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RANDOM LIFTS OF GRAPHS: PERFECT MATCHINGS

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We study random lifts of a graph G as defined in [1]. We prove a 0-1 law which states that for every graph G either almost every lift of G has a perfect matching, or almost none of its lifts has a perfect matching. We provide a precise description of this dichotomy. Roughly speaking, the a.s. existence of a perfect matching in the lift depends on the existence of a fractional perfect matching in G. The precise statement appears in Theorem 1.

1. Introduction

We begin with a brief background on random lifts of graphs[1]. A comprehensive account can be found in [1,2]. A random n-lift of a graph $G=(V,E)$ is a graph whose vertex set is $V\times[n]$. The vertex (v,j) is said to belong to the j-th *level* and to the *fiber* F_v *above* v. For each edge $e=[u,v]\in E(G)$, there is a random matching F_e between F_u and F_v. This set of edges F_e is called the fiber above e. Our discussion is mostly asymptotic, so we adopt a common abuse of the language and say that a property of lifts holds *almost surely* if it holds with probability $1-o(1)$ as $n\to\infty$. In this paper we answer the following question: Given a graph G, and n large and even, do the n-lifts of G tend to have a perfect matching?

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[1] Graphs in this paper are in fact multigraphs. Multiple edges and loops are allowed, unless otherwise stated. A random lift of a loop on a vertex v is the graph of a random permutation on the fiber of v.

1.1. Prelude

There is one case in which the answer is obviously positive. If G has a perfect matching M, then the union of fibers $\cup_{e\in M} F_e$ is a PM in every lift of G.

There is also a case where the answer is obviously negative. Let M be a matching in an n-lift of G. The *projection* of M is a mapping $\varphi: E(G) \to \mathbb{R}^+$ that is defined as $\varphi(e) := \frac{|M \cap F_e|}{n}$, the fraction of edges in the fiber F_e that are in the matching M. We recall that a nonnegative function f on G's edges is called a *fractional matching* if $\sum_{e=[v,x]} f(e) \le 1$ for every vertex $v \in V(G)$. If equality holds at every vertex, f is said to be a *fractional perfect matching* (FPM in short). Clearly the projection of a perfect matching in a lift of G is a FPM. Therefore, if G has no FPM, no lift of G can have a PM. Moreover, for every M as above, $\frac{2}{|V|}\sum \varphi(e)$ is the fraction of vertices in the lift that the matching M meets. Therefore, if every fractional matching φ in G satisfies $\sum \varphi(e) < (1-\epsilon)|V|/2$, then every matching in any lift of G must miss at least a fraction ϵ of the vertices in the lift.

These two comments cover the first and fourth cases of our main theorem.

Theorem 1. *Let G be finite connected graph, and consider $L_n(G)$, the space of its lifts of order n, where n is an even integer. Then exactly one of the following situations occurs:*

1. *Every $H \in L_n(G)$ has a perfect matching.*
2. *Not every $H \in L_n(G)$ has a PM, but almost all of them do.*
3. *In almost every $H \in L_n(G)$, the largest matching misses $\Theta(\log n)$ vertices.*
4. *Every matching, in every $H \in L_n(G)$, misses $\Omega(n)$ vertices.*

The implicit constants in the Θ and Ω terms depend only on G.

The first order of things is to characterize those graphs that have a FPM. We need some basic definitions:

Definition 1.1. An ordered pair of nonempty *disjoint* sets $(A,B) \subset V(G)$ is called a *t-split* if $N(B) \subset A$ and $|A| = |B| + t$. The smallest t for which a t-split exists is called the *excess* of G and is denoted $\xi(G)$.

Remarks: $N(B)$ is the set of all vertices in $V(G)$ adjacent to an element of B (including elements in B). Also, for graphs with a loop on each vertex the excess is left undefined.

Here is a characterization of graphs with a FPM that resembles Tutte's theorem on perfect matchings:

Lemma 1.2. *A graph has a FPM if and only if its excess is nonnegative.*

This lemma can be easily derived from material in [4], or proved directly via LP duality. ∎

Let us return to the main discussion. If every FPM vanishes on the edge $e \in E(G)$, then we say that e is a *non-FPM edge*. In this case, F_e is disjoint from every PM in any lift of G. Therefore almost every lift of G has a PM iff the same holds for $G \setminus e$. In other words, for our purposes, non-FPM edges can be always eliminated from the graph. It is also clear that lifts of G a.s. have a PM, iff this holds for every connected component of G. We only care now about graphs with a FPM, i.e., graphs of nonnegative excess. It's not difficult to see that a connected graph of excess 0 and no non-FPM edges has a PM (the Hall condition is satisfied). Our discussion is now limited to connected graphs G of positive excess (or with a loop on each vertex), having no PM, and no non-FPM edges. We want to eliminate next graphs with $e_G \leq v_G$ i.e., trees and unicyclic graphs. But it is not difficult to show that such a graph must be a odd cycle. A typical lift of an odd cycle consists of $\Theta(\log n)$ disjoint cycles, approximately half with an odd order. A matching must miss a vertex at each odd cycle, and we are led to the third case of the theorem.

Our discussion thus further narrows to graphs G that are (i) connected, (ii) have more edges than vertices, and, (iii) either have strictly positive excess, or have a loop at every vertex. The class of all such graphs is called $\mathcal{G}$ and the main technical part of our work is the proof of:

Theorem 2. *Let G be a graph in the class $\mathcal{G}$, then almost every lift of G has a perfect matching.*

The algorithmic realization of Theorem 1 is direct, as it is easy to decide whether a graph has a PM, FPM, and which are the non-FPM edges.

