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Note

# A counterexample to a conjecture of Björner and Lovász on the $\chi$-coloring complex

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**Abstract**

Associated with every graph $G$ of chromatic number $\chi$ is another graph $G'$. The vertex set of $G'$ consists of all $\chi$-colorings of $G$, and two $\chi$-colorings are adjacent when they differ on exactly one vertex. According to a conjecture of Björner and Lovász, this graph $G'$ must be disconnected. In this note we give a counterexample to this conjecture.


*Keywords:* Graph coloring; Coloring complex; Graph homomorphism

One of the most disturbing problems in graph theory is that we only have few methods to prove lower bounds on the chromatic number of graphs. A famous exception is Lovász's [3] proof of the Kneser conjecture. This paper has indeed introduced a new method into this area and is one of the first applications of topological methods to combinatorics. It shows how to use the Borsuk–Ulam Theorem to derive a (tight) lower bound for the chromatic number of the Kneser graph. Since then, the idea of finding topological obstructions to graph colorings has been extensively studied [2,4]. In particular, Björner and Lovász made a conjecture generalizing the concept of a topological obstruction to graph coloring (see [1, Conjecture 1.6]). In this note, we provide a counterexample to this general conjecture.

To state the general conjecture, we need some definitions. For two graphs $G$, $H$, an $H$-coloring of $G$ is a homomorphism from $G$ to $H$. Namely, a mapping $\phi : V(G) \to V(H)$, such that for all edges $(x, y) \in E(G)$ we have $(\phi(x), \phi(y)) \in E(H)$. The coloring

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complex $\mathrm{Hom}(G, H)$ is a CW-complex whose 0-cells are the $H$-colorings of $G$. The cells of $\mathrm{Hom}(G, H)$ are the maps $\eta$ from the vertices of $G$ to non-empty vertex subsets of $H$ such that $\eta(x) \times \eta(y) \subseteq E(H)$ for all $(x, y) \in E(G)$. The closure of a cell $\eta$ consists of all cells $\tilde{\eta}$ such that $\tilde{\eta}(v) \subseteq \eta(v)$ for all $v \in V(G)$. We say that a complex $C$ is $k$-connected if every map from $S^k$ to $C$ can be extended to a map from $B^{k+1}$ to $C$. Equivalently, if all the homotopy groups up to dimension $k$ are trivial. Specifically, $(-1)$-connected means non-empty, and 0-connected is connected. We denote the chromatic number of a graph $G$ by $\chi(G)$. In an attempt to capture some topological obstructions to low chromatic number, Björner and Lovász have made the following conjecture:

**Conjecture 1** (*Björner and Lovász*). *Let $G$, $H$ be two graphs such that the coloring complex* $\mathrm{Hom}(G, H)$ *is $k$-connected. Then $\chi(H) \geqslant \chi(G) + k + 1$.*

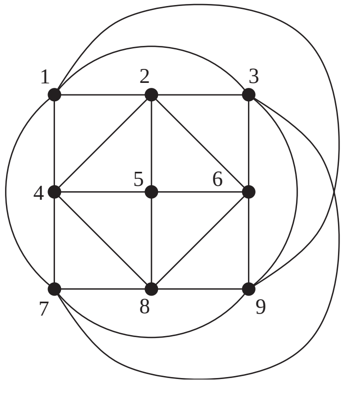
There are three special cases of this conjecture that are known to be true. One is essentially Lovász's original argument implying the Kneser conjecture. The other two are recent results of Babson and Kozlov:

**Theorem 2.** *Conjecture* 1 *holds when* (i) $G = K_2$ [3], (ii) $G = K_m$ [1], (iii) $G = C_{2r+1}$ [2].

We refute the conjecture by exhibiting an explicit graph $G$ with chromatic number 5, such that $\mathrm{Hom}(G, K_5)$ is 0-connected. For the purpose of the present note, no background is needed beyond elementary graph theory.

In the case under consideration here, $k = 0$ and the graph $G$ has chromatic number $\chi$. (In the specific counterexample we show, $\chi = 5$, but many other examples can be exhibited for other values of $\chi$). The vertex set $W$ of the complex $\mathrm{Hom}(G, K_\chi)$ consists of all $\chi$-colorings of $G$. Whether this complex is 0-connected, is determined by its 1-skeleton. This is a graph $G'$ on vertex set $W$, where two $\chi$-colorings of $G$ are adjacent in $G'$ if they differ on exactly one vertex. The complex is 0-connected iff the graph $G'$ is connected. We present here a graph $G$ of chromatic number $\chi = 5$, for which the graph $G'$ is connected, contradicting the above conjecture.

Let $G$ be the following graph with 9 vertices and 22 edges:



It is easy to check that $\chi(G) = 5$. We enumerate the 5-colorings of $G$ as follows. Since the four corners of the square form a clique, their colors must be distinct. Assume they are colored 1, 2, 3 and 4 in clockwise order starting at vertex number 1 (this is one of the 5! ways to color the corners). We would like to enumerate the ways to complete the following square:

$$\boxed{\begin{array}{ccc} 1 & - & 2 \\ - & - & - \\ 4 & - & 3 \end{array}}$$

If the center is 5, then there are exactly two colorings

$$s_{1234} = \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 5 & 4 \\ 4 & 1 & 3 \end{array}},\quad t_{1234} = \boxed{\begin{array}{ccc} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 4 & 2 & 3 \end{array}}.$$

