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Some Low Distortion Metric Ramsey Problems*

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Abstract. In this note we consider the metric Ramsey problem for the normed spaces ℓ_p. Namely, given some $1 \leq p \leq \infty$ and $\alpha \geq 1$, and an integer n, we ask for the largest m such that every n-point metric space contains an m-point subspace which embeds into ℓ_p with distortion at most α. In [1] it is shown that in the case of ℓ_2, the dependence of m on α undergoes a phase transition at $\alpha = 2$. Here we consider this problem for other ℓ_p, and specifically the occurrence of a phase transition for $p \neq 2$. It is shown that a phase transition does occur at $\alpha = 2$ for every $p \in [1, 2]$. For $p > 2$ we are unable to determine the answer, but estimates are provided for the possible location of such a phase transition. We also study the analogous problem for isometric embedding and show that for every $1 < p < \infty$ there are arbitrarily large metric spaces, no four points of which embed isometrically in ℓ_p.

1. Introduction

A Ramsey-type theorem states that large systems necessarily contain large, highly structured subsystems. Here we consider Ramsey-type problems for finite metric spaces, interpreting "highly structured" as having low distortion embedding in ℓ_p.

A mapping between two metric spaces $f : M \to X$ is called an embedding of M in X. The *distortion* of the embedding is defined as

$$\mathrm{dist}(f) = \sup_{\substack{x, y \in M \\ x \neq y}} \frac{d_X(f(x), f(y))}{d_M(x, y)} \cdot \sup_{\substack{x, y \in M \\ x \neq y}} \frac{d_M(x, y)}{d_X(f(x), f(y))}.$$

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The least distortion required to embed M in X is denoted by $c_X(M)$. When $c_X(M) \leq \alpha$ we say that M α-embeds in X. In this note we study the following notion.

Definition 1 (Metric Ramsey Function). We denote by $R_X(\alpha, n)$ the largest integer m such that every n-point metric space has a subspace of size m that α-embeds into X.

When $X = \ell_p$ we use the notations c_p and R_p. Note that for $p \in [1, \infty]$, it is always true that $R_p(\alpha, n) \geq R_2(\alpha, n)$. When $\alpha = 1$ we drop it from the notation, i.e., $R_X(n) = R_X(1, n)$.

Bourgain et al. [4] study this function for $X = \ell_2$, as a metric analog of Dvoretzky's theorem [7]. They prove:

Theorem 1 [4]. *For any $\alpha > 1$ there exists $C(\alpha) > 0$ such that $R_2(\alpha, n) \geq C(\alpha) \log n$. Furthermore, there exists $\alpha_0 > 1$ such that $R_2(\alpha_0, n) = O(\log n)$.*

In [1] the metric Ramsey problem is studied comprehensively. In particular, the following phase transition is established in the case of $X = \ell_2$.

Theorem 2 [1]. *Let $n \in \mathbb{N}$. Then*:

1. *For every $1 < \alpha < 2$: $c(\alpha) \log n \leq R_2(\alpha, n) \leq 2 \log n + C(\alpha)$, where $c(\alpha)$ and $C(\alpha)$ may depend only on α.*
2. *For every $\alpha > 2$: $n^{c'(\alpha)} \leq R_2(\alpha, n) \leq n^{C'(\alpha)}$, where $c'(\alpha)$ and $C'(\alpha)$ depend only on α and $0 < c'(\alpha) \leq C'(\alpha) < 1$. Moreover, $c'(\alpha)$ tends to 1 as α tends to ∞.*

By Dvoretzky's theorem, the lower bound in part 2 of Theorem 2 implies in particular that if $\alpha > 2$, and X is any infinite-dimensional normed space, then $R_X(\alpha, n) \geq n^{c'(\alpha)}$. Therefore, in our search for a possible phase transition for $R_p(\cdot, n)$, $p \neq 2$, it is natural to extend the upper bound in part 1 of Theorem 2 to this range. The main result proved in this note is the following:

Theorem 3. *There is an absolute constant $c > 0$ such that for every $0 < \delta < 1$:*

1. *For $1 \leq p < 2$, $R_p(2 - \delta, n) \leq e^{c/\delta^2} \log n$.*
2. *For $2 < p < \infty$, $R_p(2^{2/p} - \delta, n) \leq e^{c/p^2\delta^2} \log n$.*

Thus we extend the result of [1] to show that a phase transition occurs in the metric Ramsey problem for ℓ_p, $p \in [1, 2)$, at $\alpha = 2$. The asymptotic behavior of $R_p(\alpha, n)$ for $p > 2$, and $\alpha \in [2^{2/p}, 2]$, is left as an open problem. In particular, we do not know whether or not this function undergoes a similar phase transition. We find this problem potentially significant: if there is a phase transition at 2 also in the range $2 < p < \infty$, then this result will certainly be of great interest. On the other hand, if it is possible to improve the lower bound in part 2 of Theorem 2 for $p > 2$ and certain distortions strictly less than 2, then this would involve an embedding technique that is different from the method used in [1], which does not distinguish between the various ℓ_p spaces.