2. Theorem 2 – An overview

Let Π be the class of those graphs G such that almost every lift of G has a PM. Theorem 2 thus claims that $\mathcal{G} \subset \Pi$. Our proof is inductive. The simplest type of an induction step is a deletion of an edge. This argument works, since Π is clearly a monotone class of graphs. So, for example, if $G \in \mathcal{G}$ is 2-edge connected and has excess 3 or above, it is not hard to show that we can delete *any* edge e and argue by induction. If G has bridges, or has excess 1 or 2, a more careful analysis is required. Specifically, we introduce a reduction step that we apply only to graphs with excess 1. It is shown that if $G \in \mathcal{G}$ is reduced to the smaller graph $H \in \mathcal{G}$, and if $H \in \Pi$ then also

$G \in \Pi$. The only graphs for which neither edge deletion nor reduction work are loop stars. The membership of these graphs in Π is proved directly.

Definition 2.1. For any positive integer r, the loop star B_r is the graph that results by placing a loop on each leaf of the star $K_{1,r}$.

Definition 2.2. Reduction step on $G=(V,E)$: Assume $\xi(G)=1$ and (A,B) is a 1-split. Replace $S=A\cup B$ by a new vertex s which has a loop for each edge in $E(A)$. For each edge $[x,w]\in E(A,V\setminus S)$ there corresponds an edge $[s,w]$ (with multiplicities). If H is the resulting graph we denote $H=\mathbf{Red}(G,A,B)$ or $H=\mathbf{Red}(G,S)$. The natural bijection between $V(H)\setminus\{s\}$ and $V(G)\setminus S$, and the one between $E(H\setminus\{s\})$ and $E(G\setminus B)$ are used freely.

The proof of theorem 2 consists of the following three propositions:

Proposition 2.3. *Let G be a graph in $\mathcal{G}$ that is not a loop star, and does not have a PM. Then a series of operations can be applied to G, each of which is either an edge deletion or a reduction step, so that the components of the resulting graph still belongs to $\mathcal{G}$.*

Note: In most cases of the proof it is enough to carry out just a single operation.

Proposition 2.4. *Let G be a graph with excess 1, let (A,B) be a 1-split in G and let $H=\mathbf{Red}(G,A,B)$. If $H\in\Pi$ then also $G\in\Pi$.*

Finally, (or should we say initially?)

Proposition 2.5. *All loop stars are in Π.*

3. Some simple observations

In this section we collect several simple and useful observations that are used throughout the proof. First we show that the excess cannot decrease much under either vertex or edge deletions.

Claim 3.1. *Let G be a connected graph with excess $t\geq 2$ and let $e\in E(G)$. Then $\xi(G\setminus e)\geq t-2$. Moreover, $\xi(G\setminus e)=t-2$ iff $e\in E_G(B)$ for every $(t-2)$-split (A,B) in $G\setminus e$. If $G\setminus e$ is disconnected then $\xi(G\setminus e)\geq t-1$. For every vertex v, $\xi(G\setminus v)\geq t-1$.*

Proof. Let $\xi(G\backslash e)=k$ and consider a k-split (A,B) in $G\backslash e$, where $e=[x,y]$ is an edge in G. If $B\backslash\{x,y\}$ is nonempty, then $(A,B\backslash\{x,y\})$ is a split in G and $t\le|A|-|B\backslash\{x,y\}|\le|A|-|B|+2=k+2$, with equality iff both x,y are in B. If, say $B=\{x\}$, then $(A\cup\{y\},B)$ is a split in G, so $t\le|A\cup\{y\}|-1\le|A|=k+1$. Finally, if $B=\{x,y\}$, then $(A\cup\{y\},\{x\})$ is a split in G, so $t\le|A|=k+2$, and the condition for equality holds as well.

Consider now a bridge $e=[x_1,x_2]$ in G and let G_1,G_2 be the two components of $G\backslash e$, where x_i is in G_i. Suppose that $\xi(G\backslash e)=t-2$, and let (A,B) be a $(t-2)$-split in $G\backslash e$. Let A_1,A_2 resp. B_1,B_2 be the parts of A resp. B in G_1,G_2. By the previous observation, $x_1\in B_1$, and $x_2\in B_2$. Therefore, B_1,B_2 are nonempty, so that (A_1,B_1) and (A_2,B_2) are splits in $G\backslash e$. Again by the previous claim, they are suboptimal splits. Namely, $|A_i|-|B_i|\ge t-1$ for $i=1,2$. Add these inequalities to conclude that $t-2=|A|-|B|\ge 2t-2$, so $t\le 0$, a contradiction.

For the last part of the claim, assume $v\in V(G)$. Let (A,B) be some k-split in $G\backslash v$. Then $(A\cup\{v\},B)$ is a split in G which shows that $k+1\ge t$. ∎

As mentioned above, a connected graph of positive excess cannot have vertices of degree 1. More generally,

Claim 3.2. *If x is a loopless vertex in the graph G, then $\xi(G)\le d(x)-1$.* ∎

We say that a bipartite graph $(P,Q;E)$ has *surplus* $\ge k$ if $|N(X)|\ge|X|+k$ for every nonempty $X\subset Q$.

Claim 3.3. *Let $\xi(G)=t$, and let (A,B) be a t-split in G. Then the bipartite graph $(A,B;E_G(A,B))$ has surplus t. Equivalently, $|N(Y)|\ge|Y|$ for every $Y\subset A$ with $|Y|\le|B|$.*

Proof. $(N(P),P)$ is an $(|N(P)|-|P|)$-split for every $P\subset B$, and the conclusion follows. ∎

4. Reduction to a loop star

4.1. Graphs of excess 1

In this section we prove Theorem 2 for graphs of excess 1. In this case, proposition 2.3 simplifies to:

Lemma 4.1. *Let $G\in\mathcal{G}$ be a graph with excess 1. Then there is a 1-split (A,B) in G such that $H=\textbf{Red}(G,A,B)\in\mathcal{G}$ as well.*

