On the Number of 4-Cycles in a Tournament

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Abstract: If T is an n-vertex tournament with a given number of 3-cycles, what can be said about the number of its 4-cycles? The most interesting range of this problem is where T is assumed to have $c \cdot n^3$ cyclic triples for some $c > 0$ and we seek to minimize the number of 4-cycles. We conjecture that the (asymptotic) minimizing T is a random blow-up of a constant-sized transitive tournament. Using the method of flag algebras, we derive a lower bound that almost matches the conjectured value. We are able to answer the easier problem of maximizing the number of 4-cycles. These questions can be equivalently stated in terms of transitive subtournaments. Namely, given the number of transitive triples in T, how many transitive quadruples can it have? As far as we know, this is the first study of inducibility in tournaments. J. Graph Theory 83: 266–276, 2016

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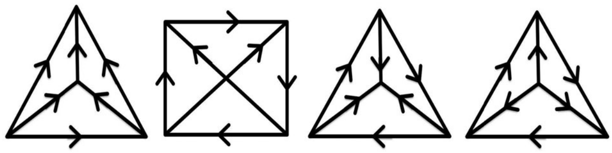

FIGURE 1. T_4, C_4, W, L

Keywords: *local graph theory; tournaments; extremal combinatorics; small cycles in tournaments*

1. INTRODUCTION AND NOTATION

A. Notation

For tournaments T, H, let $\mathbf{pr}(H, T)$ be the probability that a random set of $|H|$ vertices in T spans a subtournament isomorphic to H. For an infinite family of tournaments $\mathcal{T}$, let $\mathbf{pr}(H, \mathcal{T}) = \lim_{T \in \mathcal{T}, |T| \to \infty} \mathbf{pr}(H, T)$, assuming the limit exists. (Nonexistence of the limit may be repaired, of course, by passing to an appropriate subfamily).

We denote the transitive m-vertex tournament by T_m, and the 3-vertex cycle by C_3. There are four isomorphism types of 4-vertex tournaments, see Figure 1.

- C_4 that is characterized by having a directed 4-cycle,
- the transitive T_4,
- W, a cyclic triangle and a sink,
- L, a cyclic triangle and a source.

Denote $r(\mathcal{T}) = \mathbf{pr}(R, \mathcal{T})$ for any letter $r \in \{c_3, c_4, t_k, w, l\}$ (e.g. $c_3(\mathcal{T})$ is the limit proportion of cyclic triangles in members of $\mathcal{T}$). We omit $\mathcal{T}$ when appropriate. We will always restrict ourselves to families for which all the relevant limits exist, though we do not bother to mention this any further.

In [8] we initiated the study of 4-local profiles of tournaments, namely the set

$\mathcal{P} = \{(t_4(\mathcal{T}), c_4(\mathcal{T}), w(\mathcal{T}), l(\mathcal{T})) | \mathcal{T}$ is a family of tournaments for which all the limits *exist*$\} \subseteq \mathbb{R}^4$.

Here, we continue with these investigations.

B. Our questions

In studying the set $\mathcal{P}$ of 4-local profiles of tournaments, it is of interest to understand its projection to the first two coordinates, which raises Problems 3 and 6 below. We are, in fact, interested in all the following six problems, but as we show below, they are interdependent.

(1) Maximize $c_4(\mathcal{T})$ when $c_3(\mathcal{T})$ is set.
(2) Maximize $t_4(\mathcal{T})$ when $t_3(\mathcal{T})$ is set.
(3) Maximize $c_4(\mathcal{T})$ when $t_4(\mathcal{T})$ is set.
(4) Minimize $c_4(\mathcal{T})$ when $c_3(\mathcal{T})$ is set.
(5) Minimize $t_4(\mathcal{T})$ when $t_3(\mathcal{T})$ is set.
(6) Minimize $c_4(\mathcal{T})$ when $t_4(\mathcal{T})$ is set.

