Random Lifts of Graphs

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Plan of this talk

- A brief introduction to the probabilistic method.
- A quick review of expander graphs and their spectrum.
- Lifts, random lifts and their properties.
- Spectra of random lifts.

What is the probabilistic method? Introduction by example.

Theorem

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In other words: If you color the edges of K_6 (the complete graph on 6 vertices) blue and red, you necessarily find a monochromatic (either red or blue triangle.

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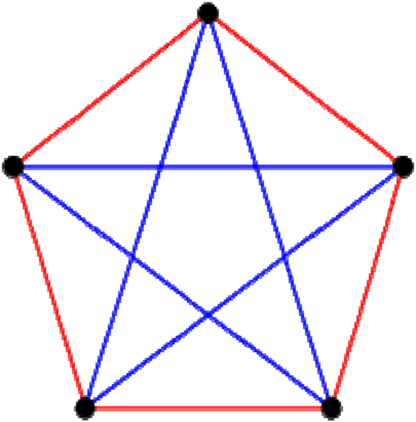

More generally,

Theorem (Ramsey; Erdős-Szekeres)

Let $N = \binom{r+s-2}{r-1}$. If you color the edges of K_N red and blue, then you necessarily get a red K_r or a blue K_s.

Diagonal Ramsey Numbers

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In every red-blue coloring of the edges of K_N there is a monochromatic complete subgraph on at least

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Every N-vertex graph contains either a clique or an anti-clique on at least

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Diagonal Ramsey Numbers (contd.)

Theorem (Erdős '49)

There are red-blue colorings of K_N where no monochromatic subgraph has more than

$$2 \log_2 N \quad \textit{vertices.}$$

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We do not know how to do this. In fact it is a major challenge to find such explicit colorings.

Instead, we use the probabilistic method.

Introducing the probabilistic method

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In other words, we give the following recipe for sampling from Ω: For each edge of K_N, flip a coin (independently from the rest). If it comes out heads color the edge red if you get tails, color it blue.

Let us consider an integer r (to be determined later) and a random variable B defined on Ω. for a given coloring $\mathcal{C}$ of K_N, we define $B(\mathcal{C})$ to be the number of sets of r vertices in $\mathcal{C}$ all of whose edges are blue. The expectation of X is:

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We likewise define a random variable R that counts red subgraphs.

Note that if $B(\mathcal{C}) = \mathcal{R}(\mathcal{C}) = 0$, there is no monochromatic set of r vertices in $\mathcal{C}$, which is just what we need.

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In every mathematical field intuition is created from examples that we know.
But it is hard to analyze large specific examples and the probabilistic method allows us to bypass this difficulty. It serves us as an observational tool, much like the astronomer's telescope or a biologist's microscope.

Models of random graphs

What we saw is a close relative of the most basic model of random graphs, Erdős-Rényi's $G(n, p)$ model. In this model we start with n vertices. For each pair of vertices x, y we decide, independently and with probability p, to put an edge between x and y.

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There are other important and interesting models of graphs. For example, we have known for 30 years now how to sample random d-regular graphs.

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A typical basic question in this area: What is the probability that an infinite connected component remains.

You could also seek models to describe natural or artificial phenomena such as the Internet graph or biological control networks.

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Random lifts of graphs, our subject today, are such a model.

A very quick review on expansion in graphs

There are three main perspectives of expansion:

- Combinatorial - isoperimetric inequalities
- Linear Algebraic - spectral gap
- Probabilistic - Rapid convergence of the random walk (which we do not discuss today)

For (much) more on this: see our survey article with Hoory and Wigderson.

The combinatorial definition

A graph $G = (V, E)$ is said to be ϵ-edge-expanding if for every partition of the vertex set V into X and $X^c = V \setminus X$, where X contains at most a half of the vertices, the number of cross edges

$$e(X, X^c) \geq \epsilon |X|.$$

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In words: in every cut in G, the number of cut edges is at least proportionate to the size of the smaller side.

The combinatorial definition (contd.)

The edge expansion ratio of a graph $G = (V, E)$, is

$$h(G) = \min_{S \subseteq V, |S| \leq |V|/2} \frac{|E(S, \overline{S})|}{|S|}.$$

The linear-algebraic perspective

The **Adjacency Matrix** of an n-vertex graph G, denoted $A = A(G)$, is an $n \times n$ matrix whose (u, v) entry is the number of edges in G between vertex u and vertex v. Being real and symmetric, the matrix A has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

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- The graph is bipartite iff $\lambda_1 = -\lambda_n$.
- $\chi(G) \geq -\frac{\lambda_1}{\lambda_n} + 1$.
- A substantial spectral gap implies logarithmic diameter.

Spectrum vs. expansion

Theorem

Let G be a d-regular graph with spectrum $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{(d+\lambda_2)(d-\lambda_2)}.$$

The bounds are tight.

What's a "large" spectral gap?

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Theorem (Alon, Boppana)

$$\lambda_2 \geq 2\sqrt{d-1} - o(1)$$

The meaning of the number $2\sqrt{d-1}$

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$$(-2\sqrt{d-1}, 2\sqrt{d-1})$$

Some questions

How tight is this bound?

