Discrete Mathematics 33 (1981) 53–56.

ON PETERSEN'S GRAPH THEOREM

Nathan LINIAL

Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024, USA

Received 17 May 1979
Revised 10 April 1980

In this paper we prove the following: let G be a graph with e_G edges, which is $(k-1)$-edge-connected, and with all valences $\geq k$. Let $1 \leq r \leq k$ be an integer, then G contains a spanning subgraph H, so that all valences in H are $\geq r$, with no more than $\lceil re_G/k \rceil$ edges. The proof is based on a useful extension of Tutte's factor theorem [4, 5], due to Lovász [3]. For other extensions of Petersen's theorem, see [6, 7, 8].

1. Notations

Our graph-theoretic terminology is quite standard, generally following Berge [1]. We add the following conventions: a graph $G=(V, E)$ has $|V|=v$ edges, and $e=|E|$ edges. For A, B disjoint subsets of V we denote by $e(A)$ the number of edges in E with both end-vertices in A, $e(A, B)$ is the number of edges in H having one vertex in A and one in B. The subgraph of G, spanned by A is denoted by $\langle A \rangle$. The set of neighbors in G, of a vertex $x \in V$ is denoted by $N(x)$. $|N(x)|$, the *valence* of x, is denoted by $d(x)$.

We sometimes add a subscript to the graph-theoretic function in order to clarify for which graph it is evaluated.

Let f be a *limiter* on G, namely, an integer-valued function defined on V, so that $d_G(x) \geq f(x) \geq 0$ $(x \in V)$. For $A \subseteq V$ define $f(A)=\sum_{x\in A} f(x)$. We define now two classes of spanning subgraphs of G, which depend on F. $\mathcal{L}=\mathcal{L}_f$ is the class of all spanning subgraphs H of G for which $f(x) \geq d_H(x)$ $(x \in V)$ holds. $\mathcal{U}=\mathcal{U}_f$ is the class of all spanning subgraphs H of G which satisfy $d_H(x) \geq f(x)$ $(x \in V)$. Define $L(f)$ to be the minimum of $\sum_{x\in V}(f(x)-d_H(x))=f(V)-2e_H$ over all $H \in \mathcal{L}$. $U(f)$ is defined as the minimum of $\sum_{x\in V}(d_H(x)-f(x))=2e_H-f(V)$ over all $H \in \mathcal{U}$.

Let $B=(S, T, U)$ be a decomposition of V into three subsets. Let h be the number of components C of $\langle U \rangle$ for which $f(C)+e(C, T)$ is odd. Define

$$n(B, f)=h+f(T)-f(S)-2e(T)-e(T, U).$$

The key lemma in proving our main theorem is the following extension of Tutte's factor theorem [4, 5], which is due to Lovász [3].

Theorem 1. *Let $G=(V, E)$ be a graph, and let f be a limiter on G. Then $U(f)=L(f)=\max\{n(B, f) \mid B=(S, T, U)$ is a decomposition of V into 3 subsets$\}$.*

2. The main theorem

Theorem 2. *Let $G=(V,E)$ be a $(k-1)$-edge-connected graph so that $d(x)\geqslant k$ for every $x\in V$, and let $1\leqslant r\leqslant k$ be an integer. then G contains a spanning subgraph H, so that $d_H(x)\geqslant r$ $(x\in V)$, and $e_H\leqslant\lceil re_G/k\rceil$.*

Proof. For $r=k$, the theorem is obvious, so we assume $1\leqslant r\leqslant k-1$. We have to show that $n(B,f)\leqslant 2\lceil re_G/k\rceil-f(V)$, where $f(x)=r$ $(x\in V)$, for every $B=(S,T,U)$, a 3-decomposition of V.

Suppose first that $B=(\emptyset,\emptyset,V)$, then $n(B,f)=h$. Namely, $n(B,f)=0$ or 1, according to the parity of $f(V)=r\cdot v$. Since $d(x)\geqslant k$ for every $x\in V$, we have $e\geqslant\frac{1}{2}kv$ and therefore $2\lceil re/k\rceil\geqslant 2\lceil\frac{1}{2}rv\rceil=rv+h$, as needed.

