What is high-dimensional combinatorics?

Nati Linial

Random-Approx, August '08

A (very) brief and (highly) subjective history of combinatorics

Combinatorics must always have been fun. But when and how did it become a serious subject? I see several main steps in this development:

- The asymptotic perspective.
- Extremal combinatorics (in particular extremal graph theory).
- The emergence of the probabilistic method.
- The computational perspective.

So, what is the next frontier?

The ubiquity of graphs

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But what if we have relations involving more than two objects at a time?

A little about simplicial complexes

This is one of the major contact points between combinatorics and geometry (more specifically - with topology).
From the combinatorial point of view, this is a very simple and natural object. Namely, a down-closed family of sets.

Definition

Let V be a finite set of *vertices*. A collection of subsets $X \subseteq 2^V$ is called a *simplicial complex* if it satisfies the following condition:

$$A \in X \ \text{ and } \ B \subseteq A \Rightarrow B \in X.$$

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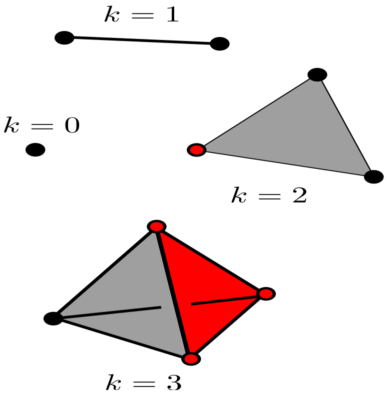
The dimension of X is the largest dimension of a face in X.

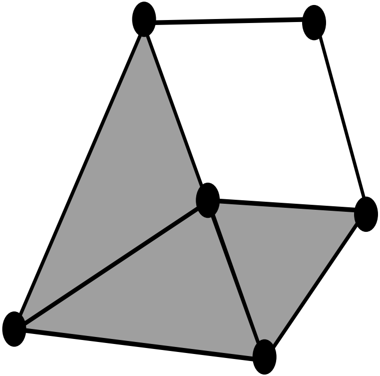
Simplicial complexes as geometric objects

We view $A \in X$ and $|A| = k + 1$ as a k-dimensional simplex.



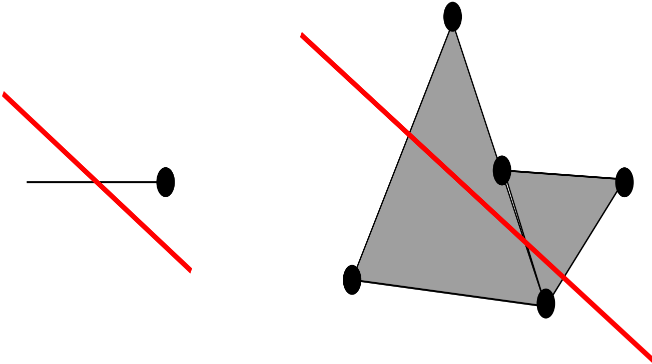
Putting simplices together properly

The intersection of every two simplices in X is a common face.



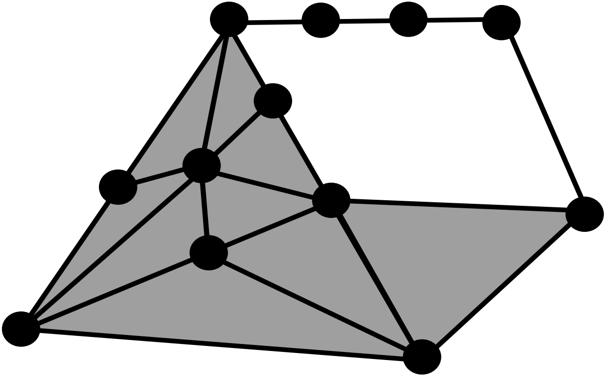
How NOT to do it

Not every collection of simplices in $\mathbb{R}^d$ is a simplicial complex



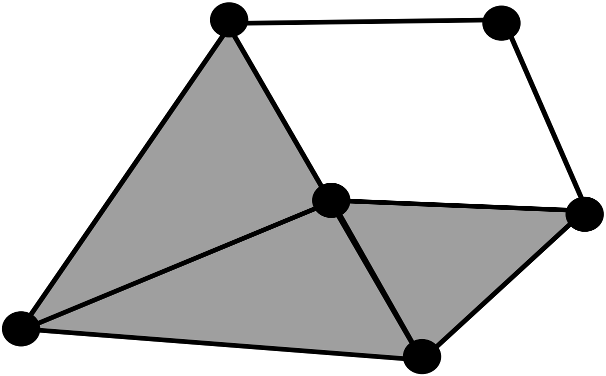
Geometric equivalence

Combinatorially different complexes may correspond to the same geometric object (e.g. via subdivision)