The proof of the upper bound on $R_2(\alpha, n)$ for $\alpha < 2$ stated in Theorem 2 uses the Johnson-Lindenstrauss dimension reduction lemma for ℓ_2 [10]. For ℓ_p, $p \neq 2$, no such dimension reduction is known to hold. (Recent work [5], [11] shows that dimension reduction does not, in general, hold in ℓ_1.) Our proof is based on a non-trivial modification of the random construction in [4], in the spirit of Erdös' upper bound on the Ramsey numbers [9], [3]. In the process we prove tight bounds on the embeddability of the metrics of complete bipartite graphs in ℓ_p. Specifically we show that

$$c_p(K_{n,n}) = \begin{cases} 2 - \Theta(n^{-1}), & p \in [1, 2], \\ 2^{2/p} - \Theta((pn)^{-1}), & p > 2. \end{cases}$$

The second part of this note addresses the isometric Ramsey problem for $p \in (1, \infty)$. It turns out that this problem is naturally tackled within the class of uniformly convex normed spaces (see Section 3 for the definition).

Theorem 4 (Isometric Ramsey Problem). *Let X be a uniformly convex normed space with* $\dim(X) \geq 2$. *Then* $R_X(1, n) = 3$ *for* $n \geq 3$.

Since ℓ_p is uniformly convex for $p \in (1, \infty)$, the conclusion of Theorem 4 holds in these cases. Note that the theorem does not apply for ℓ_1 and ℓ_∞ which are not uniformly convex. Specifically, it is known that ℓ_∞ is universal in that it contains an isometric copy of every finite metric space, whence $R_\infty(n) = n$. It is known [6] that any four-point metric space is isometrically embeddable in ℓ_1, and therefore $R_1(n) \geq 4$ for $n \geq 4$. The determination of $R_1(n)$ is left as an open problem.

2. An Upper Bound for $\alpha < 2$

In this section we prove that for any $\alpha < \min\{2, 2^{2/p}\}$, $R_p(\alpha, n) = O(\log n)$. Our technique both improves and simplifies the technique of [4], which is itself in the spirit of Erdös' original upper bound for the Ramsey coloring numbers. The basic idea is to exploit a universality property of random graphs $G \in G(n, \frac{1}{2})$. Namely, that any fixed graph of constant size appears as an induced subgraph of every induced subgraph of G of size $\Omega(\log n)$. More precisely, we define the following notion of universality.

Definition 2. Let H be a graph. A graph G is called (H, s)-universal if every set of s vertices in G contains an induced subgraph isomorphic to H.

Proposition 1. *For every k-vertex graph H there exists a constant $C > 0$ and an integer n_0 such that for any $n > n_0$ there exists an $(H, C \log n)$-universal graph on n vertices. Furthermore,*

$$C \leq O(k^2 2^{\binom{k}{2}}) \quad \text{and} \quad n_0 \leq O(k^3 2^{\binom{k}{2}}).$$

Such facts are well known in random graph theory, and similar arguments can be found for example in [13]. We sketch the standard details for the sake of completeness.

Recall that a family of sets $\mathcal{F}$ is called *almost disjoint* if $|A \cap B| \leq 1$ for every $A, B \in \mathcal{F}$. In what follows, given a set S and an integer k, we denote by $\binom{S}{k}$ the set of all k-point subsets of S.

Lemma 2. *For every integer k and a finite set S of cardinality $s = |S| > 2k^2$, there exists an almost disjoint family $K \subset \binom{S}{k}$, such that $|K| \geq \lfloor s/2k \rfloor^2$.*

Proof. Let p be a prime satisfying $s/2k \leq p \leq s/k$, and assume that

$$L = \{(i, j); i, j \in \mathbb{Z}_p,\ i \in \{0, \dots, k-1\}\} \subseteq S.$$

For each $a, b \in \mathbb{Z}_p$ (the field of residues modulo p), define

$$A_{a,b} = \{(i, j);\ j \equiv ai + b \pmod{p},\ i \in \{0, \dots, k-1\}\},$$

and take $K = \{A_{a,b} | a, b \in \mathbb{Z}_p\}$. The set K is easily checked to satisfy the requirements. □

As usual $G(n, \frac{1}{2})$ denotes the model of random graphs in which each edge on n vertices is chosen independently with probability $\frac{1}{2}$.

Lemma 3. *Let H be a k-vertex graph and let $s > 2k^2$. The probability that a random graph $G \in G(s, \frac{1}{2})$ does not contain an induced subgraph isomorphic to H, is at most $(1 - 2^{-\binom{k}{2}})^{\lfloor s/2k \rfloor^2}$.*

Proof. Construct, as in Lemma 2, an almost disjoint family $\mathcal{F}$ of $\lfloor s/2k \rfloor^2$ subsets of $\{1, \dots, s\}$, the vertex set of G. If $F_1 \neq F_2 \in \mathcal{F}$, then the event that the restriction of G to F_1 (resp. F_2) is isomorphic to H is independent. Hence, the probability that none of the sets $F \in \mathcal{F}$ spans a subgraph isomorphic to H is at most $(1 - 2^{-\binom{k}{2}})^{\lfloor s/2k \rfloor^2}$. □

Proof of Proposition 1. Let G be a random graph in $G(n, \frac{1}{2})$. By the previous lemma, the expected number of sets of s vertices which contain no induced isomorphic copy of H is at most $\binom{n}{s}(1 - 2^{-\binom{k}{2}})^{\lfloor s/2k \rfloor^2}$. If this number is < 1, then there is an (H, s)-universal graph, as claimed. It is an easy matter to check that this holds with the parameters as stated. □

A class $\mathcal{C}$ of finite metric spaces is called a *metric class* if it is closed under isometries. $\mathcal{C}$ is said to be *hereditary* if $M \in \mathcal{C}$ and $N \subset M$ imply $N \in \mathcal{C}$. We call a metric space (X, d) a $\{0, 1, 2\}$ metric space if for all $x, y \in X$, $d(x, y) \in \{0, 1, 2\}$. There is a simple 1:1 correspondence between graphs and $\{0, 1, 2\}$ metrics. Namely, associated with a $\{0, 1, 2\}$ metric space $M = (X, d)$ is the graph $G = (X, E)$ where $\{x, y\} \in E$ iff $d_M(x, y) = 1$.