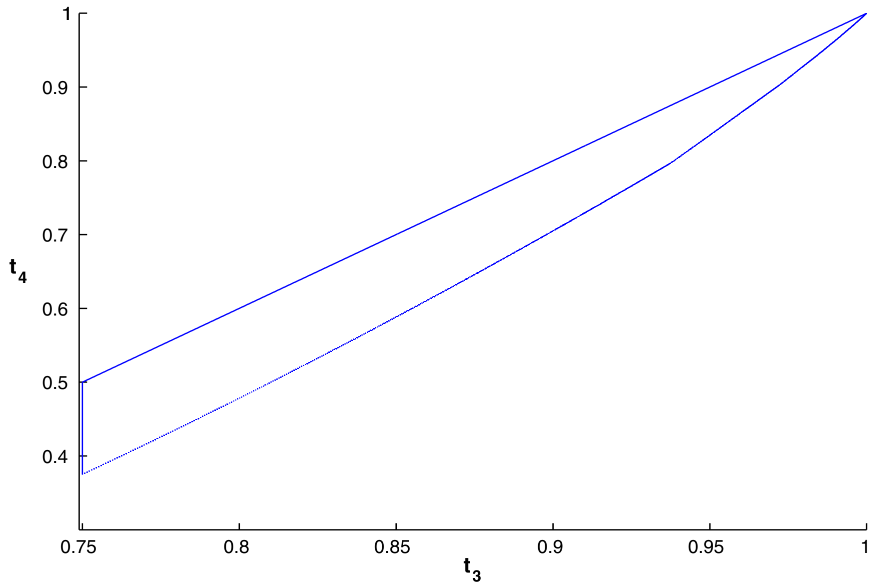

FIGURE 2. The boundary of the set $\{(t_3(\mathcal{T}), t_4(\mathcal{T}))\}$. The lower curve is conjectured.

Proposition 1.1. *Problems 1 to 3 are equivalent in the sense that the solution of any one of them can be transformed into a solution for the other two. Likewise Problems 4 to 6 are equivalent.*

To prove this proposition we need

Observation 1.2. $t_4 - c_4 = 1 - 4c_3$

Proof. We count cyclic triangles in 4-vertex tournaments. There are none in T_4, two in C_4 and one each in W and L. Therefore the number of cyclic triangles in an n-vertex tournament satisfies $c_3\binom{n}{3} = \frac{2c_4+w+l}{n}\binom{n}{4}$. The claim follows, since $t_4 + c_4 + w + l = 1$.■

We can now prove Proposition 1.1.

Proof of Proposition 1.1. Obviously, $t_3 + c_3 = 1$. Combined with Observation 1.2 this already proves that Problems 1 and 2 and Problems 4 and 5 are equivalent. To see that Problems 5 and 6 are equivalent, note that Problem 5 is equivalent to maximizing t_3 for given t_4, or, equivalently, to minimizing c_3 given t_4. The equivalence follows by Observation 1.2. A similar argument proves that Problems 2 and 3 are equivalent. ■

Problems 1 to 3 are rather straightforward and we proceed to solve them. Problems 4 to 6 are deeper. By the equivalence proved above, the discussion is restricted to problem 4. We state a conjecture on the solution of this problem and prove a lower bound. This problem raises interesting structural limitations on tournaments, on which we elaborate in Section 2. We defer the technical proofs to Section 3 and in Section 4 we offer some further directions.

The three regions $\{(t_3, t_4)\}$, $\{(c_3, c_4)\}$, $\{(t_4, c_4)\}$ of the realizable pairs of parameters are illustrated in Figures 2 to 4.

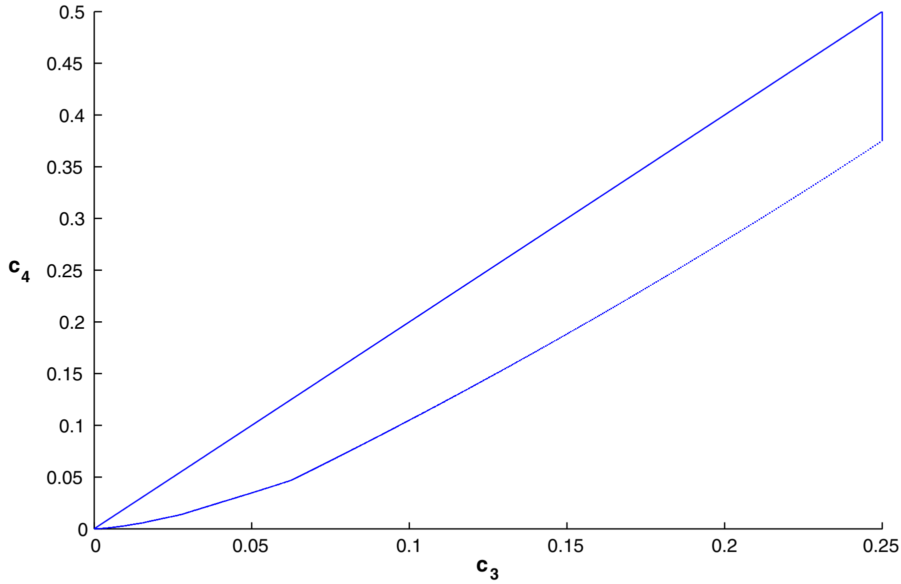

FIGURE 3. The boundary of the set $\{(c_3(\mathcal{T}), c_4(\mathcal{T}))\}$. The lower curve is conjectured.

