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HARD ENUMERATION PROBLEMS IN GEOMETRY AND COMBINATORICS*

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Abstract. A number of natural enumeration problems in geometry and combinatorics are shown to be complete in the class #P introduced by Valiant. Among others this is established for the numeration of vertices and of facets of a polytope, acyclic orientations of a graph and satisfying assignments of implicative boolean formulas.

Key words. #P, polytopes, partial orders, acylic orientations

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Introduction. This article contains a contribution to the theory of hard enumeration problems. The foundations of this area were laid by Valiant [Va1], [Va2] who defined the class #P of enumeration problems and the subclass of problems complete in #P. The most interesting of his results is the #P completeness of computing permanents of 0–1 matrices. This problem can also be stated as the problem of enumerating perfect matchings in bipartite graphs. While deciding whether a bipartite graph has a perfect matching can be done in polynomial time [H] the enumeration problem is #P-complete.

Valiant's pioneering work was continued by a recent article of Provan and Ball [PB] who prove the #P-completeness of a number of natural enumeration problems. With every enumeration problem there is an associated decision problem. Instead of asking for the number of objects in question we ask whether this number is zero or not. The decision problem associated with the computation of the permanent function is the question whether a given bipartite graph has a perfect matching. While this decision is solvable in polynomial time, it is by no means trivial. Notice, however, that for many of the problems discussed in [PB] the situation is even more extreme: Consider for example the problem of enumerating independent sets in a bipartite graph, the decision problem associated with this enumeration problem is trivial: Every graph has an independent set of vertices. So an enumeration problem can be #P-complete even if the existence problem is trivial.

This article assumes acquaintance with the theory of #P-completeness as presented in [Va1], [PB] and [GJ]. Our purpose is to present a number of natural enumeration problems which belong to the class of #P-complete problems. The problems are geometric, combinatorial and from propositional calculus.

Here is our main theorem:

THEOREM. *The following enumeration problems are* #P-*complete.*

(1) *Vertices in a polytope.*
Input: A system of linear inequalities $Ax \leqq b$ defining a polytope $P \subseteq \mathbb{R}^n$.
Output: The number of vertices of P.

(2) *d-dimensional faces of a polytope* (*fixed d*).
Input: As in (1).
Output: The number of d-dimensional faces of P.

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(3) *Facets of a polytope.*
Input: A finite set of points in $\mathbb{R}^n$.
Output: The number of facets ($(n-1)$-dimensional faces) of P.

(4) *Components of slotted space.*
Input: A set $\{H_i | i \in I\}$ of hyperplanes in $\mathbb{R}^n$.
Output: The number of connected components of $\{\mathbb{R}^n \backslash \bigcup H_i | i \in I\}$.

(5) *Acyclic orientations of a graph.*
Input: A graph $G = (V, E)$.
Output: The number of orientations of G with no directed circuit.

(6) 3*-colorings of a bipartite graph.*
Input: A bipartite graph $G = (A, B, E)$.
Output: The number of ways to properly color G with 3 colors.

(7) *Satisfying assignments of an implicative Boolean formula.*
Input: A Boolean formula B on variables $x_1, \cdots, x_n$ of the form $B = \bigwedge_{i=1}^{t} (x_{i1} \vee \bar{x}_{i2})$.
Output: Number of truth assignments for $x_1, \cdots, x_n$ which makes B true.

Proof. (1) We use the fact from [PB] that enumerating order ideals is #P-complete. Given a poset $(P, \geqq)$ with $P = [n]$ we associate with P a polytope $B = B(P)$ in $\mathbb{R}^n$ as follows:

$$B = \{x \in \mathbb{R}^n | 1 \geqq x_i \geqq 0, \ x_i > x_j \text{ if } i \geqq j \text{ in } P\}.$$

(See [St2], [Li], [KS] where use is made of this polytope.)