Geometric equivalence

So



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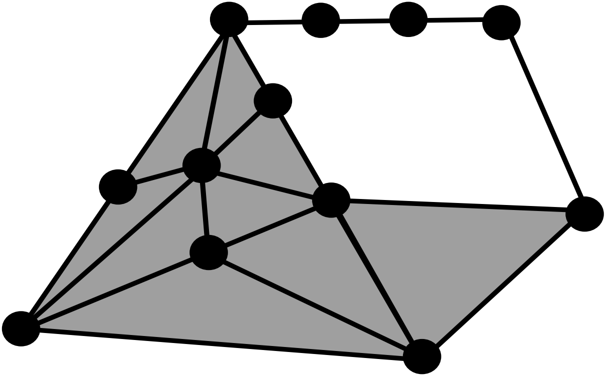
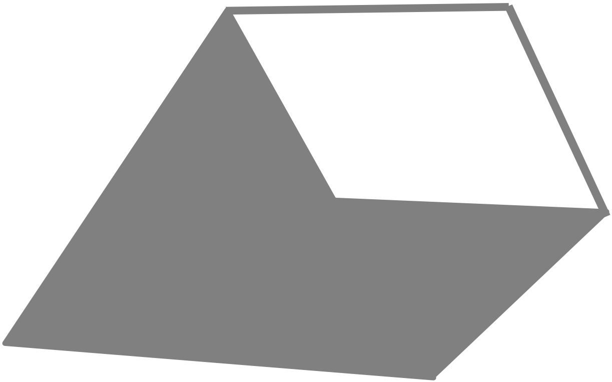


Geometric equivalence

are two different combinatorial descriptions of the same geometric object



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- Graphs need no advertising for computer scientists.
- A graph may be viewed as a one-dimensional simplicial complex.
- Higher dimensional complexes have a very geometric (mostly topological) aspect to them.
- Can we benefit from investigating higher dimensional complexes?
- How should this be attacked?
  1. Using extremal combinatorics
  2. With the probabilistic method

A disclaimer and an apology

My description of past relevant work is extremely incomplete.
My deep and sincere apology to all those whose work I have no time to mention.
This (academic) misdemeanor will be justified if the audience finds this interesting and starts learning the subject at depth.

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So far there are only very few (but very impressive) such examples. I believe that there are many important connections waiting to be discovered. Some of the past achievements:

- Applications to the evasiveness conjecture (See below).
- Impossibility theorems in distributed asynchronous computation (Starting with [Herlihy, Shavit '93] and [Saks, Zaharoglou '93]).

In combinatorics

Here the list is a bit longer, e.g.,

- To graph connectivity (Lovász's proof of A. Frank's conjecture 1977).
- Lower bounds on chromatic numbers of Kneser's graphs and hypergraphs. (Starting with [Lovász '78]).
- To matching in hypergraphs (Starting with [Aharoni Haxell '00]).

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The game ends when Alice knows with certainty whether G has property $\mathcal{P}$.

The evasiveness conjecture

Conjecture

For every monotone graph property $\mathcal{P}$, Bob has a strategy that forces Alice to query all $\binom{n}{2}$ pairs of vertices in V.

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A: Fix n, the number of vertices in the graphs we consider. Think of an n-vertex graph as a subset of $W = \binom{[n]}{2}$. (Careful: W is the set of vertices of the complex we consider).

If $\mathcal{G}$ is the collection of all n-vertex graphs that have property $\mathcal{P}$, then $\mathcal{G}$ is a simplicial complex (since $\mathcal{P}$ is monotone).

Kahn Saks and Sturtevant (contd.)

The (simple but useful) observation with which they start is

Lemma

A non-evasive complex is collapsible.

Collapsibility is a simple combinatorial property of simplicial complexes which we do not define here. It can be thought of as a higher-dimensional analogue of being a forest.

