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OPTIMA OF DUAL INTEGER LINEAR PROGRAMS

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We consider dual pairs of packing and covering integer linear programs. Best possible bounds are found between their optimal values. Tight inequalities are obtained relating the integral optima and the optimal rational solutions.

1. Introduction

It is well recognized that a large part of combinatorics can be formulated in terms of integer linear programs. A particularly satisfying situation occurs when the linear program and its dual both have integral solutions. In this case the optima of the programs are equal by the duality theorem of linear programming, and there results a min-max theorem. Apart from their aesthetic value, these cases are important in that polynomial time algorithms are available for computing the quantities involved. However, in most combinatorial problems, the solutions of the corresponding linear program or its dual are not integral. Since, by the duality theorem of linear programming the rational solutions to both programs are equal, the following question naturally arises: how much can the integral optima differ? And how far can each of them be from the rational optimum? The purpose of this paper is to obtain bounds on the distances between these quantities.

In fact, we shall be dealing only with a restricted class of linear programs, that is, with programs of the form:

$$\text{(L)} \qquad \begin{aligned} &\max x\cdot 1\\ &Ax \leqq 1\\ &x \geqq 0 \end{aligned}$$

and its dual

$$\text{(D)} \qquad \begin{aligned} &\min y\cdot 1\\ &yA \geqq 1\\ &y \geqq 0 \end{aligned}$$

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where A is a $0-1$ matrix of dimensions $n\times m$. Let z be the integral optimum of (L) (i.e. the maximum in (L) where x ranges over all nonnegative integral vectors of dimension m), Z the integral optimum of (D) and q the common rational optimum of (L) and (D). The matrix A can be viewed as the incidence matrix of a hypergraph H on n vertices having m edges.

We shall interchangeably refer to A as a matrix and as a hypergraph. Thus vertices will be interchanged with rows and edges with columns. For example, when we speak about the size of an edge in A we mean the number of 1's in the corresponding column of the matrix.

In hypergraph terms z is the *matching number* of H, namely the largest number of mutually disjoint edges. Simiarly, Z has a combinatorial interpretation as the *covering number* of H, that is the least size of a set of vertices meeting every edge. The parameter q is commonly refered to as the optimal fractional matching number or the optimal fractional covering number of H. The obvious relationship linking them is $Z\geqq q\geqq z$.

Lovász [4] found the following inequality relating Z and q:

$$Z \leqq q/(1+\ln d)$$

where d is the largest degree of a vertex in H (i.e. the largest sum of a row in A.) This result can be shown also by the methoods of [3]. The result is effective in that it is possible to find in time polynomial in m and n a set of not more than $q/(1+\ln d)$ vertices which meets all edges. It is also sharp, namely there exists a positive constant c such that for every q and d there exists a hypergraph for which $q/(1+\ln d)<cZ$. In this paper we study the two remaining relationships: between z and q, and between z and Z.

2. An inequality relating z and q

Theorem. *Let z, q, n and m be defined as above, and let f be the least size of an edge. Then*

$$z \geqq \frac{q^2}{n-\frac{f-1}{m}q^2} \geqq \frac{q^2}{n}$$

Proof. Let x be a rational vector at which the maximum in (L) is attained, that is, $Ax\leqq\bar{1}$, $x\geqq 0$ and $x\cdot\bar{1}=\sum x_i=q$. Then

$$x^TA^TAx \leqq \bar{1}\cdot\bar{1} = n. \tag{1}$$

Let C be the $m\times m$ matrix whose (i,j) entry is 1 if $A_i\cap A_j\neq\emptyset$ and 0 otherwise. The (i,j) entry of A^TA is $|A_i\cap A_j|$, and hence

$$A^TA \geqq \operatorname{diag}(|A_i|-1)+C. \tag{2}$$

By (1) and (2)

$$n \geqq x^TA^TAx \geqq x^T\operatorname{diag}(|A_i|-1)x+x^TCx. \tag{3}$$

By the Cauchy Schwartz inequality we have

$$(4)\qquad x^T \operatorname{diag}(|A_i|-1)x = \sum_{i=1}^{m}(|A_i|-1)x_i^2 \geqq$$

$$(f-1)\sum x_i^2 \geqq (f-1)\frac{1}{m}\left(\sum x_i\right)^2 = (f-1)\frac{q^2}{m}.$$

To estimate x^TCx we use a theorem of Motzkin and Straus [5].