We finally note that the planar sets $\{(t_3(\mathcal{T}), t_4(\mathcal{T}))\}$, $\{(c_3(\mathcal{T}), c_4(\mathcal{T}))\}$, $\{(t_4(\mathcal{T}), c_4(\mathcal{T}))\}$ are simply connected. This shows that these sets coincide with the bounded regions in Figures 2 to 4.

Lemma 1.3. *The set of all pairs $(c_3(\mathcal{T}), c_4(\mathcal{T}))$ is simply connected. Here $\mathcal{T}$ is an arbitrary family of tournaments for which these limits exist.*

The arguments introduced in Proposition 1.1 yield the same conclusion for the sets $\{(t_3, t_4)\}$, $\{(t_4, c_4)\}$. We prove the lemma in Section 3.

2. RESULTS AND CONJECTURES

A. The maximum

In this subsection, we solve Problems 1 to 3:

Observation 2.1. *The following inequalities hold in all tournaments. These inequalities are tight.*

- $c_4 \leq 2c_3$.
- $t_4 \leq 2t_3 - 1$.
- $c_4 \leq \min\{t_4, 1 - t_4\}$.

Proof. Clearly $t_4 + c_4 \leq 1$, since $t_4 + c_4 + w + l = 1$. Also, as we saw $t_4 - c_4 = 1 - 4c_3$ and $c_3 + t_3 = 1$. In addition it is well-known and easy to show that $c_3 \leq \frac{1}{4}$. All the inequalities follow. To show that the first two inequalities are tight, we construct (Section 3) tournaments with $w = l = 0$ for all values of $c_3 \leq \frac{1}{4}$. This shows as well the

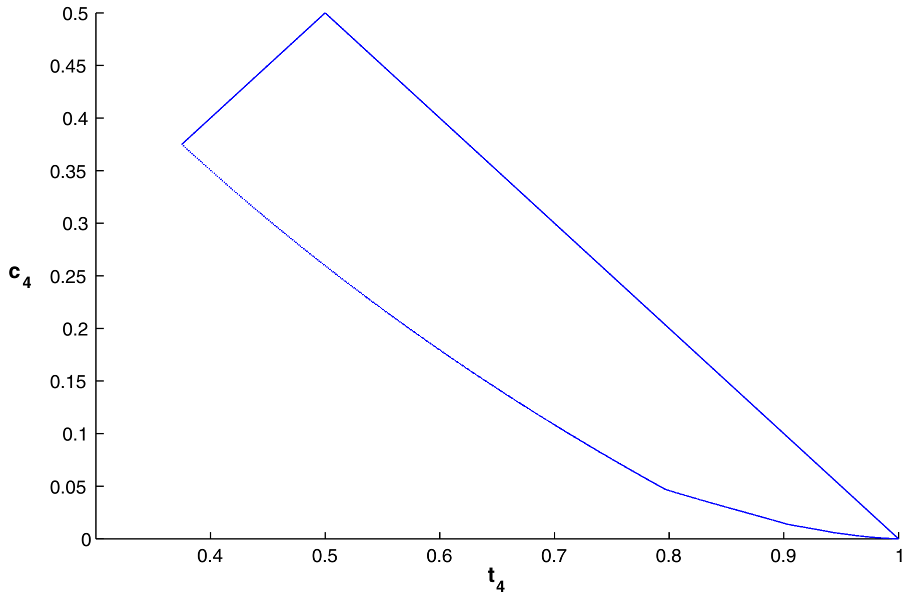

FIGURE 4. The boundary of the set $\{(t_4(\mathcal{T}), c_4(\mathcal{T}))\}$. The lower curve is conjectured.

tightness of the third inequality when $t_4 \geq \frac{1}{2}$. For $t_4 \leq \frac{1}{2}$ we need a different construction that satisfies $c_4 = t_4$, which we also do in Section 3. ■

B. The minimum

Problems 4 to 6 are more involved. We focus on Problem 4. To derive an upper bound for Problem 4 we introduce the *random blow-up* of a m-vertex tournament H. Associated with H and a probability vector $(w_1, \dots, w_m)$ is an infinite family of tournaments $\mathcal{T} = \mathcal{T}(H; w_1, \dots, w_m)$ whose n-th member has vertex set $\{V_i | i \in H\}$ where $|V_i| = \lfloor w_i n \rfloor$. If $(i \to j) \in E(H)$ then there is an edge $(u \to v)$ from every $u \in V_i$ to every $v \in V_j$. The subtournament on each V_i is random. In the *balanced* case $w_1 = w_2 = \cdots = w_m = \frac{1}{m}$, we use the shorthand $\mathcal{T}(H)$.