Proof. We consider a 1-split (A,B) where $|A|+|B|$ is a as large as possible. To establish that $H\in\mathcal{G}$, we need to verify three properties.
(i) Connectivity: Clearly H is connected since connectivity is preserved by identifying vertices in a graph, and a reduction can be realized as a series of identification steps.
(ii) Excess: Assume by contradiction that there is a t-split (X,Y) in H with $t\le 0$. Let $s\in H$ be the vertex corresponding to the reduced set S. If $s\notin Y$, then $(A\cup X, B\cup Y)$ is a τ-split where $\tau\le(|A|+|X|)-(|B|+|Y|)=1+|X|-|Y|\le 1$. (The first inequality follows since B and Y are disjoint.) This contradicts the maximality of $|A|+|B|$.
If $s\in Y$ we claim that $(B\cup X, A\cup Y\setminus\{s\})$ is a τ-split in G with $\tau\le 0$. Since $s\in Y$, it follows that $N(A)\subset B\cup X$. So this is indeed a τ-split in G with $\tau=|B|+|X|-(|A|+|Y|-1)\le 0$.
(iii) $e(H)>v(H)$: We need to preclude the possibility that H is a tree or unicyclic. By Claim 3.2, H cannot have a vertex of degree one unless it is a loop, so H is either a single vertex with a loop or a cycle. If it is a cycle, it must be an odd cycle, since an even cycle has excess 0 while we already know $\xi(H)\ge 1$. If H is an odd cycle of order >1, say $x_1,x_2,\dots,x_{2k+1}$, let us assume that the reduced vertex s is x_{2k+1}. Define $X=x_1,x_3,\dots,x_{2k-1}$, $Y=x_2,x_4,\dots,x_{2k}$; then $(X\cup A, Y\cup B)$ is a 1-split in G contrary to the maximality of $|A|+|B|$. We are left with the case where H is a single vertex with a loop. That is, $A\cup B=V(G)$ and $|E_G(A)|=1$. In this case we exhibit another 1-split $V=A'\cup B'$ such that $\mathbf{Red}(G,A',B')$ is not a loop. This single edge $\{e\}=E(A)$ cannot be a loop, say in $a\in A$, for then $(B,A\setminus\{a\})$ is a 0-split in G. Thus $e=[x_1,x_2]$ for some $x_1,x_2\in A$. Note that $(B\cup\{x_2\},A\setminus\{x_2\})$ is a 1-split since $N(A\setminus\{x_2\})\subset B\cup\{x_2\}$. We can repeat the above arguments for this 1-split and conclude that $E(B\cup\{x_2\})$ consists of a single edge. Since $d_B(x_2)>0$, this must be an edge (x_2,x_3) where $x_3\in B$. This shows $d_B(x_2)=1$ and thus that $d_G(x_2)=2$ (and the same for x_1 by symmetry). This process may continue to create a sequence of vertices $x_1,x_2,\dots,x_k$ each with degree 2 in G. Since G is connected this shows that G is a cycle. This contradicts the assumption $e(G)>v(G)$. ∎

We turn to prove proposition 2.4:

Proof of proposition 2.4. By claim 3.3 the bipartite graph $(A,B,E_G(A,B))$ has surplus ≥ 1, so $A\cup B\setminus x$ has a perfect matching for every $x\in A$. We recall the following result from [4]:

Lemma 4.2. *Let $K=(P,Q;F)$ be a bipartite graph such that for every $x\in P$ there is a perfect matching in $K\setminus\{x\}$. Then there is a spanning tree $T\subset F$ such that $(P,Q;T)$ has the same property.*

We use this lemma to select a spanning tree T in the induced bipartite subgraph $(A,B;E_G(A,B))$ such that for every $x \in A$, the bipartite graph $(A \setminus \{x\},B;T)$ has a PM. Let L be a spanning subgraph of G, which is attained by replacing $E_G(A,B)$ by its subset T. We show that if $H \in \Pi$, then $L \in \Pi$ as well. Since L is a subgraph of G, this clearly implies that $G \in \Pi$.

We need an observation from [1]: Recall that a graph $\tilde{G} \in L_n(G)$ is defined by assigning a random permutation $\sigma_e \in S_n$ to each $e \in E(G)$. An edge $e \in E(G)$ is called *flat in* $\tilde{G}$ if σ_e is the identity. Recall also that a *graph property* is a property of graphs that does not depend on vertex labeling.

Proposition 4.3. *Let G be a graph and let $F \subset E(G)$ be a forest. The probability of every graph property in $L_n(G)$ stays unchanged under the conditioning that all edges in F are flat.*

We actually exhibit a bijection between PM's in n-lifts of L and n-lifts of H. Since $H = \mathbf{Red}(L,A,B)$, there is a natural bijection between $E(L \setminus B)$ and $E(H)$. We assume that the edges of the tree T are flat, which by the previous claim has no effect on any probabilities we consider. Thus, lifts of $L \setminus B$ and those of L are in 1:1 correspondence, so finally there is a bijection of lifts of H and L.

It will suffice to show now that if a lift $\tilde{H}$ of H has a PM then the corresponding lift $\tilde{L}$ of L has a PM as well. This will show that if $H \in \Pi$, then the same holds for L.

Let M_H be a PM of $\tilde{H}$. We create a PM of the corresponding lift $\tilde{L}$ as follows: M_H naturally induces a (not necessarily perfect) matching on $\tilde{L}$ by the bijection $E(\tilde{H})$ to $E(\tilde{L} \setminus \tilde{B})$ where $\tilde{B}$ is the lift of $B \subset S$. This matching uses exactly one vertex in every level of $\tilde{S}$, the lift of S, namely the vertex which corresponds to $s \in H$. Now the edges $E(\tilde{A},\tilde{B})$ are flat and by the choice of T, there is a perfect matching of $A \setminus \{x\}$ into B for every $x \in A$. This allows us to extend the induced matching from $\tilde{H}$ to a PM in $\tilde{L}$. ∎

4.2. Reprise: Graphs with excess ≥ 2

We now turn to complete the proof of proposition 2.3, by proving it for graphs of excess ≥ 2. Let $\mathcal{H}$ be the family of graphs $G \in \mathcal{G}$ with ≥ 3 vertices and $\xi(G) \geq 2$.