Problem

Are there d-regular graphs with second eigenvalue

$$\lambda_2 \le 2\sqrt{d-1} \quad ?$$

When such graphs exist, they are called Ramanujan Graphs.

What is the typical behavior?

Problem

How likely is a (large) random d-regular graph to be Ramanujan?

What is currently known about Ramanujan Graphs?

Margulis; Lubotzky-Phillips-Sarnak; Morgenstern:

d-regular Ramanujan Graphs exist when $d-1$ is a prime power. The construction is easy, but the proof uses a lot of heavy mathematical machinery.

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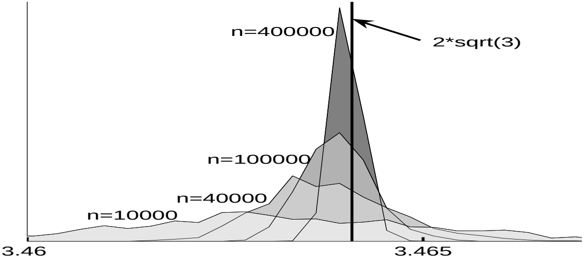
Friedman: If you are willing to settle for $\lambda_2 \leq 2\sqrt{d-1}+\epsilon$, they exist. Moreover, almost every d-regular graph satisfies this condition.

The distribution of the second eigenvalue



Some open problems on Ramanujan Graphs

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- Can we find combinatorial/probabilistic methods to construct graphs with large spectral gap (or even Ramanujan)? As we'll see random lifts of graphs (Bilu-L.) yield graphs with

$$\lambda_2 \leq O(\sqrt{d}\log^{3/2} d).$$

Covers and lifts - the abstract approach

Definition

A map $\varphi : V(H) \to V(G)$ where G, H are graphs is a covering map if for every $x \in V(H)$, the neighbor set $\Gamma_H(x)$ is mapped $1:1$ onto $\Gamma_G(\varphi(x))$.

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This is a special case of a fundamental concept from topology. Recall that a graph is a one-dimensional simplicial complex, so covering maps can be defined and studied for graphs.

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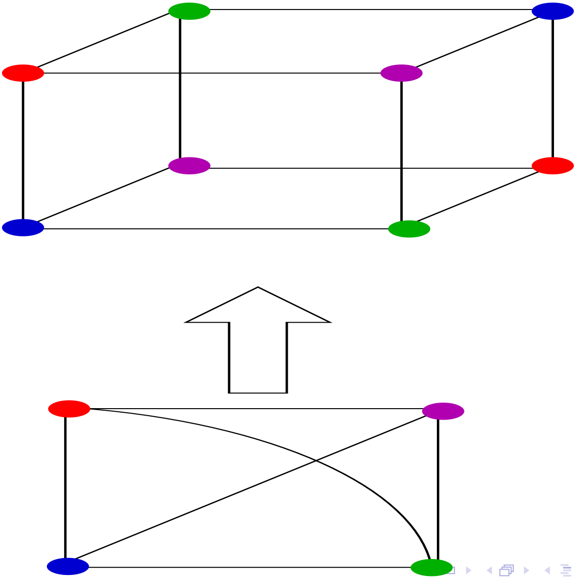
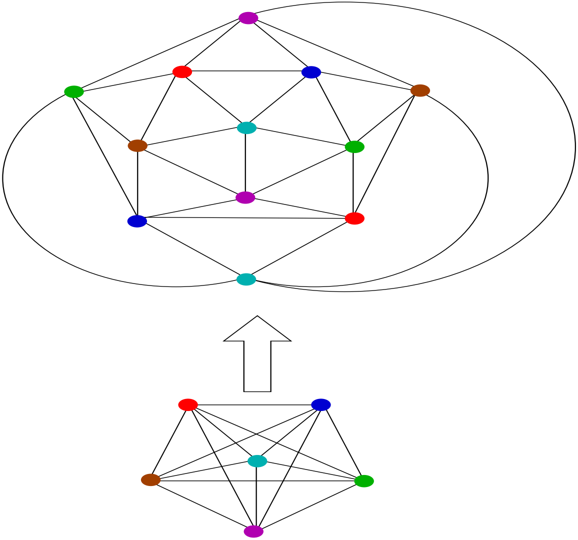
Convention: We will always assume that the base graph is connected. This creates no loss in generality.

An example - The 3-cube is a 2-lift of K_4



The icosahedron is a 2-lift of K_6



Making this definition more concrete

We see in the previous examples that the covering map φ is $2:1$.

- The 3-cube is a 2-lift of K_4.
- The graph of the icosahedron is a 2-lift of K_6.

In general, if G is a connected graph, then every covering map $\varphi : V(H) \to V(G)$ is $n:1$ for some integer n (easy).

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- The set of those graphs that are n-lifts of G is denoted by $L_n(G)$.

A direct, constructive perspective

- Every $H \in L_n(G)$, has vertex set $V(H) = V(G) \times [n]$.