We can now state our conjecture.

Conjecture 2.2. *The minimum of c_4, given c_3 is attained by a random blow-up of a transitive tournament T_m.*

Lemma 2.7 below says that among all such tournaments of given c_3, the smallest c_4 is attained by taking m as small as possible and $w_1 = w_2 = \cdots = w_{m-1} \geq w_m$.

When $c_3 = \frac{1}{4r^2}$ the random blow-up that minimizes c_4 is the balanced blow-up of T_r. It is conceivable that this case of the conjecture should be easier to handle. When $c_3 = \frac{1}{4}$ and $m = 1$ this reduces to the well-known fact that t_4 is minimized by a random tournament. (Recall that given c_3, minimization of c_4 and of t_4 are equivalent). In this article, we settle as well the case $c_3 = \frac{1}{16}$ and $m = 2$.

Note that, if the conjecture is indeed true, then there is no simple expression for min c_4 in terms of c_3.

Remark 2.3. The problem and the structure of the construction in Conjecture 2.2 resemble a well-studied problem in graph theory. The problem statement is as follows: For $r > s$ and a family of graphs with given K_s-density (that is the asymptotic probability for s randomly chosen vertices to form a clique), how small can the K_r density be?

The recent works of Razborov [7], Nikiforov [9], and Reiher [10] solved the problem for $s = 2$ (with $r = 3$, $r = 4$, and all $r \geq 5$, respectively). For all we know the problem is still open for $s = 3$ and any r.

The optimal construction (Razborov [7]) for $s = 2$, $r = 3$ is obtained by blowup of complete graphs of the smallest possible size. The blowup weights are all equal except (maybe) one smaller part. There are no edges inside the blowup sets (unlike our random tournament inside each set).

We reproduce a proof from [8], that we later (Lemma 2.5) improve.

Proposition 2.4. $c_4 \geq 6c_3^2$.

Proof. For an edge $e = uv$ in a tournament T, let x_e be the probability that the triangle uvw is cyclic when the vertex w is selected uniformly at random. We define the random variable X on $E(T)$ with uniform distribution that takes the value x_e at $e \in E(T)$. Clearly $\mathbb{E}X = c_3 + o_{|T|}(1)$. But a 4-vertex tournament is isomorphic to C_4 iff it contains two cyclic triangles with a common edge. Consequently, $\mathbb{E}(X^2) = \frac{c_4}{6} + o_{|T|}(1)$. The proposition simply says that $Var(X) \geq 0$. ■

Consequently, our main problem is to find the smallest possible variance $Var(X)$ for given $\mathbb{E}(X)$. Conjecture 2.2 and Lemma 2.5 below are some quantitative forms of the assertion that when $0 < c_3 < \frac{1}{4}$, cyclic triangles cannot be uniformly distributed among the edges. We presently have no conceptual proof of this claim, and we must resort to flag algebra methods, which unfortunately offer no intuition as to the reason that this statement is true.

Here is another curious aspect of this problem. Define $\varphi_T(x) := \mathbf{pr}(X \geq x)$ and let $f := \limsup_{|T|\to\infty} \varphi_T$. For all we know, f may be discontinuous. To see this note that $f(\frac{1}{3}) \geq \frac{2}{3}$ where $\frac{2}{3}$ is the value that is attained by balanced blow-ups of C_3. We suspect that $f(\frac{1}{3}+)$ is strictly smaller than $\frac{2}{3}$. In fact, the best lower bound that we have is $f(\frac{1}{3}+) \geq \frac{4}{9}$ that is attained by an imbalanced blow-up of C_3 (e.g. a $(\frac{1}{3}+\epsilon, \frac{1}{3}+\epsilon, \frac{1}{3}-2\epsilon)$ blowup with $\epsilon \to 0^+$).

