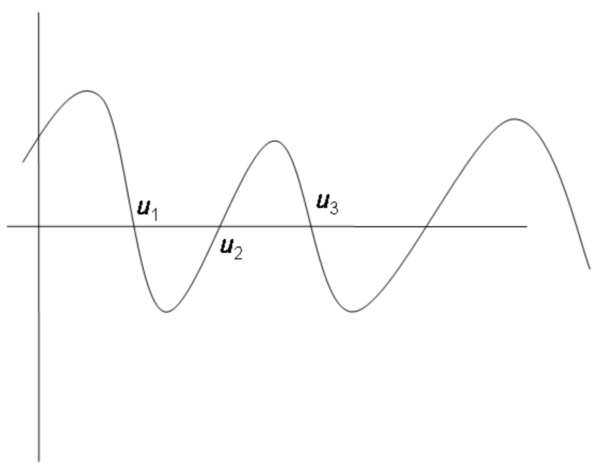
$$x \cdot P_j = \sum_{i=0}^{j+1} \lambda_i P_i \tag{5.1}$$

We now claim that $\lambda_0 = \lambda_1 = \cdots = \lambda_{j-2} = 0$. Let's take in (5.1) an inner product with P_l,$l < j - 1$.

$$\langle xP_j, P_l \rangle = \sum_{i=0}^{j+1} \lambda_i \langle P_i, P_j \rangle = \lambda_l \|P_l\|^2$$

$$\langle P_j, xP_l \rangle = \lambda_l \|P_l\|^2$$

which is 0 for P_l of degree $\leq j - 1$. □



If u_i's are the zeros of P_j of odd multiplicity then

$$0 = \langle P_j, \prod (x - u_i) \rangle = P_j \prod (x - u_j) > 0$$