We claim that the vertices of B are in 1:1 correspondence with the order ideals of $(P, \geqq)$. First we prove that all vertices of B have 0–1 coordinates. Let $x \in B$ have some $0 < x_i < 1$. If $\alpha = \max \{x_j | 0 < x_j < 1\}$, then by replacing all coordinates $x_j = \alpha$ by $\alpha + \varepsilon$ or by $\alpha - \varepsilon$ we will get a point of B. This implies that x is not a vertex of B. The correspondence between vertices and ideals is as follows:

$$x \in \operatorname{vert}(B) \leftrightarrow S = \{1 \leqq j \leqq n | x_j = 0\}.$$

It is easily verified that S is an ideal and that this correspondence is bijective.

(2) Suppose that for some fixed d we can find $f_d(K)$ the number of d-dimensional faces of a polytope K. Consider r-fold pyramids P_r with K as basis. In [Gru, p. 55] one finds

$$(*) \qquad f_d(P_r) = \sum_i \binom{r}{i} f_{d-i}(K).$$

If we write (*) for $r = 0, \cdots, d$ and have all $f_d(P_r)$ evaluated, then we obtain a system of equations in unknown

$$f_0(K), \cdots, f_k(K).$$

This system of equations has a triangular matrix and so they can be solved successively and $f_0(K)$ can be determined in polynomial time. Since evaluating $f_0(K)$ = the number of vertices of K is #P-complete by (1), our claim follows.

(3) This is just the dual of (1): See [Gru, p. 46] for polytope duality.

(5) The proof here is based on two observations.

PROPOSITION [St1]. *Let $G = (V, E)$ be a graph with n vertices and let $P(G, \lambda)$ be its chromatic polynomial. Then $(-1)^n P(G, -1)$ equals the number of acyclic orientations of G.*

For the other observation we have to define the operation of *join* of two graphs $G=(V_1, E_1)$, $H=(V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$. The join $G+H$ has $V_1 \cup V_2$ as its vertex set and

$$E_1 \cup E_2 \cup \{[x, y] | x \in V_1, y \in V_2\}$$

as its edge set. The following observation is immediate.

PROPOSITION. $P(G+K_t, \lambda) = \lambda(\lambda-1), \cdots, (\lambda-t+1)P(G, \lambda-t)$.

Now we can combine these two facts as follows. Being able to enumerate acyclic orientations is equivalent to computing $P(G, -1)$ for the graph. But if we have the values of $P(G+K_t, -1)$ for $t=1, \cdots, n$, that means we can calculate the integers

$$P(G, -j) \qquad (n+1 \geqq j \geqq 2).$$

But P is a monic polynomial of degree n so from these numbers we can compute $P(G, \lambda)$, the chromatic polynomial of G. This is a #P-complete problem because the reduction to coloring is parsimonious [GJ, p. 169].

(4) The proof here makes use of (5) that enumerating acyclic orientations is #P-complete and on the following result of Greene. $H_{ij} \subseteq \mathbb{R}^n$ is the hyperplane given by $\{x \in \mathbb{R}^n | x_i = x_j\}$.

PROPOSITION [Gre]. *Let $G=(V, E)$ be a graph on n vertices and consider*

$$S(G) = \mathbb{R}^n \backslash \cup H_{ij}$$

where the union is over all i, j such that $[i, j] \in E$. The number of connected components of $S(G)$ equals the number of acyclic orientations of G.

(6) We base this proof on the #P-completeness of enumerating independent sets in bipartite graphs [PB]. Let $G=(A, B, E)$ be a partite graph for which we want to find $I(G)$ the number of independent sets. Consider a graph H which is obtained by adding two new vertices a, b with a being adjacent to all vertices in $B \cup \{b\}$ and b to all vertices of $A \cup \{a\}$. Now let us compute $\chi(H, 3)$, the number of 3-colorings of H. Suppose w.l.o.g. that a, b are colored 1, 2 respectively. The 3-coloring is now uniquely defined by the set of vertices colored 3. This can be any independent set of G and so

$$\chi(H, 3) = 6I(G).$$

This proves the #P-completeness of computing $\chi(H, 3)$.