We turn next to apply Razborov's flag-algebra method [6] that yields a lower bound that is not far from the conjectured value. In particular, it proves Conjecture 2.2 for $c_3 = \frac{1}{16}$.

Lemma 2.5. $c_4 \geq \frac{18c_3^2}{1+8c_3}$.

Corollary 2.6. *If* $c_3 \geq \frac{1}{16}$ *then* $c_4 \geq \frac{3}{64}$. *In particular, the construction in Conjecture 2.2 is optimal for* $c_3 = \frac{1}{16}$.

See Figure 5 for a comparison between the bound in Lemma 2.5 and Conjecture 2.2.

Using available computer software, we were able to get further numerical evidence that indicates that Lemma 2.5 is not tight for $c_3 \neq 0, \frac{1}{16}, \frac{1}{4}$, and the true minimum of c_4 is closer to the conjectured value. The results are graphically presented in Figure 5 and the method of computation is explained in Appendix A.

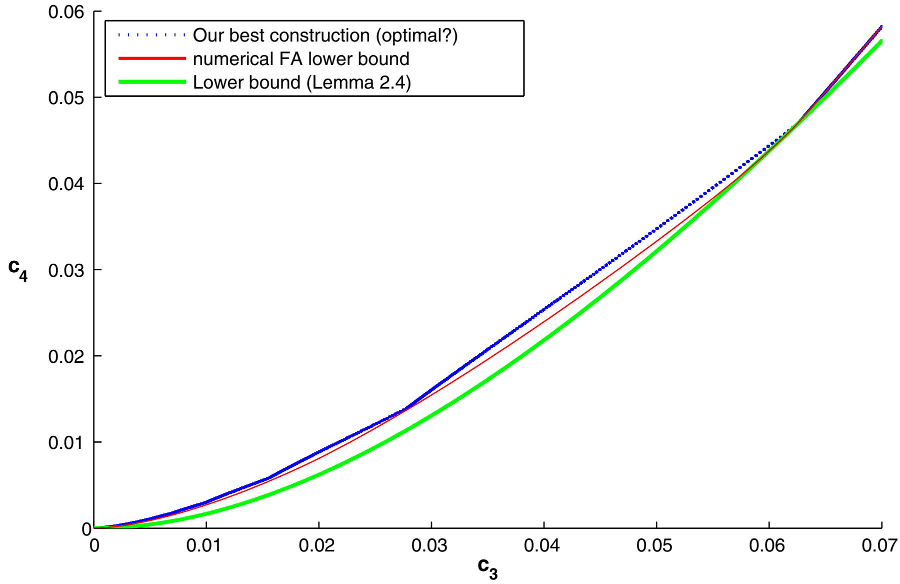

FIGURE 5. Lower bound from numerical application of flag algebras, compared with proven lower bound and our best construction, which is conjectured to be optimal. To improve visibility we present only the range $c_3 \in [0, 0.07]$. Numerical lower bound and the construction seem to coincide for $c_3 \geq \frac{1}{16}$.

Concluding this section, we formulate the following analytic lemma. It states that among all blow-ups considered in Conjecture 2.2 the best one is a blow-up of a transitive tournament of least possible order, with equal vertex weights, except possibly one smaller weight.

Lemma 2.7. *Fix any $0 < C < 1$ and consider all probability vectors w satisfying $\sum w_i^3 = C$. The minimum of $\sum w_i^4$ among such vectors is attained by letting $w_1 = \ldots = w_{m-1} \geq w_m > 0$ with the smallest possible m.*

The relevance of the lemma in the setting of Conjecture 2.2, is that $c_3(\mathcal{T}) = \frac{1}{4}\sum w_i^3$, and $c_4(\mathcal{T}) = \frac{3}{8}\sum w_i^4$ where $\mathcal{T} = \mathcal{T}(T_m; w_1, \ldots, w_m)$.

3. PROOFS

Proof of Lemma 1.3. We will show that the set $\{(c_3, c_4)\}$ is vertically convex. Let $\mathcal{T}_1, \mathcal{T}_2$ be two families with $c_3(\mathcal{T}_1) = c_3(\mathcal{T}_2)$ and $c_4(\mathcal{T}_1) < c < c_4(\mathcal{T}_2)$. We construct an n-vertex tournament T with $c_3(T) = c_3(\mathcal{T}_1) + o_n(1)$ and $c_4(T) = c + o_n(1)$. Let $0 \leq p, \alpha \leq 1$ be two constant parameters. Choose $T_1 \in \mathcal{T}_1$ on αn vertices (we can choose a random subtournament of a larger member if $\mathcal{T}_1$ has no member of this order). Let $T_2 \in \mathcal{T}_2$ of order $(1-\alpha)n$. Let $T = T_1 \cup T_2$, where for $x \in T_1$ and $y \in T_2$ there is an edge $x \to y$ with probability p and $y \to x$ with probability $1 - p$.