Kahn Saks and Sturtevant

The additional ingredient is that $\mathcal{P}$ is a graph property. Namely, it does not depend on vertex labeling. This implies that the complex $\mathcal{G}$ is highly symmetric. Using some facts from group theory they conclude:

Theorem (KSS '83)

The evasiveness conjecture holds for all graphs of order n when n is prime.

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- Fixed-point theorems (Borsuk-Ulam, Sperner's Lemma...).
- Collapsibility, contractibility
- The "size" of homology, Betti numbers...
- Topological connectivity.

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- Introduce ideas from topology into computational complexity

So now that we are highly motivated...

We want to develop a theory of random complexes, similar to random graph theory. Specifically we seek a higher-dimensional analogue to $G(n, p)$. We consider (Though this is by no means the only sensible model).

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We denote by $X(n, p)$ this probability space of two-dimensional complexes.

The art of asking good questions

What properties of these random complexes should we investigate?
Perhaps the simplest nontrivial property of a graph is being connected. Here is what Erdős-Rényi showed nearly 50 years ago:

Theorem (ER '60)

The threshold for graph connectivity in $G(n, p)$ is

$$p = \frac{\ln n}{n}$$

When is a simplicial complex connected?

Unlike the situation in graphs, this question has many (in fact infinitely many) meaningful answers when it comes to complexes.

- The vanishing of the first homology (with any ring of coefficients).
- Being simply connected (vanishing of the fundamental group).

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- Likewise, if S is the vertex set of a connected component of G, then $\mathbf{1}_S M = 0$.
- It is not hard to see that G is connected iff the only vector x that satisfies $xM = 0$ is $x = \mathbf{1}$.

This brings us back to topology

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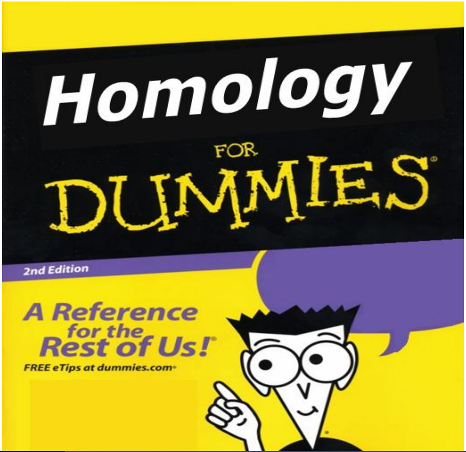

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- The transformations associated with A_1 resp. A_2 are called *the boundary operator* (of the appropriate dimension) and are denoted ∂ (perhaps with an indication of the dimension).

It is an easy exercise to verify that $A_1 A_2 = 0$ (in general there holds $\partial\partial = 0$, a key fact in homology theory).

Vanishing of the first homology

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To a two-dimensional complex in $X(n, p)$ corresponds a random matrix B that we obtain by selecting a random subset of the $\binom{[n]}{3}$ columns in A_2 where each column is selected independently with probability p.

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To a two-dimensional complex in $X(n, p)$ corresponds a random matrix B that we obtain by selecting a random subset of the $\binom{[n]}{3}$ columns in A_2 where each column is selected independently with probability p.
We can now consider our high-dimensional notion of being connected: The first homology with $\mathbb{F}_2$ coefficients vanishes.

Concretely, we ask

Question

What is the critical p at which

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Theorem (L. + Meshulam '06)

The threshold for the vanishing of the first homology in $X(n,p)$ over $\mathbb{F}_2$ is

$$p = \frac{2 \ln n}{n}$$

A bit more accurately...

If M_1 (resp. M_2) are the inclusion matrices of the $(d-1)$-dimensional vs. d-faces (resp. d-faces vs. $(d+1)$-faces). Again the relation $M_1 M_2 = 0$ holds.

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- In topological parlance this is cycles modulo boundaries but we do not go into this important geometric point of view.
- Here you can also see why our "dummies" version does not apply when the coefficients come from a ring (such as $\mathbb{Z}$) and not from a field.

The vanishing of the $(d-1)$-st homology

The above result extends to d-dimensional simplicial complexes with a full $(d-1)$-st dimensional skeleton. Also, for other coefficient groups. (Most of this was done by Meshulam and Wallach).
We still do not know, however:

Question

What is the threshold for the vanishing of the $\mathbb{Z}$-homology?