(7) This follows from #P-completeness of enumerating ideals in posets [PB]: Let $(P, \geqq)$ be a poset with $P=\{p_1, \cdots, p_n\}$. Associate with it the Boolean expression

$$B = \wedge\{x_i \vee \bar{x}_j | p_j > p_i \text{ in } P\}.$$

It is fairly easy to verify that the set of x_i which are assigned a true value in any assignment satisfying B is an ideal in $(P, \geqq)$ and that all ideals are obtained in this way.

Let us mention in closing a most intriguing problem in this field: For a poset $(P, \geqq)$ a *linear extension* is a 1:1 mapping $f: P \to \{1, \cdots, |P|\}$ such that if $x<y$ in P then $f(x)<f(y)$. Consider the problem:

Enumeration of linear extensions.
Input: A poset $(P, \geqq)$.
Output: $L(P)$, the number of linear extensions of $(P, \leqq)$.

Conjecture. The enumeration of linear extensions is a #P-complete problem.

A proof of this conjecture will provide a first explicit statement to the effect that computing the volume of a convex polytope is a hard computational problem. To see this we remind the reader about the polytope $B(P)$ which was used in proving part 1 of our main theorem. We quote without proof of the following fact from [Li]:

PROPOSITION. *For a poset* $(P, \geqq)$ *on* $|P| = p$ *elements* $L(P)$ *the number of linear extensions of* P *satisfies*

$$L(P) = p!\operatorname{vol}(B(P)).$$

The connection between the #P-completeness of enumerating linear extensions and the complexity of evaluating the volume of a convex polytope is now clear.

Let us also comment about the relationship between the number of linear extensions of a poset and enumerating order ideals. We use $I(P)$ to denote the number of ideals in the poset P. For posets P, Q we define their *product* $P \times Q$ to be a partial order on the cartesian product of P and Q with $(x_1, y_1) \geqq (x_2, y_2)$ if $x_1 \geqq x_2$ in P and $y_1 \geqq y_2$ in Q. A mapping $f: P \to Q$ is *order preserving* if $x \geqq y$ in P implies $f(x) \geqq f(y)$ in Q.

PROPOSITION. *Let* $(P, \geqq)$ *be a poset and let* C_t *be the chain on* t *elements. Then* $I(P \times C_t)$ *equals the number of order preserving maps* $f: P \to \{0, 1, \cdots, t\}$.

Proof. With an ideal $J \subseteq P$ we associate a function $f: P \to \{0, \cdots, t\}$ as follows: For every $x \in P$ there is unique $t \geqq j \geqq 0$ such that $(x, j) \in J$ and $(x, j-1) \notin J$. Let $f(x) = t - j$ for that value j. Since J is an ideal, f is well defined and easily seen to be order preserving. It is also a routine matter verifying that this correspondence is bijective.

Now we come to the expression for the number of linear extensions of a poset.

THEOREM. *For a poset* $(P, \geqq)$ *on* $|P| = n$ *elements, the number of linear extensions* $L(P)$ *satisfies*

$$L(P) = I(P \times C_{n-1}) - nI(P \times C_{n-2}) + \binom{n}{2} I(P \times C_{n-3})$$
$$- + \cdots \pm \binom{n}{n-2} I(P \times C_1) \mp \binom{n}{n-1}.$$

Proof. This follows from the previous proposition and Inclusion-Exclusion. Classify order preserving maps $f: P \to \{0, \cdots, n-1\}$ according to their range. There are $I(P \times C_{n-1})$ such mappings altogether. Say f has property t $(n-1 \geqq t \geqq 0)$ if t is not in the range of f. $L(P)$ is the number of order preserving maps which are onto, i.e., have no property and there are

$$\binom{n}{j} I(P + C_{n-j-1})$$

maps having a given set of j properties.

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