Lemma 4. *Let $\mathcal{C}$ be a hereditary metric class of finite metric spaces, and suppose that there exists some finite $\{0, 1, 2\}$ metric space M_0 which is not in P. Then there exist metric spaces $M = M_n$ of arbitrarily large size n such that every subspace $S \subset M_n$ with at least $C \log n$ points is not in $\mathcal{C}$. The constant C depends only on the cardinality of M_0.*

Proof. Let H_0 be the graph corresponding to the metric space M_0. We apply Proposition 1, to construct arbitrarily large graphs $G_n = (V_n, E_n)$ with $|V_n| = n$, in which every set of $\geq C \log n$ vertices contains an induced subgraph isomorphic to H_0. Let M_n be the n-point metric space corresponding to G_n. It follows that every subspace of M_n of size $\geq C \log n$ contains a metric subspace that is isometric to M_0. Since $\mathcal{C}$ is hereditary, $S \notin \mathcal{C}$. □

Note that $\{M; M$ is a metric space, $c_p(M) \leq \alpha\}$ is a hereditary metric class. Therefore, in order to show that for $\alpha < 2$, $R_p(\alpha, n) = O(\log n)$, it is enough to find a $\{0, 1, 2\}$ metric space whose ℓ_p distortion is greater than α. We use the complete bipartite graphs $K_{n,n}$. The ℓ_p-distortion of $K_{n,n}$, $1 \leq p < \infty$, is estimated in the following proposition.

Proposition 5. *For every* $1 \leq p \leq 2$,

$$2\left(\frac{n-1}{n}\right)^{1/p} \leq c_p(K_{n,n}) \leq 2\sqrt{\frac{n-1}{n}}.$$

For every $2 \leq p < \infty$,

$$2^{2/p}\left(\frac{n-1}{n}\right)^{1/p} \leq c_p(K_{n,n}) \leq 2^{2/p}\left(1-\frac{1}{2n}\right)^{1/p}.$$

Before proving Proposition 5, we deduce the main result of this section:

Theorem 5. *There is an absolute constant* $c > 0$ *such that for every* $0 < \delta < 1$, *if* $1 \leq p \leq 2$, *then*

$$R_p(2-\delta, n) \leq e^{c/\delta^2} \log n,$$

and if $2 < p < \infty$, *then*

$$R_p(2^{2/p}-\delta, n) \leq e^{c/p^2\delta^2} \log n.$$

Proof. Proposition 1 implies that there is an absolute constant C such that for every $n \geq 2^{Ck^3}$ there exists a $\{0, 1, 2\}$ metric space M_n such that any subset $S \subset M_n$ of cardinality at least $2^{Ck^2} \log n$ contains an isometric copy of $K_{k,k}$.

We start with $1 \leq p \leq 2$. Let $k = \lfloor 2/\delta \rfloor + 1$. By Proposition 5,

$$c_p(K_{k,k}) \geq 2\left(1-\frac{1}{k}\right)^{1/p} > 2\left(1-\frac{\delta}{2}\right) = 2-\delta,$$

so that for n large enough ($\geq e^{C'/\delta^3}$), and hence for all n (by proper choice of constants),

$$R_p(2-\delta, n) \leq e^{C'/\delta^2} \log n.$$

When $p > 2$ take $k = 2\lfloor 4/p\delta \rfloor$. In this case one easily verifies that

$$c_p(K_{k,k}) \geq 2^{2/p}\left(1-\frac{1}{k}\right)^{1/p} \geq 2^{2/p}-\delta,$$

from which the required result follows as above. □

In order to prove Proposition 5, we need some preparation.

Lemma 6. *Let $A = (a_{ij})$ be an $n \times n$ matrix and $2 \le p < \infty$. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|^{p} + \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|^{p}\right) \le \frac{(2n)^p}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^p.$$

Proof. We identify $\ell_p^{n^2}$ with the space of all $n \times n$ matrices $A = (a_{ij})$, equipped with the ℓ_p norm:

$$\|A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p}.$$

Define a linear operator $T : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \oplus \mathbb{R}^{n^2}$ by

$$T(a_{ij}) = \left(\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right)_{ij} \oplus \left(\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right)_{ij}.$$

For $q \ge 1$ denote $\|T\|_{q\to q} = \max_{A\neq 0}\|T(A)\|_q/\|A\|_q$. Our goal is to show that $\|T\|_{p\to p} \le 2^{1-1/p}n$. By a result from the complex interpolation theory for linear operators (see [2]), for $2 \le p \le \infty$, $\|T\|_{p\to p} \le \|T\|_{2\to 2}^{2/p}\cdot\|T\|_{\infty\to\infty}^{1-2/p}$. It is therefore enough to prove the required estimate for $p = 2$ and $p = \infty$. The case $p = \infty$ is simple:

$$\|T(A)\|_\infty = \max_{1\le i,j\le n}\max\left\{\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|, \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|\right\} \le 2n\|A\|_\infty.$$