Note 4.4. *A graph $H \in \mathcal{H}$ satisfies $e(H) \geq v(H)+2$, since by claim 3.2, every loopless vertex in H has degree ≥ 3. If l out of the v vertices have loops, then $e(H) \geq l + (3(v-l)+l)/2 = 3v/2$. Now $\lceil \frac{3v}{2} \rceil \geq v+2$ for $v \geq 3$, as claimed.*

Note 4.5. *Each of the additional graphs that are still to be checked, contains either B_1 or B_2 (the smallest loop stars) as spanning subgraphs. These cases will be considered at the beginning of section 5.*

4.2.1. Exceptional edges We consider first "exceptional" edges: Parallel edges, loops and bridges and show that if they exist, the theorem follows by a simple inductive argument.

Claim 4.6. *Let $G \in \mathcal{H}$ have two parallel edges e, f. Then $G \setminus f \in \mathcal{G}$.*

Proof. Obviously $G \setminus f$ is connected, $\xi(G) = \xi(G \setminus f)$ and $e(G \setminus f) > v(G \setminus f)$. ∎

Henceforth we may, therefore, consider only graphs without parallel edges. We now characterize those loops which cannot be deleted.

Claim 4.7. *Let $G \in \mathcal{H}$ have no parallel edges. Suppose that $G \setminus e \notin \mathcal{G}$ for some loop $e = [v, v] \in E(G)$. Then $d_{G \setminus e}(v) = 1$.*

Proof. Obviously $G \setminus e$ is connected and has more edges than vertices. So $G \setminus e$ can fail to be in $\mathcal{G}$ only if $\xi(G \setminus e) = 0$. Let (A, B) be a 0-split in $G \setminus e$. Now $v \in B$, or else (A, B) is a 0-split in G as well, contrary to the assumption that $G \in \mathcal{H}$. If $B = \{v\}$ then indeed $d_{G \setminus e}(v) = 1$ since (A, B) is a 0-split and there are no parallel edges. If $|B| \geq 2$ then $(A, B \setminus \{v\})$ is a 1-split in G contrary to the assumption that $G \in \mathcal{H}$. ∎

We next turn our attention to bridges:

Proposition 4.8. *Let $G \in \mathcal{H}$ have no parallel edges, and let e be a bridge in G. Let C_1, C_2 be the components of $G \setminus e$. If $C_1 \notin \mathcal{G}$ then C_1 is a single vertex with a loop.*

Proof. By claim 3.1, C_1 has excess ≥ 1. It is also connected, so it can fail to be in $\mathcal{G}$ only because $e(C_1) = v(C_1)$, i.e., C_1 is an (odd) cycle. If C_1 has length ≥ 3 then it has a loopless vertex of degree 2. This contradicts the assumption that $\xi(G) \geq 2$. ∎

If there is a bridge e such that both components of $G \setminus e$ are in $\mathcal{G}$ then we are done by induction. So we assume from now on that at least one of the components of $G \setminus e$ is not in $\mathcal{G}$, and is therefore a loop according to the previous claim.

If G has bridges, then another kind of simple inductive argument applies. Say that $e = [u, v]$ is a bridge, and consider the graph $G \setminus v$. By claim 3.1 $\xi(G \setminus v) \geq 1$. Let $C_1, C_2, \ldots, C_l$ be the components of $G \setminus v$. Say $C_1 \ldots C_k$ are loops, and $C_{k+1} \ldots C_l$ are not. (Note that $k \geq 1$, by proposition 4.8, since

e is a bridge.) If each of $C_{k+1} \dots C_l$ is in $\mathcal{G}$ or has a PM, we argue thus: $\{v\} \cup C_1, \dots, \cup C_k$ is a loop star hence in Π, by proposition 2.5 (to be proved below). By induction each of $C_{k+1} \dots C_l$ is in Π and all told, $G \in \Pi$ as well. The same holds if those of $C_{k+1} \dots C_l$ that are not in $\mathcal{G}$ have a PM.

So assume $C = C_{k+1} \notin \mathcal{G}$ and has no PM. As in the proof of proposition 4.8, C must be an odd cycle, and not a loop. Also, every vertex w of C is a neighbor of v, since otherwise $d(w) = 2$, contrary to $\xi(G) \geq 2$. In other words, each of $C_{k+1} \dots C_l$ is an odd wheel centered at v. In G we can replace such an odd cycle by a cycle shorter by two, to obtain the graph H and show that $H \in \Pi$ implies $G \in \Pi$. This process terminates when we arrive at a loop star.

This shortening of cycles can be carried out in three steps:
(i) Remove an edge $e = [v, w]$ between the cut vertex v and a vertex w of the odd cycle C.
(ii) Now $d_{G \setminus e}(w) = 2$. We apply a reduction step to the 1-split $(N(w), \{w\})$.
(iii) Remove one of the two parallel edges between the vertex representing $N(w)$ and v.

4.2.2. The 2-edge-connected case In view of the previous subsection, to complete the proof of proposition 2.3, it suffices to establish it for simple (no loops, no parallel edges) 2-edge-connected graphs $G \in \mathcal{H}$. In this case we show that either it is possible to delete an edge and stay in $\mathcal{G}$, or G has a PM. Now $G \setminus e$ can fail to be in $\mathcal{G}$ only if the removal of edge e reduces the excess from 2 to 0. However, in this case we show that G has a PM.

Lemma 4.9. *Let $G \in \mathcal{H}$ be a simple 2 edge-connected graph. Suppose that $\xi(G \setminus e) = 0$ for every edge $e \in E(G)$. Then G has a perfect matching.*

Proof. The proof follows by showing that such a graph G is 3-regular and quoting Petersen's theorem. We must have $\xi(G) = 2$, for $\xi(G) \geq 3$ implies, by claim 3.1 $\xi(G \setminus e) \geq 1$ for every edge $e \in E(G)$ whence $G \setminus e \in \mathcal{G}$.