Otherwise, there are exactly 32 colorings. To see this, observe that if the center is different from 5, then the occurrences of 5 are either restricted to the central row or the central column but not to both. This allows us to partition these colorings to $h$- and $v$-type, respectively. Therefore, the colorings where the center is different from 5 are the possible completions of the following eight squares (the plus sign marks vertices that must be colored 5):

$$h_{1234,a} = \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ + & - & - \\ 4 & 2 & 3 \end{array}},\quad h_{1234,b} = \boxed{\begin{array}{ccc} 1 & 4 & 2 \\ - & - & + \\ 4 & 1 & 3 \end{array}},\quad v_{1234,a} = \boxed{\begin{array}{ccc} 1 & + & 2 \\ 3 & - & 4 \\ 4 & - & 3 \end{array}},\quad v_{1234,b} = \boxed{\begin{array}{ccc} 1 & - & 2 \\ 2 & - & 1 \\ 4 & + & 3 \end{array}},$$

$$h_{1234,c} = \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ - & - & - \\ 4 & 1 & 3 \end{array}},\quad h_{1234,d} = \boxed{\begin{array}{ccc} 1 & 4 & 2 \\ - & - & - \\ 4 & 2 & 3 \end{array}},\quad v_{1234,} = \boxed{\begin{array}{ccc} 1 & - & 2 \\ 3 & - & 1 \\ 4 & - & 3 \end{array}},\quad v_{1234,d} = \boxed{\begin{array}{ccc} 1 & - & 2 \\ 2 & - & 4 \\ 4 & - & 3 \end{array}}.$$

One can verify that each of these eight squares have four possible completions. For example, the central row of $h_{1234,a}$ colorings must be in {514, 515, 545, 541} and the central row of $h_{1234,c}$ colorings must be in {245, 545, 525, 524}. Since all resulting colorings are distinct, we deduce that the total number of 5-colorings is $5! \cdot (32 + 2) = 4080$.

To prove that the graph $G'$ is connected, first note that the 4 colorings represented by each of the eight squares above, are connected. Indeed, the restriction of $G'$ to each of these four vertex sets is a path. For example

$$\boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 4 & 2 & 3 \end{array}} \rightarrow \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 1 & 5 \\ 4 & 2 & 3 \end{array}} \rightarrow \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 4 & 5 \\ 4 & 2 & 3 \end{array}} \rightarrow \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 4 & 1 \\ 4 & 2 & 3 \end{array}}.$$

We denote an edge in the graph $G'$ as a coloring, where one of the vertices is given two colors. Using this notation, consider the following edges:

$$\boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 4 & 2 & 3,5 \end{array}}, \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 2 & 4 \\ 4 & 1 & 5,3 \end{array}}; \boxed{\begin{array}{ccc} 1 & 3 & 2,5 \\ 5 & 4 & 1 \\ 4 & 2 & 3 \end{array}}, \boxed{\begin{array}{ccc} 1 & 4 & 5,2 \\ 5 & 3 & 1 \\ 4 & 2 & 3 \end{array}}; \boxed{\begin{array}{ccc} 1,5 & 4 & 2 \\ 2 & 3 & 5 \\ 4 & 1 & 3 \end{array}}, \boxed{\begin{array}{ccc} 5,1 & 3 & 2 \\ 2 & 4 & 5 \\ 4 & 1 & 3 \end{array}}.$$

These edges connect $(h_{1234,a}, v_{1254,a})$, $(v_{1254,a}, h_{1234,c})$; $(h_{1234,a}, v_{1534,b})$, $(v_{1534,b}, h_{1234,d})$; and $(h_{1234,b}, v_{5234,b})$, $(v_{5234,b}, h_{1234,c})$, respectively. Therefore, $h_{1234,a}$, $h_{1234,b}$, $h_{1234,c}$, $h_{1234,d}$ are connected, and we denote this connected set of colorings by $h_{1234}$. By symmetry, the same applies to the $v$-type colorings. Therefore, if the four corners are colored *a*, *b*, *c* and *d* (in clockwise order starting at vertex 1) then we have four connected sets of colorings $h_{abcd}$, $v_{abcd}$, $s_{abcd}$ and $t_{abcd}$. This reduces the number of connected components to $5! \cdot 4 = 480$.

Next, consider the following edges:

$$\boxed{\begin{array}{ccc} 1 & 3 & 2,5 \\ 5 & 1 & 4 \\ 4 & 2 & 3 \end{array}}, \boxed{\begin{array}{ccc} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 4 & 2 & 3,5 \end{array}}, \boxed{\begin{array}{ccc} 1,5 & 4 & 2 \\ 2 & 3 & 5 \\ 4 & 1 & 3 \end{array}}, \boxed{\begin{array}{ccc} 1 & 4 & 2 \\ 2 & 3 & 5 \\ 4,5 & 1 & 3 \end{array}}.$$

These edges connect $h_{1234}$ to $v_{1534}$, $v_{1254}$, $v_{5234}$ and $v_{1235}$. By symmetry, $v_{1234}$ is connected to $h_{1534}$, $h_{1254}$, $h_{5234}$ and $h_{1235}$. Therefore, we can get from any $h_{abcd}$ either to $h_{1234}$ or $v_{1234}$. Since $s_{1234}$ is connected to $v_{5234}$ and $h_{1534}$, we have the path $v_{5234}, s_{1234}, h_{1534}, v_{1234}, h_{5234}$, and hence that all the $h$ and $v$ colorings are connected. Finally, we are done, since $t_{1234}$ is connected to $h_{5234}$. Therefore, the 5-colorings graph $G'$ is connected as claimed.

## References


[1] E. Babson, D. Kozlov, Complexes of graph homomorphisms, http://arxiv.org/abs/math.CO/0310056, 2003.

[2] E. Babson, D. Kozlov, Proof of the Lovász conjecture, http://arxiv.org/abs/math.CO/0402395, 2004.

[3] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Combin. Theory Ser. A 25 (3) (1978) 319–324.

[4] J. Matoušek, Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler, Using the Borsuk–Ulam Theorem, Universitext, Springer, Berlin, 2003.