For $p = 2$ we have to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|^{2} + \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|^{2}\right) \le 2n^2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2.$$

This inequality follows from the following elementary identity:

$$\begin{aligned} 2n^2\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2 \;&=\; \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right)^2 + \left(\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right)^2\right] \\ &\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(na_{ij} - \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{kj}\right)^2. \end{aligned}$$

□

Corollary 7. *Let $1 \le p < \infty$ and $x_1, \dots, x_n, y_1, \dots, y_n \in \ell_p$. Then if $2 \le p < \infty$,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p\right) \le 2^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - y_j\|_p^p.$$

If $1 \le p \le 2$, *then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p\right) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - y_j\|_p^p.$$

Proof. By summation it is clearly enough to prove these inequalities for $x_1, \dots, x_n$, $y_1, \dots, y_n \in \mathbb{R}$. If $2 \le p < \infty$, then the required result follows from an application of Lemma 6 to the matrix $a_{ij} = x_i - y_j$. If $1 \le p \le 2$, then consider ℓ_p equipped with the metric $d(x, y) = \|x - y\|_p^{p/2}$. It is well known (see [14]) that (ℓ_p, d) embeds isometrically in ℓ_2, so that the case $1 \le p \le 2$ follows from the case $p = 2$. □

Remark. In [8] Enflo defined the notion on generalized roundness of a metric space. A metric space (M, d) is said to have generalized roundness $q \ge 0$ if for every $x_1, \dots, x_n, y_1, \dots, y_n \in M$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(d(x_i, x_j)^q + d(y_i, y_j)^q) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n} d(x_i, y_j)^q.$$

Enflo proved that Hilbert space has generalized roundness 2 and in [12] the concept of generalized roundness was investigated and was shown to be equivalent to the notion of negative type (see [6] and [14] for the definition). Particularly, it was proved in [12] that for $1 \le p < 2$, ℓ_p has generalized roundness p, which is precisely the second statement in Corollary 7. For the case $p = 1$, simpler more direct proofs can be given which do not use reduction to the case $p = 2$, see, e.g., [6]. Observe that Lemma 6 would follow simply by convexity had it not been for the additional factor $\frac{1}{2}$ on the right-hand side. This factor is crucial for our purposes, and this is why the interpolation argument was needed.

Proof of Proposition 5. We identify $K_{n,n}$ with the metric on $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ where $d(u_i, u_j) = d(v_i, v_j) = 2$ for all $i \ne j$, and $d(u_i, v_j) = 1$ for every $1 \le i, j \le n$. Fix some $1 \le p < \infty$ and let $f : \{u_1, \dots, u_n, v_1, \dots, v_n\} \to \ell_p$ be an embedding such that for every $x, y \in K_{n,n}$, $d(x, y) \le \|f(x) - f(y)\|_p \le Ld(x, y)$. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\|f(u_i) - f(u_j)\|_p^p + \|f(v_i) - f(v_j)\|_p^p) \ge 2n(n-1)2^p$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\|f(u_i) - f(v_j)\|_p^p \le n^2L^p.$$

For $1 \le p \le 2$ Corollary 7 gives

$$2n(n-1)^p 2^p \le 2n^2L^p \quad \Longrightarrow \quad L \ge 2\left(\frac{n-1}{n}\right)^{1/p}.$$

For $2 \le p < \infty$ we get that

$$2n(n-1)2^p \le 2^{p-1}n^2L^p \quad \Longrightarrow \quad L \ge 2^{2/p}\left(\frac{n-1}{n}\right)^{1/p}.$$

This proves the required lower bounds on $c_p(K_{n,n})$.

To prove the upper bound assume first that $p = 2$ and denote by $\{e_i\}_{i=1}^{\infty}$ the standard unit vectors in ℓ_2. Define $f : K_{n,n} \to \ell_2^{2n}$ by

$$f(u_i) = \sqrt{2}\left(e_i - \frac{1}{n}\sum_{j=1}^{n} e_j\right),$$
$$f(v_i) = \sqrt{2}\left(e_{n+i} - \frac{1}{n}\sum_{j=1}^{n} e_{n+j}\right).$$

Then for $i \ne j$, $\|f(u_i) - f(u_j)\|_2 = \|f(v_i) - f(v_j)\|_2 = 2 = d(u_i, u_j) = d(v_i, v_j)$. On the other hand,

$$\|f(u_i) - f(v_j)\|_2 = \sqrt{\|f(u_i)\|_2^2 + \|f(v_j)\|_2^2}$$
$$= \sqrt{4\left(1 - \frac{1}{n}\right)^2 + 4(n-1)\cdot\frac{1}{n^2}} = 2\sqrt{\frac{n-1}{n}}.$$

This finishes the calculation of $c_2(K_{n,n})$. For $1 \le p < 2$, since for every $\varepsilon > 0$ and for every k, ℓ_p contains a $(1+\varepsilon)$ distorted copy of ℓ_2^k, we get the estimate $c_p(K_{n,n}) \le 2\sqrt{(n-1)/n}$.