By Claim 3.2, in a simple graph with excess ≥ 2 all vertex degrees are ≥ 3, and we want to show equality. We employ a theorem of Lovász and Las Vergnas on bipartite graphs (see [4]): Let $H = (P, Q; E)$ be a bipartite graph of surplus k. Say that an edge $e \in E(H)$ is *redundant* if $H \setminus e$ still has surplus k.

Proposition 4.10 (Lovász, Las Vergnas). *Let $H = (P, Q; E)$ be a bipartite graph with surplus $k \geq 1$. If H has no redundant edges, then every vertex in Q has degree k.*

In a general graph G, an edge e is said to be (A,B)-*redundant* if it is redundant for the induced bipartite graph $(A,B;E_G(A,B))$. We will later prove:

Claim 4.11. *Let G be as in Lemma 4.9. If an edge e is (A,B)-redundant for some 2-split (A,B) then $\xi(G\setminus e)\geq 1$.*

We are assuming that G has no edges with $\xi(G\setminus e)\geq 1$. Hence, no 2-split (A,B), has (A,B)-redundant edges, so by Lovász-Las Vergnas all vertices in the B side have degree 3. To show that G is 3-regular, it suffices, then, to prove that every vertex $v\in V(G)$ belongs to the B-side of some 2-split (A,B). Consider an edge $e=[v,w]$ incident with v. Since $\xi(G\setminus e)=0$, by claim 3.1, there is a 0-split (A,B) in $G\setminus e$ with $v,w\in B$. Now $(A\cup\{w\},B\setminus\{w\})$ is a 2-split in G in which v is in the disconnected side as desired. This concludes the proof. ∎

Proof of claim 4.11. Let e be a (A,B)-redundant edge for some 2-split in G. We want to prove that $\xi(G\setminus e)\geq 1$. We argue by contradiction and consider a 0-split (X,Y) in $G\setminus e$. We classify the vertices in X and Y according to the partition $V(G)=A\cup B\cup(V(G)\setminus(A\cup B))$. Namely, $X=A_X\cup B_X\cup Q_X$ and $Y=A_Y\cup B_Y\cup Q_Y$, where $A_X=A\cap X$, $B_X=B\cap X$, $Q_X=X\setminus(A\cup B)$, and likewise A_Y,B_Y,Q_Y. Now $e\in E(Y)$, by claim 3.1 so $e\in E(A_Y,B_Y)$, whence A_Y,B_Y are both non-empty.

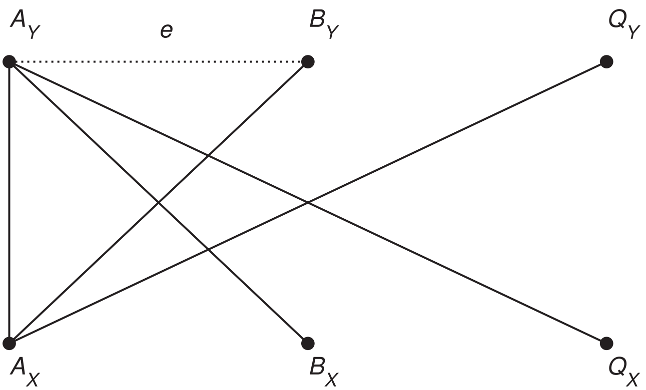

Fig. 1. The split X,Y. Solid lines connect sets that may have edges between them.

We prove the following three inequalities:
$|A_X|\geq|B_Y|+2$**:** Note that $N_{G\setminus e}(B_Y)\subset A$, since (A,B) is a split. Also, $N_{G\setminus e}(B_Y)\subset X$ since (X,Y) is a split. Thus $N_{G\setminus e}(B_Y)\subset X\cap A=A_X$. On the

other hand, since e is (A,B)-redundant, $|N_{G\setminus e}(B_Y)| \geq |B_Y|+2$, which gives the needed inequality.
$|B_X| \geq |A_Y|$**:** Since(X,Y) is a split, $N_{G\setminus e}(A_Y) \subset X$. Now $|A_Y| \leq |A|-2$ since $A_Y \subset A \setminus A_X$ and the previous inequality shows that $|A_X| \geq 2$. Now we can use claim 3.3 on the bipartite graph (A,B) induced by $G\setminus e$ and deduce that $|N_{G\setminus e}(A_Y) \cap B| \geq |A_Y|$. Since $N_{G\setminus e}(A_Y) \cap B \subset X \cap B = B_X$ we deduce the inequality.
$|Q_X| \geq |Q_Y|$**:** Otherwise, there is a ≤ 1-split in G.
Now add these inequalities to conclude that $|X| \geq |Y|+2$, contrary to the assumption that (X,Y) is a 0-split. ∎

4.3. Graphs with undefined excess

Recall that the excess is undefined for graphs with a loop on each vertex. Such graphs are in $\mathcal{G}$ if they are connected and have more edges than vertices. This case is easy to deal with by induction:

Proposition 4.12. *Let $G \in \mathcal{G}$ be a graph with a loop on each vertex, and $v(G) > 1$. Then either G has a PM, or we may remove a loop e of G such that $G \setminus e \in \mathcal{G}$.*

Proof. Removing a loop cannot disconnect the graph. Take the vertex v with degree $d(v)$ maximal in G (disregarding loops in the degree). If v has more than one loop on it then remove one of the loops e and we are done, since still $e(G\setminus e) > v(G)$. So assume there is only one loop e on v. If $d(v) > 1$ then $\xi(G \setminus e) = d(v) - 1 > 0$ as required. Otherwise G must consist of two vertices connected by an edge, with a loop on each one – a graph with a PM. ∎

The only case left is a graph with one vertex and two loops (or more) on it. This will be addressed in section 5.

5. The case of loop stars

In this section we show that all loop stars B_r belong to Π. We first deal with the cases $r = 1, 2$. Since B_1 has a PM, each of its lifts does as well. In B_2, perform the reduction that corresponds to the 1-split of the two leaves vs. the root. The reduced graph G has a single vertex with two loops. This G is in Π, because the edge set of every graph in $L_n(G)$ is the union of two random independent permutations. It is shown in [3] that such graphs a.s. contain a Hamiltonian cycle, and hence an PM. From now on, we consider only $r \geq 3$.