Theorem MS. *Let* $G=(W, F)$ *be a graph. The maximum of* $\sum_{[i,j]\in F} x_i x_j$, *where* x_i $(i\in W)$ *are nonnegative and* $\sum x_i=1$, *is* $(1-1/k)/2$, *where* k *is the size of the largest clique in* G *(it is attained by putting* $x_i=1/k$ *on the vertices of a largest clique and zero elsewhere.)*

Let now $W=E$ and $F=\{[i,j]\colon A_i\cap A_j=\emptyset\}$, and let $G=(W, F)$. The size of the largest clique in G is z. The adjacency matrix of G is $B=J-C$, where J is the all 1's $m\times m$ matrix. Hence

$$\sum_{[i,j]\in F} x_i x_j = 1/2x^TBx.$$

Thus Theorem MS yields:

$$(5)\qquad x^TBx \leqq \left(1-\frac{1}{z}\right)\left(\sum x_i\right)^2 = q^2\left(1-\frac{1}{z}\right).$$

Combining (3), (4) and (5) and noting that $x^TJx=(\sum x_i)^2=q^2$, we have

$$(6)\qquad n \geqq x^TA^TAx \geqq q^2\left(\frac{f-1}{m}+1-\left(1-\frac{1}{z}\right)\right) = q^2\left(\frac{f-1}{m}+\frac{1}{z}\right)$$

which is the desired inequality.

In fact, the inequality in the theorem can be slightly improved, using the following idea of Z. Füredi. The optimum of (L) is attained at a vector x with at most n nonzero coordinates. Hence, if $m>n$, we can omit an edge without changing q, while z can only decrease by such a step. Thus we may replace m in the theorem by $\min(m, n)$, to obtain:

$$z \geqq \frac{q^2}{n-\dfrac{f-1}{\min(m,n)}q^2}$$

If h is a projective plane of order p then there holds $m=n=p^2+p+1$, $z=1$ (every two lines meet), $f=p+1$ and $q=m/f$. In this case equality holds in the theorem, showing the tightness of our inequality. But we suspect that the inequality can be improved when the size of the edges or the degree of each vertex are small. For example, if H is a graph, then by [3] the maximal rational mathcing x takes values 0, 1/2 and 1. Let a be the number of edges at which x has value 1/2 and b the number of edges on which x is 1. Then $q=a/2+b$, while $n\geqq a+2b$. The edges on which x has value form chains and circuits, and by taking edges on each chain and circuit alternately, an integral matching can be formed whose value is at least $a/3+b$. Thus $z \geqq 2q/3$, and it is easy to deduce that always $z \geqq 4q^2/(3n)$.

Finally, let us note that an integral matching satisfying the inequality in the theorem can be found in polynomial time. A rational vector x at which the optimum of (L) is attained can be found in polynomial time, using the ellipsoid method. Once x is found, the way to find a large matching is exactly as in [MS]: If $x_i, x_j > 0$ with $A_i \cap A_j \neq \emptyset$, then consider the vectors u, v defined by

$$u_k = \begin{cases} x_k & k \neq i, j \\ x_i + x_j & k = i \\ 0 & k = j \end{cases} \qquad v_k = \begin{cases} x_k & k \neq i, j \\ 0 & k = i \\ x_i + x_j & k = j. \end{cases}$$

It is easily verified that $\sum u_i = \sum v_i = \sum x_i$ and

$$\max (u^T B u, v^T B v) \geqq x^T B x.$$

By continuing in this fashion we will produce a vector w which is supported on a matching and which satisfies $w^T B w \geqq x^T B x$, and $\sum w_i = \sum x_i$.

It is again clear that if the support of w has p members, then

$$w^T B w \leqq \left(1 - \frac{1}{p}\right) (\sum x_i)^2 .$$

Thus the matching which supports w has

$$p \geqq \frac{q^2}{h - \frac{f-1}{m}}$$

edges and it is clearly found in polynomial time. ∎

3. An inequality relating z and Z

The question we shall consider in this section is: given m, n and z, how large can Z be? Whenever $m > e\sqrt{nz}$, define: $g(m, n, z) = \sqrt{nz \ln (m/\sqrt{nz})}$. We can then prove the following:

Theorem 2. *Let m, z, Z and $g = g(m, n, z)$ be defined as above. Then*

$$Z \leqq \begin{cases} \min (n, 3g) & \text{if} \quad m > e\sqrt{nz} \\ m & \text{if} \quad m < e\sqrt{nz} \end{cases} \tag{7}$$

Moreover, these inequalities are best possible in the regions $m < e\sqrt{nz}$ and $m > e\sqrt{nz}$ in the following sense: there exists a constant $\varepsilon > 0$ ($\varepsilon = 1/5$ will do) such that for m, n, z in these ranges, there exists a matrix A of dimensions $m \times n$, for which $\varepsilon \cdot z(A) \leqq z$ and

$$(1/\varepsilon) Z(A) \geqq \begin{cases} \min (n, 3g) & m > en \\ m & m < e\sqrt{nz} . \end{cases}$$

(Note: Z. Füredi and J. Pach [2] have informed us that they obtained a similar result for $z=1$.)