The case $2 < p < \infty$ requires a different embedding. We begin by describing an embedding with distortion $2^{2/p}$ and then explain how to modify it so as to reduce the distortion by a factor of $(1 - 1/2n)^{1/p}$. Let $z_1, \ldots, z_n$ be a collection of n mutually orthogonal ± 1 vectors of dimension $m = O(n)$. (For example, the first n rows in an $m \times m$ Hadamard matrix.) In our first embedding we define $f(u_i)$ as the $(2m)$-dimensional vector $(z_i, 0)$, namely, z_i concatenated with m zeros. Likewise, $f(v_i) = (0, z_i)$ for all i. Now $\|f(u_i) - f(u_j)\|_p = 2(m/2)^{1/p}$ and $\|f(u_i) - f(v_j)\|_p = (2m)^{1/p}$, and so f has distortion $2^{2/p}$. To get the $(1 - 1/2n)^{1/p}$ improvement, note that for some $m \le 4n$ it is possible to select the z_i so that the mth coordinate in all of them is $+1$. Modify the previous construction to an embedding into $2m - 1$ dimensions as follows: now $g(u_i)$ is z_i concatenated with $m - 1$ zeros, whereas $g(v_i)$ has zeros in the first $m - 1$ coordinates, 1 in the mth and this is followed by the first $m - 1$ coordinates of the vector z_i. The easy details are omitted. □

Remark. The upper bounds in Proposition 5 were not used in the proof of Theorem 5. Apart from their intrinsic interest, these upper estimates show that the above technique cannot prove an upper bound of $O(\log n)$ on $R_2(2 - \varepsilon, n)$ which is independent of ε. In fact, this can never be achieved using $\{0, 1, 2\}$ metric spaces due to the following proposition.

Proposition 8. *Let X be an n-point $\{0, 1, 2\}$ metric space. Then* $c_2(X) \le 2\sqrt{(n-1)/n}$.

Proof. We think of X as a metric on $\{1,\dots,n\}$ and denote $d(i,j)=d_{ij}$. Define an $n\times n$ matrix $A=(a_{ij})$ as follows:

$$a_{ij}=\begin{cases}2 & \text{if } \ i=j,\\ 0 & \text{if } \ d_{ij}=2,\\ \dfrac{2}{n} & \text{if } \ d_{ij}=1.\end{cases}$$

We claim that A is positive semidefinite. Indeed, for any $z\in\mathbb{R}^n$,

$$\begin{aligned}\langle Az,z\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}z_iz_j\\ &\ge \sum_{i=1}^{n}2z_i^2-\sum_{i\neq j}\frac{2}{n}|z_i|\cdot|z_j|\\ &\ge \sum_{i=1}^{n}2z_i^2-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{2}{n}|z_i|\cdot|z_j|\\ &= 2\|z\|_2^2-\frac{2}{n}\|z\|_1^2\ge 2\|z\|_2^2-\frac{2}{n}n\|z\|_2^2=0.\end{aligned}$$

In particular, it follows that A has a square root, denoted $A^{1/2}$. Let $e_1,\dots,e_n$ be the standard unit vectors in $\mathbb{R}^n$. Define $f:X\to\mathbb{R}^n$ by $f(i)=A^{1/2}e_i$. Now,

$$\|f(i)-f(j)\|_2^2=\langle Ae_i,e_i\rangle+\langle Ae_j,e_j\rangle-2\langle Ae_i,e_j\rangle=a_{ii}+a_{jj}-2a_{ij},$$

so that if $d_{ij}=1$, then $\|f(i)-f(j)\|_2=\sqrt{4-4/n}$ and if $d_{ij}=2$, then $\|f(i)-f(j)\|_2=2$. It follows that

$$\mathrm{dist}(f)=2\sqrt{\frac{n-1}{n}}.\qquad\square$$

3. The Isometric Ramsey Problem

In this section we prove that for $n\ge 3$, $1<p<\infty$, $R_p(n)=R_p(1,n)=3$. In fact, we show that this is true for any uniformly convex normed space. We begin by sketching an argument that is specific to ℓ_2:

Proposition 9. *$R_2(n)=3$ for $n\ge 3$.*

Proof. That $R_2(n)\ge 3$ follows since any metric space on three points embeds isometrically in ℓ_2^2. To show that $R_2(n)\le 3$, we construct a metric space on $n>3$ points, no four-point subspace of which embeds isometrically in ℓ_2. Fix an integer $n>3$ and let $\{a_i\}_{i=0}^n$ be an increasing sequence such that $a_0=0$, $a_1=1$, and for $1\le i<n$, $a_{i+1}\ge 2(n+1)a_i$. Fix some $0<\varepsilon<1/(2a_n)$. It is easily verified that $d(i,j)=|i-j|-\varepsilon a_{|i-j|}$ is a metric on $\{1,2,\dots,n\}$. We show that for ε small enough no four points in $(\{1,\dots,n\},d)$ embed isometrically in ℓ_2. Fix four integers $1\le i_1<i_2<i_3<i_4\le n$ and set $j=i_2-i_1$,