We compute $c_3(T) = \alpha^3 c_3(T_1) + (1-\alpha)^3 c_3(T_2) + 3\alpha(1-\alpha)p(1-p) + o(1)$. Choose p such that $p(1-p) = c_3(\mathcal{T}_1) = c_3(\mathcal{T}_2)$ and then $c_3(T) = c_3(\mathcal{T}_1) + o(1)$.

In computing $c_4(T)$, several terms come in, each up to $+o(1)$ error

- $\alpha^4 c_4(T_1)$ for quadruples contained in T_1
- $(1-\alpha)^4 c_4(T_2)$ all in T_2
- $6\alpha^2(1-\alpha)^2(p(1-p) + 2p^2(1-p)^2)$ two in each
- $4\alpha^3(1-\alpha)(c_3(\mathcal{T}_1)3p(1-p) + (1-c_3(\mathcal{T}_1))p(1-p))$ three in T_1 and one in T_2
- $4\alpha(1-\alpha)^3(c_3(\mathcal{T}_2)3p(1-p) + (1-c_3(\mathcal{T}_2))p(1-p))$ one and three.

Consequently $c_4(T)$ is expressed (up to an additive $o(1)$ term) as a degree four polynomial in α which for $1 \geq \alpha \geq 0$ takes every value between $c_4(\mathcal{T}_1)$ and $c_4(\mathcal{T}_2)$. ■

Completing the proof of tightness in Observation 2.1. Let us recall the well-known *cyclic tournaments* (see e.g. [8]). Place an odd number of vertices equally spaced along a circle, and $x \to y$ is an edge if the shorter arc from x to y is clock-wise. We are now ready to construct tournaments with the desired parameters.

- Tournaments with arbitrary $0 \leq c_3 \leq \frac{1}{4}$, and $w = l = 0$:
  Fix some $\frac{n}{2} \leq s \leq n$. Let T be the tournament with vertex set $1, 2, \ldots, n$, where $x \to y$ for $1 \leq x < y \leq n$, iff $y \leq x + s$. We claim that $w(T) = l(T) = 0$. For suppose that $x \to y \to z \to x$ is a cyclic triangle in T and there is some vertex w with either $w \to x, y, z$ or $w \leftarrow x, y, z$. w.l.o.g. $x < y, z$ and it follows that $x < y \leq x + s < z \leq y + s$. If $w < x$, then $w \to x$ since $s \geq \frac{n}{2}$, but $z \to w$. Likewise we rule out the possibility that $w > z$, i.e. necessarily $x < w < z$. If $x < w < y$ then necessarily $x \to w \to y$. Likewise, $y < w < z$ implies $y \to w \to z$.
  For $n \to \infty$ odd and $s = \frac{n}{2}$ this yields the cyclic tournaments and $c_3 = \frac{1}{4}$. when $s = n$ we obtain transitive tournaments. As s varies we cover the whole range $0 \leq c_3 \leq \frac{1}{4}$.
- For $t \in [\frac{3}{8}, \frac{1}{2}]$, we construct a family $\mathcal{T}$ with $t_4(\mathcal{T}) = c_4(\mathcal{T}) = t$ (recall that no family of tournaments can have $t_4 < \frac{3}{8}$):
  Fix some $0 \leq p \leq \frac{1}{2}$. We construct $\mathcal{T}$ from the cyclic tournaments by flipping each edge independently with probability p. As we show below, $c_3(\mathcal{T}) = \frac{1}{4}$, so by Observation 1.2, $t_4(\mathcal{T}) = c_4(\mathcal{T})$. When $p = 0$ we have the cyclic tournament with $t_4 = c_4 = \frac{1}{2}$ and when $p = \frac{1}{2}$ we have a random tournament with $t_4 = c_4 = \frac{3}{8}$. The claim follows by continuity.
  To see that $c_3(\mathcal{T}) = \frac{1}{4}$, note that almost surely all vertex outdegrees in $T \in \mathcal{T}$ equal $\frac{n}{2} \pm o(n)$. The claim follow by a standard Goodman-type argument. ■

Proof of Lemma 2.5. We define the random variables X and Y over $E(T)$ with uniform distribution. For $e = \{v_1 \to v_2\} \in E(T)$ we define:

- $X(e)$ is the probability that $\{v_1, v_2, v_3\}$ is a cyclic triangle in T, where the vertex v_3 is chosen uniformly at random.
- $Y(e)$ the probability that $\{v_1 \to v_3\} \in E(T)$ and $\{v_3 \to v_2\} \in E(T)$, where the vertex v_3 is chosen uniformly at random.