Our strategy is to apply, to every $G \in L_n(B_r)$, a random mapping ψ so that $\psi(G)$ is another random graph. We show that for every $G \in L_n(B_r)$ and every possible mapping ψ, if $\psi(G)$ has a PM, then so does G. Finally, we show that for almost every such G and a.e. mapping ψ, indeed $\psi(G)$ has a PM. The latter claim works only for $r \geq 3$, but that is enough.

5.1. Variations on a theme

We denote the vertex set of B_r as $\{0,1,\dots,r\}$, where 0 represents the central vertex. Recall that the vertex set of every $G \in L_n(B_r)$ is $\{0,1,\dots,r\} \times [n]$, where the vertex (i,j) is said to be in the i-th *fiber* F_i and in the j-th *level* of G. It is convenient to use proposition 4.3 again, and assume that the permutation corresponding to each non-loop edge of B_r is the identity (these edges are "flat"). Thus for every $G \in L_n(B_r)$, and every $j \in [n]$, the graph induced on the j-th level of G is the r-star $K_{1,r}$. Each loop $[i,i]$ of B_r is mapped to the graph of a random permutation σ_i on F_i. That is, each vertex (i,j) is adjacent to $(i,\sigma_i(j))$, and to $(i,\sigma_i^{-1}(j))$. (The graph corresponding to each permutation σ_i is a set of disjoint cycles in the i-th fiber.)

We now explain, for a given $G \in L_n(B_r)$ what a random $\psi(G)$ looks like. For each $i \in [r]$ we consider the cycles of the permutation σ_i. In each of these cycles we randomly select a maximal matching, the edges of which we *retain*. For an even cycle there are, of course, just two choices. For an odd cycle of length l, each of the l possible choices leaves exactly one *missed* vertex that is not covered by the matching. We note that a.s., it does not happen that both (i,j) and (i',j) are missed for two indices $i \neq i'$, since a.s. altogether only $O(r \log n)$ vertices are missed. In the (unlikely) event that some (i,j) and (i',j) are both missed, $\psi(G)$ remains undefined, but this does not affect our analysis.

Suppose that the vertices (i,α) and (i,β) are adjacent in G for some $r \geq i \geq 1$. The *projection* of this edge is $[\alpha,\beta]$. To construct the graph $\psi(G)$ we start with the graph on vertex set $[n]$ whose edges are the projections of all retained edges. Then we delete the projections of all missed vertices, i.e., all $n \geq j \geq 1$ such that (i,j) is missed for some $r \geq i \geq 1$. Note that $\psi(G)$ has no loops, since every vertex with a loop in G is necessarily missed.

We now show:

Claim 5.1. *If $\psi(G)$ has a PM then so does G.*

Proof. We observe that the set of all retained edges is a matching, so in our attempt to construct M, a PM in G, we first place those edges in M. Also, if (i,j) is missed, we add to M the edge connecting (i,j) to $(0,j)$. These two vertices are adjacent in G by our flatness assumption. So far, M fails to cover only vertices in the central fiber F_0. Now we assume that K is a PM in $\psi(G)$. For every edge $[\alpha,\beta]$ in K we modify M as follows: Say that $[\alpha,\beta]$ was projected from fiber F_i. Then we remove the (retained) edge between (i,α) and (i,β) from M. Instead, we add the edge that connects $(0,\alpha)$ and (i,α) as well as the edge between $(0,\beta)$ and (i,β). It is easy to check that the new M is indeed a PM. ∎

5.2. Finale

Fix some $r \geq 3$. In the previous subsection we have considered the distribution over graphs $\psi(G)$ that is defined by picking uniformly at random $G \in L_n(B_r)$, as well as a mapping ψ. We wish to show that the resulting graph a.s. has a PM. Here is an alternative, more convenient recipe for generating the same distribution of random graphs. For $i = 1,\dots,r$ we let E_i be a random perfect matching on the set $[n] \setminus H_i$. The H_i are i.i.d. random sets with cardinalities that are distributed as the number of odd cycles in a random permutation in S_n. In particular, $|H_i|$ is even. Finally, the vertices in $\cup H_i$ are omitted. Our aim is to show that these graphs a.s. have a PM.

It is not hard to see that it suffices to prove a.s. existence of a PM in the class of graphs $\mathcal{M}_{\mathrm{r,n}}$. These graphs have n vertices and an edge set of the form $\cup_1^r E_i$ where each E_i is a random perfect matching on a randomly selected set of $n - C_1 \log n$ vertices where $C_1 \leq 2(r-1)$. Actually, it will suffice to prove the result for $r = 3$, since the proof will not rely on this specific bound on C_1. Therefore we can reduce the general case to the case $r = 3$ by ignoring the matchings E_k with $k > 3$ in the edge set of $\mathcal{M}_{\mathrm{r,n}}$. In the following discussion let $\mathcal{M} = \mathcal{M}_{\mathrm{3,n}}$.

Claim 5.2. *Almost every $G \in \mathcal{M}$ has a perfect matching.*

The proof is based on the Tutte criterion (e.g. [4]). If G has no PM, then there exists a set $T \subset V(G)$ such that $G \setminus T$ has $\geq |T| + 1$ odd connected components. We show that this almost surely cannot occur. If A is a set of vertices in a graph G, let $f(A) = |E_G(A, V \setminus A)|$ be the size of the corresponding cut. We also define $f_s = \min_{A \subset V, |A| = s} f(A)$ and note that $f(A) = f(A^c)$ and $f_s = f_{n-s}$ always hold. So let A_i be the connected components of $G \setminus T$, of which more than $|T|$ have odd cardinality. Also $a_s = |\{j : |A_j| = s\}|$ is the number of components of order s. Observe that $3|T| \geq f(T) = |E_G(T, V \setminus T)| = \sum f(A_i)$ and since there are more than $|T|$ odd components:

$$-3 \geq \sum_{|A_i| \text{ is odd}} (f(A_i) - 3) + \sum_{|A_i| \text{ is even}} f(A_i) \tag{1}$$

In the standard proof of Petersen's Theorem, $f(A_i) \geq 3$ whenever $|A_i|$ is odd, and we get a contradiction. The complications that arise here are solved by means of the following lemma.