$k = i_3 - i_2$, $l = i_4 - i_3$. Suppose that for every $\varepsilon > 0$ there exists an isometric embedding $f : (\{i_1, i_2, i_3, i_4\}, d) \to \ell_2^3$. Without loss of generality we may assume that $f(i_1) = (\alpha, \beta, \gamma)$, $f(i_2) = (0, 0, 0)$, $f(i_3) = (k - \varepsilon a_k, 0, 0)$, and $f(i_4) = (p, q, 0)$. Then

$$\begin{aligned} 2\alpha(k - \varepsilon a_k) &= 2\langle f(i_1), f(i_3)\rangle \\ &= \|f(i_1) - f(i_2)\|_2^2 + \|f(i_3) - f(i_2)\|_2^2 - \|f(i_3) - f(i_1)\|_2^2 \\ &= (j - \varepsilon a_j)^2 + (k - \varepsilon a_k)^2 - (j + k - \varepsilon a_{j+k})^2. \end{aligned}$$

Hence,

$$\alpha \le -j + \frac{\varepsilon}{k}[(k + j)a_{k+j} - ja_j - ka_k - ja_k] + O(\varepsilon^2).$$

Similarly,

$$p \ge (k + l) + \frac{\varepsilon}{k}[(k + l)a_k - (k + l)a_{k+l} - ka_k + la_l] + O(\varepsilon^2).$$

Now

$$\begin{aligned} & j + k + l - \varepsilon a_{j+k+l} \\ &\quad = \|f(i_4) - f(i_1)\|_2 \\ &\quad \ge p - \alpha \\ &\quad \ge j + k + l \\ &\quad\quad + \frac{\varepsilon}{k}[(k + l)a_k - (k + l)a_{k+l} + la_l - (k + j)a_{k+j} + ja_j + ja_k] \\ &\quad\quad + O(\varepsilon^2). \end{aligned}$$

Letting ε tend to zero we deduce that

$$\begin{aligned} a_{j+k+l} &\le \left(1 + \frac{j}{k}\right)a_{k+j} + \left(1 + \frac{l}{k}\right)a_{k+l} - \frac{l}{k}a_l - \frac{j}{k}a_j - \frac{j + k + l}{k}a_k \\ &< 2(n + 1)a_{j+k+l-1}, \end{aligned}$$

which is a contradiction. □

The argument above is quite specific to ℓ_2, and so we now consider any uniformly convex normed space. The modulus of uniform convexity of a normed space X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|a + b\|}{2};\ \|a\|, \|b\| \le 1 \text{ and } \|a - b\| \ge \varepsilon\right\}.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $0 < \varepsilon \le 2$. The L_p spaces $1 < p < \infty$ are known to be uniformly convex. For a uniformly convex space X, δ_X is known to be continuous and strictly increasing on $(0, 2]$.

Assume that X is a uniformly convex normed space and $a, b \in X\setminus\{0\}$. Then

$$\begin{aligned}\left\|\frac{a}{\|a\|}+\frac{b}{\|b\|}\right\| &= \left\|\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(a+b)-\frac{a}{\|b\|}-\frac{b}{\|a\|}\right\| \\ &\geq \left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)\|a+b\|-\frac{\|a\|}{\|b\|}-\frac{\|b\|}{\|a\|} \\ &= 2-\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|).\end{aligned}$$

Now,

$$\begin{aligned}\delta_X\left(\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\right) &\leq 1-\frac{1}{2}\cdot\left\|\frac{a}{\|a\|}+\frac{b}{\|b\|}\right\| \\ &\leq \frac{1}{2}\cdot\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|).\end{aligned}$$

Hence

$$\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\| \leq \delta_X^{-1}\left(\frac{1}{2}\cdot\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|)\right).$$

Take $x, y, z \in X$ and apply this inequality for $a = x - y$, $b = y - z$. It follows that

$$\begin{aligned}&\left\|y-\left(\frac{\|y-z\|}{\|x-y\|+\|y-z\|}\cdot x+\frac{\|x-y\|}{\|x-y\|+\|y-z\|}\cdot z\right)\right\| \\ &\qquad\leq \frac{\|x-y\|\cdot\|y-z\|}{\|x-y\|+\|y-z\|}\cdot\delta_X^{-1}\left(\frac{\|x-y\|+\|y-z\|-\|x-z\|}{\min\{\|x-y\|,\|y-z\|\}}\right). \qquad (1)\end{aligned}$$

This inequality is the way uniform convexity is going to be applied in the sequel. Indeed, we have the following "metric" consequence of it:

Lemma 10. *Let X be a uniformly convex normed space and let $x_1, x_2, x_3, x_4 \in X$ be distinct. Then*

$$\begin{aligned}&\frac{\|x_1-x_2\|+\|x_2-x_3\|-\|x_1-x_3\|}{2\|x_2-x_3\|} \\ &\qquad\leq \delta_X^{-1}\left(\frac{\|x_1-x_3\|+\|x_3-x_4\|-\|x_1-x_4\|}{\min\{\|x_1-x_3\|,\|x_3-x_4\|\}}\right) \\ &\qquad\quad+\delta_X^{-1}\left(\frac{\|x_2-x_3\|+\|x_3-x_4\|-\|x_2-x_4\|}{\min\{\|x_2-x_3\|,\|x_3-x_4\|\}}\right).\end{aligned}$$

Lemma 10 is a quantitative version of the fact that in a uniformly convex space, if $\|x_1 - x_4\|$ is approximately $\|x_1 - x_3\| + \|x_3 - x_4\|$ and $\|x_2 - x_4\|$ is approximately $\|x_2 - x_3\| + \|x_3 - x_4\|$, then $\|x_1 - x_3\|$ is approximately $\|x_1 - x_2\| + \|x_2 - x_3\|$. This fact is geometrically evident since the first assumption implies that x_3 is almost on the line segment connecting x_1 and x_4 and x_2 is almost on the line segment connecting x_1 and x_3. It follows that x_2 is almost on the line segment connecting x_1 and x_3, as required. Since we are dealing with bi-Lipschitz embeddings, we must formulate this phenomenon without referring to "line segments."