Now we show that if $B=(S,T,U)$ is a 3-decomposition of V different from $(\emptyset,\emptyset,V)$, then

$$n(B,f)\leqslant 2\frac{re}{k}-rv.$$

Note that the square brackets are missing and this statement is stronger than that of the theorem. So we show

$$h+f(T)-f(S)-2e(T)-e(T,U)\leqslant\frac{2re}{k}-f(V).$$

Substituting $f(x)=r$, and rearranging this is the same as:

$$h+2r|T|+r|U|-2e(T)-e(T,U)\leqslant\frac{2re}{k},$$

or

$$kh+2kr|T|+kr|U|\leqslant 2re+2ke(T)+ke(T,U). \tag{1}$$

Since $d(x)\geqslant k$ for every $x\in V$, we have

$$k|T|\leqslant\sum_{x\in T}d(x)=2e(T)+e(T,U)+e(S,T).$$

So instead of (1) we shall show:

$$\begin{aligned}&k(h+r|U|)+2r(2e(T)+e(T,U)+e(S,T))\\&\quad\leqslant 2re+2ke(T)+ke(T,U)\\&\quad=2r(e(S)+e(T)+e(U)+e(S,T)+e(S,U)+e(T,U))\\&\qquad+2ke(T)+ke(T,U).\end{aligned}$$

That is

$$k(r|U|+h)\leqslant 2re(U)+2re(S,U)+ke(T,U)+2(k-r)e(T)+2re(S). \tag{2}$$

Consider a component C of $\langle U\rangle$. If $f(C)+e(C,T)$ is even, we show

$$kr\,|C|\leq 2re(C)+2re(C,S)+ke(C,T), \tag{3.1}$$

and if $f(C)+e(C,T)$ is odd, we show

$$k(r\,|C|+1)\leq 2re(C)+2re(C,S)+ke(C,T). \tag{3.2}$$

Summing (3.1) and (3.2) for all components C of $\langle U\rangle$ we shall obtain

$$k(r\,|U|+h)\leq 2re(U)+2re(S,U)+ke(T,U),$$

proving (2).

Now we prove (3.1) and (3.2). For every component C of $\langle U\rangle$, we have

$$k\,|C|\leq \sum_{x\in C} d(x)=2e(C)+e(C,T)+e(C,S). \tag{4}$$

We multiply (4) by r and (3.1) follows.

Since G is $(k-1)$-edge-connected, and $U\neq V$ we have

$$k-1\leq e(C,T)+e(C,S). \tag{5}$$

We multiply (4) by r and add (5) to get:

$$k(r\,|C|+1)-1\leq 2re(C)+(r+1)e(C,T)+(r+1)e(C,S),$$

and since $1\leq r\leq k-1$, also

$$k(r\,|C|+1)-1\leq 2re(C)+ke(C,T)+2re(C,S). \tag{6}$$

To prove (3.2) we show that if $f(C)+e(C,T)$ is odd, then equality cannot hold in (6). If, on the contrary

$$k(r\,|C|+1)-1=2r(e(C)+e(C,S))+ke(C,T),$$

then

$$k(r\,|C|+e(C,T)+1)-1=2r(e(C)+e(C,S))+2ke(C,T).$$

But this is impossible, because the right-hand side is even and the left-hand side is odd. This proves (3.2) and the proof of Theorem 2 is complete.

From Theorem 2 we infer a corollary on regular graphs:

Corollary 1. *Let $G=(V,E)$ be a $(k-1)$-edge-connected, k-regular graph on v vertices, and let $1\leq r\leq k$ be an integer. If rv is even, then G contains a spanning subgraph which is r-regular. If rv is odd, then G contains a spanning subgraph in which all vertices have valence r, except for one vertex whose valence is $r+1$.*

This corollary is an immediate consequence of Theorem 2: G has $\frac{1}{2}kv$ edges so $re/k=\frac{1}{2}rv$. A spanning subgraph in which all valences are $\geq r$ has at least $\frac{1}{2}rv$ edges, and the results follows.

References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1976).

[2] T. Gallai, Über extreme Punkt- und Kantenmengen, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. 2 (1959) 133–138.

[3] L. Lovász, Subgraphs with prescribed valencies, J. Combinatorial Theory (B) 8 (1970), 391–416.

[4] W.T. Tutte, A short proof of the factor theorem for finite graphs, Canad. J. Math. 6 (1954) 347–352.

[5] W.T. Tutte, Spanning subgraphs with specified valencies, Discrete Math. 9 (1974) 97–108.

[6] J. Plesnik, Connectivity in regular graphs and the existence of 1-factors. Mat. Cas. Slov. Akad. Vied. 22 (1972) 310–318.

[7] C.H.C. Little, D.D. Grant and D.A. Holten, On defect-*d* matchings in graphs, Discrete Math. 13 (1975) 41–54.

[8] T. Nishiseki, Amer. Math. Soc. Notices 753A-40 (1978) 25