Proof. Since $Z \leqq m$ is trivially valid, for the proof of the first part we have to show that $Z \leqq 3g$ for $m > e\sqrt{nz}$. Let $r = g/z = \sqrt{(n/z)\ln(m/\sqrt{nz})}$. We construct a cover (i.e. a set of vertices meeting all edges) of size not greater than $3g$, using the following strategy.

Step a. We first take a maximal (with respect to containment) collection of mutually disjoint edges of size not exceeding r. The union C_1 of this collection meets all edges of size r or less. Since the collection is of size at most z, there holds $|C_1| \leqq r \cdot z = g$.

Step b. Delete from H all edges of size r or less. Since all remaining edges are of size larger than r, there exists an element v_1 belonging to a fraction of at least r/n of the edges. Delete v_1 and the edges containing it. There exists a vertex v_2 belonging to a fraction of at least r/n of the remaining edges. We delete v_2 and the edges incident with it. We repeat this process (if possible) g times. Let $C_2 = (v_1, \ldots, v_g)$. By the choice of the $v_i s$, after removing all edges incident with $v_1, \ldots, v_g$ the number of edges remaining is at most

$$m \cdot (1 - r/n)^g < m \exp(-rg/n) = m \exp(-g^2/nz) = \sqrt{nz} \leqq g,$$

where the last inequality follows from the assumption that $m > e\sqrt{nz}$.

Step c. Choose one vertex from each of the remaining edges, and name C_3 the set of vertices chosen in this way. Clearly, $C = C_1 \cup C_2 \cup C_3$ is a cover and $|C| \leqq 3g$.

To show the sharpness of the theorem in the ranges of m, n and z mentioned above, we first show that for every z and m there exists an n and a matrix A of dimensions $m \times n$ such that $z(A) = z$ and $Z(A) \geqq (1/3)m$. For $z=1$ and m general, let $n = \binom{m}{2}$ and let A be the $n \times m$ matrix whose rows are all possible 0, 1 vectors containing precisely two 1's. Then it is easy to see that $z(A) = 1$ and $Z(A) = \lceil m/2 \rceil$. For general z, take direct sum of z such matrices, duplicating some (at most z) columns of A to deal with the case that m is not a multiple of z. Then $Z \geqq (1/3)m$ clearly holds.

To show the sharpness of our inequality in the region $m > en$ we use a random construction. Given m, n, z such that $m > en$, define $g = g(n, m, z)$ and $r = g/z = \sqrt{(n/z)\ln(m/\sqrt{nz})}$. We shall show that for a random hypergraph A on n vertices having m distinct edges each of size r there holds

$$\Pr(z(A) \geqq 5z) < 1/3 \tag{8}$$

$$\Pr(Z(A) \leqq (1/5)\min(n, g)) < 1/e. \tag{9}$$

($\Pr(x)$ denotes the probability of the event x). It will follow that there exists a hypergraph A with $z(A) < 5z$ and $Z(A) > (1/5)\min(n, g)$, which is the desired conclusion.

To show (8), note that the probability of t r-sets to be disjoint is

$$\frac{\binom{n-r}{r}\binom{n-2r}{r}\cdots\binom{n-(t-1)r}{r}}{\binom{n}{r}^{t-1}} \leqq \prod_{k=1}^{t-1}\left(1-\frac{kr}{n}\right)^r \leqq$$

$$\leqq \prod_{k=1}^{t-1}\exp\left(\frac{-kr^2}{n}\right) = \exp\left(\frac{-r^2}{n}\binom{t}{2}\right).$$

Hence the probability that a matching of size t can be formed from the m edges of A is less than

$$\binom{m}{t}\exp\left(\frac{-r^2}{n}\binom{t}{2}\right).$$

For the proof of (8) it suffices to show that this number is less than 1/3 for $t=5z$. Using the inequality

$$\binom{m}{t} < \left(\frac{me}{t}\right)^t$$

and extracting t-th root, we see that it suffices to show

$$(em/t)\exp\left[(-r^2/n)(t-1)/2\right] < 1/2.$$

But $(r^2/n)[(t-1)/2]=[(5z-1)/2z]\ln(m/\sqrt{nz})>2\ln(m/\sqrt{nz})$, hence we have to show that $(em/5z)(nz/m^2)<1/2$ which follows since $m>en$.