Proof of Lemma 10. Define

$$\lambda = \frac{\|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \quad \text{and} \quad \mu = \frac{\|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|}.$$

An application of (1) twice gives

$$\begin{aligned}\|x_3 - (\lambda x_1 + (1-\lambda)x_4)\| \ \le\ & \frac{\|x_1 - x_3\| \cdot \|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \\ & \cdot \delta_X^{-1}\left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}}\right)\end{aligned}$$

and

$$\begin{aligned}\|x_3 - (\mu x_2 + (1-\mu)x_4)\| \ \le\ & \frac{\|x - 2 - x_3\| \cdot \|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|} \\ & \cdot \delta_X^{-1}\left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}}\right).\end{aligned}$$

By symmetry, we may assume without loss of generality that $\lambda \le \mu$. Now,

$$\begin{aligned}&\left\|x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu - \lambda}{\mu(1-\lambda)}x_3\right\| \\ &\quad = \frac{1}{\mu}\left\|\mu x_2 + (1-\mu)x_4 - x_3 + \frac{1-\mu}{1-\lambda}(x_3 - \lambda x_1 - (1-\lambda)x_4)\right\| \\ &\quad \le \frac{1}{\mu}\|x_3 - \mu x_2 - (1-\mu)x_4\| + \frac{1-\mu}{\mu(1-\lambda)} \cdot \|x_3 - \lambda x_1 - (1-\lambda)x_4\| \\ &\quad \le \frac{\|x_2 - x_3\| + \|x_3 - x_4\|}{\|x_3 - x_4\|} \cdot \frac{\|x_2 - x_3\| \cdot \|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|} \\ &\qquad \cdot \delta_X^{-1}\left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}}\right) \\ &\qquad + \frac{\|x_2 - x_3\|}{\|x_3 - x_4\|} \frac{\|x_1 - x_3\| + \|x_3 - x_4\|}{\|x_1 - x_3\|} \frac{\|x_1 - \|x_3\| \cdot \|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \\ &\qquad \cdot \delta_X^{-1}\left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}}\right) \\ &\quad = \|x_2 - x_3\|\delta_X^{-1}\left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}}\right) \\ &\qquad + \|x_2 - x_3\|\delta_X^{-1}\left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}}\right).\end{aligned}$$

Additionally,

$$\begin{aligned}\|x_2 - x_1\| \ \le\ & \left\|x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu - \lambda}{\mu(1-\lambda)}x_3\right\| \\ & + \left\|x_1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu - \lambda}{\mu(1-\lambda)}x_3\right\| \\ =\ & \left\|x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu - \lambda}{\mu(1-\lambda)}x_3\right\| + \frac{\mu - \lambda}{\mu(1-\lambda)}\|x_1 - x_3\|\end{aligned}$$

and

$$\begin{aligned}\|x_2 - x_3\| &\le \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ &\quad + \left\| x_3 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ &= \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| + \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\|x_1 - x_3\|.\end{aligned}$$

Summing up these estimates gives the required result. □

We can now prove the main result of this section:

Theorem 6. *Let X be a uniformly convex normed space with $\dim(X) \ge 2$. Then for every $n \ge 3$, $R_X(n) = 3$. Moreover, for every $\delta : (0, 2] \to (0, \infty)$ which is continuous, increasing, and $\delta \le \delta_{\ell_2}$, let UC_δ be the class of all normed spaces X with $\delta_X \ge \delta$. Then for each $n \ge 3$ there is a constant $\varepsilon_n(\delta) > 0$ such that $R_{UC_\delta}(1 + \varepsilon_n(\delta), n) = 3$.*

The proof of Theorem 6 proceeds by constructing a space in which each quadruple violates the conclusion of Lemma 10. The construction is done iteratively, by adding one point at a time.

Proof of Theorem 6. That $R_X(n) \ge 3$ follows since any three-point metric embeds isometrically into any two-dimensional normed space, by a standard continuity argument.

Fix some $\delta : (0, 2] \to (0, \infty)$ which is continuous, increasing, and $\delta \le \delta_{\ell_2}$. We shall construct inductively a sequence $\{M_n\}_{n=3}^\infty$ of metric spaces and numbers $\{\eta_n\}_{n=3}^\infty$ such that:

(a) For every $n \ge 3$, $\eta_n > 0$. Each M_n is a metric on $\{1, \ldots, n\}$, and we denote $d_{ij}^n = d_{M_n}(i, j)$.
(b) For every $1 \le i < j < k \le n$,

$$\begin{aligned}&d_{i,j}^n + d_{jk}^n - d_{i,k}^n - \eta_n \\ &\qquad \ge 2d_{j,k}^n \left[\delta^{-1}\left(\frac{d_{i,k}^n + d_{k,n}^n - d_{i,n}^n}{\min\{d_{i,k}^n, d_{k,n}^n\}} \right) + \delta^{-1}\left(\frac{d_{j,k}^n + d_{k,n}^n - d_{j,n}^n}{\min\{d_{j,k}^n, d_{k,n}^n\}} \right) \right].\end{aligned}$$

Lemma 10 immediately implies that there is a constant $\varepsilon_n(\delta) > 0$ such that for every $1 \le i < j < k < l \le n$ and for every normed space X with $\delta_X \ge \delta$,

$$c_X(\{i, j, k, l\}, d_{M_n}) \ge 1 + \varepsilon_n(\delta),$$

as required.