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- For every edge $e = xy \in E(G)$ we have to select some perfect matching between the fibers F_x and F_y, i.e., a permutation $\pi = \pi_e \in S_n$ and connect (x, i) with $(y, \pi(i))$ for $i = 1, \ldots, n$.

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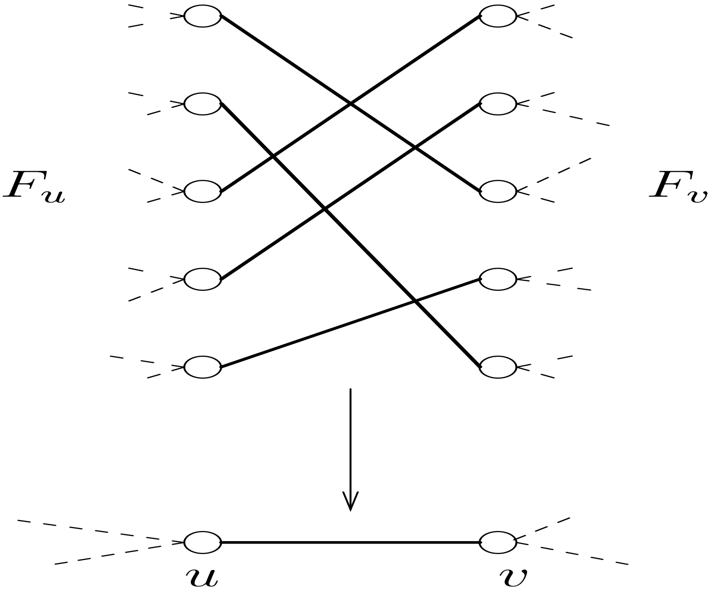
- This set of edges is denoted by F_e, the fiber over e.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  - To construct graphs with certain desirable properties. In our case, to achieve large spectral gaps.
  - To model various phenomena.
  - To study their typical properties.

A few more general properties of lifts

- Vertex degrees are maintained. If x has d neighbors, then so do all the vertices in the fiber of x. In particular, a lift of a d-regular graph is d-regular.

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- Vertex degrees are maintained. If x has d neighbors, then so do all the vertices in the fiber of x. In particular, a lift of a d-regular graph is d-regular.
- The cycle C_n is a lift of C_m iff $m|n$.
- The d-regular tree covers every d-regular graph. This is the *universal cover* of a d-regular graph. Every connected base graph has a universal cover which is an infinite tree.

Old vs. New Eigenvalues

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The lifted graph inherits every eigenvalue of the base graph.

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Namely, if H is a lift of G, then every eigenvalue of G is also an eigenvalue of H(Pf: Pullback, i.e., take any eigenfunction f of G, and assign the value $f(x)$ to every vertex in the fiber of x. It is easily verified that this is an eigenfunction of H with the same eigenvalue as f in G).

Old vs. New Eigenvalues (contd.)

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This suggests the following approach to the construction of d-regular Ramanujan Graphs by repeated lifts:

- Start from a small d-regular Ramanujan Graph (e.g. K_{d+1}).
- In every step apply a lift to the previous graph while keeping all new eigenvalues in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$

How to think about 2-lifts

In an n-lift of a graph G we associate with every edge $e = xy$ of G a permutation $\pi_e \in S_n$ which tells us how to connect the n vertices in the fiber F_x with the n vertices of F_y.

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Alternatively, we sign the edges of G where $+1$ stands for **id** and -1 for σ.

Signing and spectra

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Proposition

The new eigenvalues of a 2-lift of G are the eigenvalues of the corresponding signing matrix.

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Conjecture

Every d-regular Ramanujan Graph has a signing with spectral radius $\leq 2\sqrt{d-1}$.

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This conjecture, if true, is tight.

What is known

Theorem (Yonatan Bilu + L.)

By repeated application of 2-lifts it is possible to explicitly construct d-regular graphs ($d \geq 3$) whose second eigenvalue

$$\lambda_2 \leq O(\sqrt{d} \log^{3/2} d)$$

A highlight of the proof

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Our new lemma says that λ_2 is controlled by the extent to which G is pseudo-random.

What's involved is the graph's discrepancy, i.e. the maximum of

$$\frac{e(A,B) - \frac{d}{n}|A||B|}{\sqrt{|A||B|}}$$

A few more things about random lifts

A matching M in a graph G is a collection of disjoint edges. If the edges in M meet every vertex in G, we say that M is a perfect matching=PM. The defect of G is the number of vertices missed by the largest matching in G. (So the existence of a PM is the same as zero defect).

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Question: Given a base graph G and a large even integer n, how likely is an n-lift of G to contain a perfect matching?

Theorem (L. + Rozenman)

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- *The probability that H has a PM is $1 - o(1)$ (but not 1).*
- *Almost surely H has defect $\Theta(\log n)$.*

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- The typical degree of connectivity of a lift of G.
- The chromatic numbers of typical lifts.
- The typical distribution of new eigenvalues.
- ... and more ....

… and much more that we do not know…

Open Problem

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- *Is there a zero-one law for Hamiltonian cycles?*
- *What is the typical chromatic number of an n-lift of K_5? Is it 3 or 4, perhaps each with positive probability?*