It is not hard to verify the following expectations: $\mathbb{E}(X) = c_3$, $\mathbb{E}(Y) = \frac{t_3}{3}$. $\mathbb{E}(X^2) = \frac{c_4}{6}$, $\mathbb{E}(Y^2) = \frac{t_4}{6}$ and $\mathbb{E}(X \cdot Y) = \frac{c_4}{6}$.

We define $Z = 1 + 2(X - Y)$ and conclude that $\mathbb{E}(Z^2) = \frac{1+8c_3}{3}$ and $\mathbb{E}(X \cdot Z) = c_3$. By Cauchy–Schwarz $c_3^2 = \mathbb{E}^2(X \cdot Z) \le \mathbb{E}(X^2)\mathbb{E}(Z^2) = c_4 \frac{1+8c_3}{18}$. ■

Remark 3.1. Proper disclosure: The above derivation could not have been carried out without seeing what flag-algebra calculations yield.

Proof of Lemma 2.7. We wish to minimize $\sum_1^m w_i^4$ under the constraints $\sum_1^m w_i = 1$ and $\sum_1^m w_i^3 = C$ (for given m). We assume that all w_i are positive, since zero w_i's can be removed with smaller m. A Lagrange multipliers calculation yields that $w_i^3 = \lambda w_i^2 + \mu$ for all i and for some constants λ and μ. The cubic polynomial $x^3 - \lambda x^2 - \mu$ has at most two positive roots since the linear term in x vanishes. Therefore the coordinates of the optimal w must take at most two distinct values.

Assume towards contradiction that $x > y > 0$ appear as coordinates in w with y repeated at least twice. We will replace three of w's coordinates (x, y, y) while preserving $\sum_1^m w_i$ and $\sum_1^m w_i^3$, and reducing $\sum_1^m w_i^4 = C$.

We replace (x, y, y) by either $(s, t, 0)$ or (s, s, t) where $s \ge t \ge 0$. First, if $y \le \frac{\sqrt{5}-1}{4} \cdot x$, we prove the existence of $s \ge t \ge 0$ s.t.

- $x + 2y = s + t$.
- $x^3 + 2y^3 = s^3 + t^3$.
- $x^4 + 2y^4 > s^4 + t^4$.

Substitute $t = x + 2y - s$ in the second equation: $x^3 + 2y^3 = s^3 + (x + 2y - s)^3$, which can be rewritten as $(x + 2y)s^2 - (x + 2y)^2 s + 2y(x + y)^2 = 0$. This quadratic has real roots iff $D = (x + 2y)^4 - 8y(x + y)^2(x + 2y) \ge 0$ that holds iff $x \ge (1 + \sqrt{5})y$, the range that we consider. Moreover, when $D \ge 0$, both roots are positive, since the quadratic has a positive constant term and a negative linear term. This proves the existence of $s \ge t \ge 0$ satisfying the first two conditions.

The sum of the fourth powers of the roots of this quadratic is $s^4 + t^4 = \frac{(x+2y)^8 + 6(x+2y)^4 D + D^2}{8(x+2y)^4}$. Thus, it suffices to show that $8(x^4 + 2y^4)(x + 2y)^2 > 8(x + 2y)^6 - 64y(x + 2y)^3(x + y)^2 + 64y^2(x + y)^4$ that is easily verified by expanding all terms.

In the complementary range $x > y \ge \frac{\sqrt{5}-1}{4} \cdot x$ we find $s \ge t \ge 0$ s.t.

- $x + 2y = 2s + t$.
- $x^3 + 2y^3 = 2s^3 + t^3$.
- $x^4 + 2y^4 > 2s^4 + t^4$.