Lemma 5.3. *Almost every $G \in \mathcal{M}_n$ has the following properties:*

1. $f_1 \geq 2$, *i.e., all vertices have degree* ≥ 2.
2. *At most $\gamma_1 \log n$ vertices have degree* 2.
3. $f_3 \geq 3$.
4. *There are at most $\omega(n)$ connected sets $A \subset V$ with $|A| = 3$ and $f(A) = 3$.*
5. $f_s \geq 4$ *for every* $n - 5 \geq s \geq 5$.
6. $f_s \geq (1 + \gamma_1) \log n$ *for every* $n - \gamma_2 \log n \geq s \geq \gamma_2 \log n$.
7. *The independence number* $\alpha(G) < (1/2 - \epsilon)n$.

Here ω is any function that tends to ∞ with n, $\gamma_1 = 4C_1 = 12$, $\gamma_2 = 200\gamma_1$, and $\epsilon = 0.04$.

This lemma (whose proof is in section 5.3), has several useful consequences. For example:

Claim 5.4. *A.s. $a_s = 0$ for every $\gamma_2 \log n \leq s \leq n - \gamma_2 \log n$.*

Proof. Inequality (1) clearly yields:

$$-3 \geq -b_1 + (a_3 - b_3) + \sum_{s \geq 5 \text{ odd}} a_s(f_s - 3) + \sum_{s \text{ even}} a_s f_s \tag{2}$$

where $b_1 \leq a_1$ is the number of isolated vertices in $G \setminus T$ that have degree 2. Similarly, $b_3 \leq a_3$ is the number of components A_i with 3 vertices, and $f(A_i) = 3$. By lemma 5.3.2, $b_1 \leq \gamma_1 \log n$. Therefore, $\gamma_1 \log n \geq \sum_{s \geq 5 \text{ odd}} a_s(f_s - 3) + \sum_{s \text{ even}} a_s f_s$. Now lemma 5.3.6 implies that a_s must vanish for every $\gamma_2 \log n \leq s \leq n - \gamma_2 \log n$. ∎

We now show that (exactly) one large component must exist:

Claim 5.5. *A.s. for sets T and A_i as above, there holds $\max_i |A_i| \geq n - \gamma_2 \log n$.*

Proof. A simple count of all vertices yields $n = |T| + \sum s a_s$. The number of connected components is $\sum a_s > |T|$. Therefore, $n \leq \sum (s+1) a_s$. But lemma 5.3.7 implies that $a_1 \leq (1/2 - \epsilon)n$, since the singleton A_i's form an independent set. Therefore, $2\epsilon n \leq \sum_{s \geq 2} (s+1) a_s$. If all a_s vanish for $s \geq$

$n-\gamma_2\log n$, then by the previous claim a_s vanishes for every $s>\gamma_2\log n$, whence $\epsilon n \le \gamma_2 \log n \sum_{s\ge 2} a_s$. Now let us return to inequality (2) and conclude that

$$\gamma_1 \log n + \omega(n) \ge b_1 + b_3 \ge \sum_{s\ge 2} a_s$$

where we have used lemma 5.3.2 and 5.3.4 to estimate b_1, b_3 respectively and 5.3.5 to conclude that $f_s - 3 \ge 1$ for the relevant range. This is a contradiction, since the l.h.s is $O(\log n)$, while the r.h.s., as we saw, is $\Omega(\frac{n}{\log n})$. ∎

To complete the proof of claim 5.2 it suffices to show that almost surely, no set $T\subset V(G)$ exists so that A_i, the connected components of $G\setminus T$ satisfy: (i) $|A_1|\ge n-O(\log n)$, and (ii) At least $|T|$ of the A_i for $i\ge 2$ have odd cardinality. Fix a partition $V(G)=T\cup(\cup A_i)$ as above, and consider a perfect matching E on a random set $V\backslash H$ of vertices. What is the probability that E is consistent with the partition? Fix an ordering on the vertices of each A_i, and construct the matching E sequentially. This probability can be expressed as a product of conditional probabilities: Each vertex $x\in A_i$, at its turn, is either already matched with a previous vertex (and then the conditional probability is 1) or must be matched with a vertex in $A_i\cup H\cup T$ (and then the conditional probability is $O(\frac{\log n}{n})$). The latter case occurs in at least $\lceil\frac{|A_i|}{2}\rceil$ of the steps, so the probability that E is consistent with this given partition does not exceed $(O(\frac{\log n}{n}))^{\sum_{j\ge 2}\lceil\frac{|A_j|}{2}\rceil}$. There are no more than $n^{|T|+\sum_{j\ge 2}|A_j|}|T|^{\sum_{j\ge 2}|A_j|}$ partitions to consider (the first factor bounds the number of choices for the total set $T\cup\cup_{i\ge 2}A_i$. The second bounds the number of ways to partition this total set to the subsets T and the A_i, assuming without loss of generality that there are exactly $|T|$ odd sets A_i). There are three independent matchings, so a Tutte decomposition of the type under consideration exists with probability at most

$$n^{(|T|+\sum_{j\ge 2}|A_j|)} n^{\frac{\log|T|}{\log n}\sum_{j\ge 2}|A_j|}\Big(O\Big(\frac{\log n}{n}\Big)\Big)^{3\sum_{j\ge 2}\lceil\frac{|A_j|}{2}\rceil}$$

But now we can use the fact that $|A_j|$ is odd for at least $|T|$ of the indices $j\ge 2$, to conclude that this probability does not exceed (ignoring some logarithmic factors)

$$O\Big(\frac{\log n}{n}\Big)^{-|T|/2-\sum_{j\ge 2}|A_j|/2+\left(\frac{\log\log n}{\log n}\right)\sum_{j\ge 2}|A_j|} = o(1)$$

as claimed.