The vanishing of the fundamental group

Theorem (Babson, Hoffman, Kahle '09 ?)

The threshold for the vanishing of the fundamental group in $X(n, p)$ is near

$$p = n^{-1/2}.$$

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In other words, an n-vertex graph with more than $(\frac{r-1}{r}+o(1))\frac{n^2}{2}$ edges must contain a K_{r+1}, and the bound is tight (up to the $o(1)$ term).

Extremal combinatorics of simplicial complexes

Theorem (Brown, Erdős, Sós '73)

Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a two-sphere. The bound is tight.

A word on the upper bound

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You can also consider links of larger sets (not just singleton), the definition is essentially the same.

A word on the upper bound (contd.)

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- Consequently, there are two vertices say x and y such that their links have $\Omega(n)$ edges in common.
- In particular, there is a cycle that is included in the link of x as well as in the link of y. This gives a double pyramid which is homeomorphic to a two-sphere.

But many extremal questions on simplicial complexes remain widely open

Conjecture

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- We can show that if true this bound is tight.
- This may be substantially harder than the BES theorem, since a "local" torus need not exist.
- (With Friedgut:) $\Omega(n^{8/3})$ simplices suffice.

Closer to extremal graph theory

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With L. Aronshtam (work in progress) we can show:

Theorem

For every two integers g and k there exist two-dimensional complexes with a full one-dimensional skeleton, such that for every vertex x, the link of x is a graph with girth $\geq g$ and chromatic number $\geq k$.

Much more remains to be done here.

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- How many they are: $n!$
- How to sample a random permutation.
- Numerous *typical* properties of random permutations e.g.,:
  - Number of fixed points.
  - Number of cycles.

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The definition naturally suggests itself: It's an $n \times n \times n$ array of zeros and ones A where every line (now with three types of lines) contains exactly a single 1.

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An alternative description: An $n \times n$ array M where m_{ij} gives the unique k for which $a_{ijk} = 1$. It is easy to verify that M is defined by the condition that every row and column in M is a permutation of $[n]$.

What is a three-dimensional permutation and how many are there?

The definition naturally suggests itself: It's an $n \times n \times n$ array of zeros and ones A where every line (now with three types of lines) contains exactly a single 1.
An alternative description: An $n \times n$ array M where m_{ij} gives the unique k for which $a_{ijk} = 1$. It is easy to verify that M is defined by the condition that every row and column in M is a permutation of $[n]$. Such a matrix is called a Latin square.

Some challenges

So this raises

Question

Determine or estimate $\mathcal{L}_n$, the number of $n \times n$ Latin squares.

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Currently the best known bound is:

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The (fairly easy) proof uses two substantial facts about permanents: The proof of the van der Waerden conjecture and Brégman's Theorem. This raises:

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- Solve the even higher dimensional cases.
- Factorials are, of course, closely related to the Gamma function. Are there higher dimensional analogues of Γ?

...and a few words on tensors...

Let us quickly recall the notion of tensor rank. But first a brief reminder of matrix rank. A matrix A has rank one iff there exist vectors x and y such that $a_{ij} = x_i y_j$.

Proposition

The rank of a matrix M is the least number of rank-one matrices whose sum is M.

More on tensors...

All of this extends to tensors almost verbatim:
A three-dimensional tensor A has rank one iff there exist vectors x, y and z such that $a_{ijk} = x_i y_j z_k$.

Definition

The rank of a three-dimensional tensor Z is the least number of rank-one tensors whose sum is Z.

Can you believe that this question is open?

Open Problem

What is the largest rank of an $n \times n \times n$ real tensor.

It is only known (and easy) that the answer is between $\frac{n^2}{3}$ and $\frac{n^2}{2}$. With A. Shraibman we have constructed a family of examples which suggests

Conjecture

The answer is $(1 + o(1))\frac{n^2}{2}$

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Our ignorance may be somewhat justified since tensor rank is NP-hard to determine (Hastad '90).

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The complexity of matrix multiplication is given by the rank of certain tensors (well, border rank, but we do not go into this).
Singular Value Decomposition (SVD) of matrices is one of the most practically important algorithms in computational linear algebra. It is a fascinating challenge to develop an analogous theory for higher-dimensional tensors.

THAT'S ALL, FOLKS....