We now prove (9). Let t be a natural number satisfying $t\leqq(1/5)\min(n,g)$. The probability that a given set of t vertices is a cover is:

$$\frac{\binom{\binom{n}{r}-\binom{n-t}{r}}{m}}{\binom{\binom{n}{r}}{m}} \leqq \exp\left[-m\frac{\binom{n-t}{r}}{\binom{n}{r}}\right] \leqq \exp\left[-m\exp\left(\frac{-2rt}{n}\right)\right]$$

(in the last inequality we used the fact that $t\leqq(1/5)n$). Therefore the probability that a cover of size t exists is no more than

$$\binom{n}{t}\exp\left[-m\exp\left(\frac{-2rt}{n}\right)\right]. \tag{10}$$

It would suffice to show that this number is less than $1/e$. Since this number increases with t, it suffices to prove the inequality for $t=(1/5)g$. For this value of t there holds:

$$2rt/n = (1/5)\ln(m^2/nz).$$

Taking natural logarithm of the number in question and using the inequality

$$\ln\binom{n}{t} \leqq t\ln(ne/t)$$

it remains to verify that $t\ln(m/t)-(m^3zn)^{1/5}<-1$ which, since $t\ln(m/t)>1$, would follow if we prove $2t\ln(m/t)<(m^3zn)^{1/5}$. Write $x=m/\sqrt{nz}$. The last inequality can then be written: $(2/5)\sqrt{\ln x}\cdot\ln(5x/\sqrt{\ln x})<x^{3/5}$, which clearly holds since $x>e$.

Now we turn to the computational aspect of the theorem. To transform the proof into an algorithm notice that we do construct a matching and a cover in the proof. We start by finding a maximal matching among the small sets and then continue to construct a cover by using the vertices covered by that matching plus a cover for the large sets, which we constuct greedily. If we know where to switch from the construction of the matching to the construction of the cover our problem is solved. The answer to this is implicit in the above proof. Order the edges of the hupergraph $A_1, \ldots, A_m$ so that their sizes are non decreasing. Define a mathcing (M_i) as follows: $M_1=A_1$ and $M_j=A_i$ where A_i is disjoint from $\bigcup_{\alpha=1}^{j-1} M_\alpha$ and i is the least index with this property.

Let t be the index for which

$$t|A_t|\exp(|A_t|^2t/m)\leqq m<(t+1)|A_{t+1}|\exp\left(|A_{t+1}|^2(t+1)/m\right).$$

The case where no such t exists will be considered later.

Our construction proceeds as follows. We consider the matching $M_1, \ldots, M_t$ and a cover which consists of $\bigcup_1^t M_i$ and the cover for the family $\{A_j|A_j\cap(\bigcup_{i=1}^t M_i)=\emptyset\}$ which is produced by the greedy algorithm. The same calculation which was used to prove the upper bound on Z applies now with z replaced by t.

We come back to the case where no such t exists. If already

$$|A_1|\exp(|A_1|^2/n)>m$$

then we apply the greedy algorithm for a cover immedeately and get the upper bound for Z valid already with $z=1$. At the other extreme, if $k|A_k|\exp(k|A_k|^2/n)<m$ and $(M_1, \ldots, M_k)$ is a maximal matching then considering $\bigcup_1^k M_i$ as a cover the upper bound holds.

4. Further directions for research

Our results are not complete, in that the bounds obtained are not tight in the range $O(\sqrt{nz})<m<O(n)$. It should be pointed out that only a slight modification is required to cover the range $m>\varepsilon n$ rather than $m>en$ and $m<\sqrt{nz}/\varepsilon$ rather than $m<e\sqrt{nz}$, but the constants involved depend on ε. We have not touched upon the case that A and the constraint vectors have general itegral elements (rather than 0,1). Interesting questions which suggest themselves in the general case are:

1) Solve the problem which we considered in this article under the assumption that the elements of A and the constraint vectors are bounded by some constant M.
2) The case of total unimodularity suggests that some positive results could follow from an assumption that all minors in A have a bounded determinant. Is this true?

3) As an aside of the previous question, the following problem seems very intriguing. What is the computational complexity of the following problem: given an integral matrix A and an integer k, decide whether all minors in A have a determinant not exceeding k in absolute value. The case $k=1$ is the case of total unimodularity where a polynomial time algorithm was given by Seymour [6]. The general case seems open.
4) In the course of our investigation the following class of problems came up: We have found very tight bounds on z, Z if A is a random 0—1 matrix. What can be said about q then? The answer depends of course on the probability distribution from which A is drawn. Making the appropriate choices interesting applications can result from an answer to this question.
5) The common approach to solving integer programming problems is via introducing cuts. These are inequalities which must hold due to the integrality of the solution. It would be interesting to investigate the improvement of the approximation of the rational relaxations as more cuts are introduced.

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