We substitute $t = x + 2y - 2s$ in the second equation: $x^3 + 2y^3 = 2s^3 + (x + 2y - 2s)^3$, or, equivalently, $0 = s^3 - 2(x + 2y)s^2 + (x + 2y)^2 s - y(x + y)^2 = (s - y)(s^2 - 2xs - 3ys + (x + y)^2)$. Thus, $s^2 - (2x + 3y)s + (x + y)^2 = 0$, and $s = \frac{2x+3y-\sqrt{y(4x+5y)}}{2} > 0$. Now $t = x + 2y - 2s \ge 0$ iff $y \ge \frac{\sqrt{5}-1}{4} \cdot x$. Clearly, $s \ge t$.

It remains to compute

$$\begin{aligned} x^4 + 2y^4 - 2s^4 - t^4 = {} & (8x^3 + 50x^2y + 86xy^2 + 45y^3)\sqrt{y(4x + 5y)} \\ & -(2x^4 + 52x^3y + 180x^2y^2 + 232xy^3 + 101y^4) \end{aligned}$$

and show that this is positive. To this end we must prove that

$$\begin{aligned} y(4x + 5y)(8x^3 + 50x^2y + 86xy^2 + 45y^3)^2 - {} & (2x^4 + 52x^3y + 180x^2y^2 \\ & +232xy^3 + 101y^4)^2 > 0. \end{aligned}$$

This expression can be written as

$$4(x-y)^3(19y^5 + 73xy^4 + 98x^2y^3 + 54x^3y^2 + 9x^4y - x^5)$$

that is positive, since $9x^4y - x^5 > 0$. ■

4. FURTHER DIRECTIONS

- Many basic open questions on the local profiles of combinatorial objects are still open. Thus, it is still unknown whether the set of k-profiles of graphs is a simply connected set. Similar issues were already raised in the pioneering work of Erdős et al. [3], and remains open. The analogous question for tournaments is open as well. We do not know if the set of k-local profiles of tournaments is *convex*. We do not know it even for $k = 4$, and we are not sure what the right guess is. As first observed in [3] the analogous question is answered negatively for graphs. On the other hand, for trees the answer is positive [1].
- We recall the random variable X - the fraction of cyclic triangles containing a randomly chosen edge. It would be desirable to give a direct proof that $\mathrm{Var}(X) > 0$ for all $0 < c_3 < \frac{1}{4}$.
- In Section 2, we defined the function $f(x) = \limsup_{|T|\to\infty} \mathbf{pr}(X \geq x)$. What can be said about f? In particular, is it continuous? Is it continuous at $\frac{1}{3}$?
- The conjectured extreme construction for Problem 4 is particularly simple when $c_3 = \frac{1}{4r^2}$ for integer r. We were able to settle this case for $r = 1, 2$. Thus, the first open case is $c_3 = \frac{1}{36}$.
- To what extent can the lower bound in Lemma 2.5 be improved using higher order flags? In particular, Figure 5 suggests that our construction is optimal for $c_3 \geq \frac{1}{16}$. Can the optimum for this range be established using flags of order 6?
- Here we have studied the set $\{(t_3(\mathcal{T}), t_4(\mathcal{T}))\}$. We would like to understand the relationships among higher t_k's as well.
- Obviously, we would be interested in further describing the set of 4-profiles of tournaments.
- The powerful method of flag algebras remains mysterious, and it would be desirable to have more transparent local methods. Lemma 2.5 and the stronger Conjecture 2.2 offer concrete challenges for such methods.
- Associated with every tournament T is a 3-uniform hypergraph whose faces are the cyclic triangles of T. This hypergraph clearly does not contain a 4-vertex clique and this was used in [4] to deduce a lower bound on some hypergraph Ramsey numbers. We wonder about additional structural properties of such 3-uniform hypergraphs. Specifically,
  - Can such hypergraphs be recognized in polynomial time?
- Lemma 2.7 is the case $p = 3, q = 4$ of the following natural sounding question. Find the smallest q-norm among all probability vectors w of given p-norm, where $q > p \geq 2$ are integers. Is it true that all optimal vectors have the form $w_1 = \ldots = w_{m-1} \geq w_m$, with the least possible m? Clearly our method of proof is too ad-hoc to apply in general.

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APPENDIX A: COMPUTER GENERATED LOWER BOUNDS

We have used the flag algebra method as explained in Section 4 of [5]. Using flags of size 3 over the (only) type of order 2 yields Lemma 2.5. Using flags of size 4 over the same type we get a 16×16 PSD matrix whose entries are bilinear expressions in the coordinates of a large tournament's 6-profile. We used the cvx SDP-solver [2] to obtain the results presented in Figure 5. Working with larger flags may clearly yield better estimates but limited computational resources have stopped us from reaching beyond size 4-flags.

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