5.3. Proof of Lemma 5.3

We skip the proof of the first four parts of Lemma 5.3, which are rather easy. For the rest of the proof, the following observation is useful:

Proposition 5.6. *For n even, there are exactly $(n-1)!! := (n-1)(n-3)\cdots 1$ perfect matchings on a set of n vertices. Also, $n!! = \Theta(n^{-1/4}\sqrt{n!})$.*

The proof follows by Stirling's formula, or by noting that $\phi(n) = \frac{(n!!)^4}{n!(n-1)!}$ is increasing and bounded.

The following two claims imply 5.3.5.

Claim 5.7. *A.s. $f(A) \geq 4$ for every $A \subseteq V(G)$ with cardinality $5 \leq |A| \leq n^{1/7}$.*

Proof. We first note that the upper bound on $|A|$ is arbitrary and anything $\gg \log n$ would do. Fix a set A of size q. As before, let H_i be the set of $h = C_1 \log n$ vertices missed by E_i. Let $P(q,l,u)$ be the probability that $|A \cap H_1| = u$ and $|E_1(A, V\setminus A)| = l$.

$$P(q,l,u) = \frac{\binom{q}{u}\binom{n-q}{h-u}\binom{q-u}{l}\binom{n-q-h+u}{l} l!(n-q-h+u-l-1)!!(q-u-l-1)!!}{\binom{n}{h}(n-h-1)!!} \tag{3}$$

Explanation: $\binom{q}{u}$ is the number of choices for $A \cap H_1$,
$\binom{n-q}{h-u}$ – the choices for $H_1 \setminus A$,
$\binom{q-u}{l}$ – choices for the set A_l of l vertices in A that are matched to $V \setminus A$,
$\binom{n-q-h+u}{l}$ – choices for the set B_l of l vertices outside A that are matched to A_l,
$l!$ – number of ways to match (A_l, B_l),
$(q-u-l-1)!!$ – matchings of $A \setminus (H_1 \cup A_l)$,
$(n-q-h+u-l-1)!!$ – matchings of $V \setminus (A \cup H_1 \cup B_l)$.

It is easy to estimate:

$$\begin{aligned} &P(q,l,u) \\ &\quad \leq O\left(\frac{\binom{q}{u}\binom{n-q}{h-u}\binom{q-u}{l}\binom{n-q-h+u}{l} l!}{\binom{n}{h}} \cdot \sqrt{\left(\frac{(n-q-h+u-l-1)!(q-u-l)!}{(n-h-1)!}\right)}\right) \\ &\quad \leq O\left(q! \cdot n^{(h-u)+l-h-(q-u+l)/2+o(1)}\right) \leq O\left(q! \cdot n^{-(q+u-l)/2+o(1)}\right). \end{aligned}$$

The probability that such a set A, of cardinality q, exists for which the claim fails does not exceed

$$\sum_{|A|=q=5}^{n^{1/7}} \binom{n}{q} \sum_{u_1,u_2,u_3\le h} \sum_{l_1+l_2+l_3\le 4} \prod_{i=1}^{3} P(q,l_i,u_i)$$

$$= \sum_{q=5}^{n^{1/7}} \sum_{u_1,u_2,u_3\le h} \sum_{l_1+l_2+l_3\le 4} O\left(q!^3\cdot\binom{n}{q} n^{(\sum l_i-3q-\sum u_i)/2+o(1)}\right)$$

$$\le \sum_{q=5}^{n^{1/7}} O\left(q!^3\cdot\binom{n}{q} n^{-3q/2+4/2+o(1)}\right) \le \sum_{q=5}^{n^{1/7}} O\left(q!^3\cdot n^{-q/2+2+o(1)}\right) \le o(1)$$

as claimed. ∎

Claim 5.8. *A.s.* $f(A)\ge|A|/100$ *whenever* $100\gamma_1\log n\le|A|\le n/2$.

Proof. We omit the proof, whose general structure us similar to that of claim 5.7. ∎

Proof of Lemma 5.3.7. We start again from equation (3), and argue as in the proof of 5.8 (with $l=q$). Consider a given set A of q and a random matching E as above. The probability that no edge of E is contained in A is $P(q)\le O\left(\sqrt{\frac{\binom{n-q}{q}}{\binom{n}{q}}}\right)$. Therefore, the probability that there exists any independent set of order q does not exceed $\binom{n}{q}(P(q))^3\le\frac{\binom{n-q}{q}^{3/2}}{\binom{n}{q}^{1/2}}$. Now let $\lambda=\frac{q}{n}$. By standard estimates, this probability does not exceed

$$2^{(1+o(1))\frac{n}{2}(3(1-\lambda)H(\frac{\lambda}{1-\lambda})-H(\lambda))}.$$

But $3(1-\lambda)H(\frac{\lambda}{1-\lambda})-H(\lambda)$ is negative for $\lambda>0.4591$ and the claim follows. ∎

6. Further work

There are two obvious questions to consider next. (i) Does the theorem stay essentially unchanged for odd values of n as well? Of course if there is an odd number of vertices in the lift we can only hope for a matching that misses only a single vertex. (ii) Under what conditions on G does a.e. lift of G have a *Hamiltonian circuit*?

While the first question may not be too difficult to solve now, the other one surely requires some completely new ideas.

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References

[1] Alon Amit and Nathan Linial: Random lifts of graph I: General theory and connectivity; to appear.

[2] A. Amit, N. Linial, J. Matousek and E. Rozenman: Random Lifts of Graphs, *Proceedings of the Symposium on Discrete Algorithms (SODA)* 2001, 883–895.

[3] A. M. Frieze: Hamilton cycles in the union of random permutations, *Random Structures and Algorithms* **18** (2001), 83–94.

[4] L. Lovász and M. D. Plummer: *Matching Theory*, North Holland, 1986.

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