M_3 is the equilateral metric on $\{1, 2, 3\}$, in which case $\eta_3 = 1$. We construct $M_{n+1} = (\{1, \ldots, n+1\}, d^{n+1})$ as an extension of M_n, by setting

$$d_{n,n+1}^{n+1} = 1 - s/2 \quad \text{and} \quad \forall 1 \le i < n, \qquad d_{i,n+1}^{n+1} = d_{in}^n + 1 - s.$$

This is indeed a definition of a metric as long as $0 < s \le \min\{1, 2\min_{1\le i<n} d^n_{i,n}\}$ (this fact follows from a simple case analysis).

We are left to check condition (b). Fix $1 \le i < j < k \le n$. If $k \ne n$, then

$$\begin{aligned}
&d^{n+1}_{i,j} + d^{n+1}_{j,k} - d^{n+1}_{i,k} - \eta_n \\
&\quad = d^n_{i,j} + d^n_{j,k} - d^n_{i,k} - \eta_n \\
&\quad \ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k}+d^n_{k,n}-d^n_{i,n}}{\min\{d^n_{i,k}, d^n_{k,n}\}}\right) + \delta^{-1}\left(\frac{d^n_{j,k}+d^n_{k,n}-d^n_{j,n}}{\min\{d^n_{j,k}, d^n_{k,n}\}}\right)\right] \\
&\quad \ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k}+(d^n_{k,n}+1-s)-(d^n_{i,n}+1-s)}{\min\{d^n_{i,k}, d^n_{k,n}+1-s\}}\right)\right. \\
&\qquad \left. + \delta^{-1}\left(\frac{d^n_{j,k}+(d^n_{k,n}+1-s)-(d^n_{j,n}+1-s)}{\min\{d^n_{j,k}, d^n_{k,n}+1-s\}}\right)\right] \\
&\quad = 2d^{n+1}_{j,k}\left[\delta^{-1}\left(\frac{d^{n+1}_{i,k}+d^{n+1}_{k,n+1}-d^{n+1}_{i,n+1}}{\min\{d^{n+1}_{i,k}, d^{n+1}_{k,n+1}\}}\right)\right. \\
&\qquad \left. + \delta^{-1}\left(\frac{d^{n+1}_{j,k}+d^{n+1}_{k,n+1}-d^{n+1}_{j,n+1}}{\min\{d^{n+1}_{j,k}, d^{n+1}_{k,n+1}\}}\right)\right].
\end{aligned}$$

It remains to check (b) for the quadruple $\{i, j, n, n+1\}$. Condition (b) for M_n implies that

$$d^{n+1}_{ij} + d^{n+1}_{jn} - d^{n+1}_{in} \ge \eta_n.$$

On the other hand,

$$\begin{aligned}
&2d^{n+1}_{j,n}\left[\delta^{-1}\left(\frac{d^{n+1}_{i,n}+d^{n+1}_{n,n+1}-d^{n+1}_{i,n+1}}{\min\{d^{n+1}_{i,n}, d^{n+1}_{n,n+1}\}}\right) + \delta^{-1}\left(\frac{d^{n+1}_{j,n}+d^{n+1}_{n,n+1}-d^{n+1}_{j,n+1}}{\min\{d^{n+1}_{j,n}, d^{n+1}_{n,n+1}\}}\right)\right] \\
&\quad = 2d^n_{j,n}\left[\delta^{-1}\left(\frac{s/2}{\min\{d^n_{i,n}, 1-s/2\}}\right) + \delta^{-1}\left(\frac{s/2}{\min\{d^n_{j,n}, 1-s/2\}}\right)\right],
\end{aligned}$$

so that condition (b) will hold when s is small enough such that the quantity above is at most $\eta_n/2$ and with $\eta_{n+1} = \eta_n/2$. □

Corollary 11. *For all* $1 < p < \infty$, $R_p(n) = 3$ *for* $n \ge 3$.

We end this section with a simple lower bound for the isometric Ramsey problem for graphs. We do not know the asymptotically tight bound in this setting.

Proposition 12. *Let* G *be an unweighted graph of order* n. *Then there is a set of* $\Omega\left(\sqrt{\log n/\log\log n}\right)$ *vertices in* G *whose metric embeds isometrically into* ℓ_2.

Proof. Let Δ be the diameter of G. The shortest path between two diameterically far vertices is isometrically embeddable in ℓ_2. On the other hand, the Bourgain et al. theorem

[4] yields, for every $0 < \varepsilon < 1$, a subset $N \subset V$ which is $(1+\varepsilon)$ embeddable in Hilbert space and $|N| = \Omega\left((\varepsilon/\log(2/\varepsilon))\log n\right)$. When $\varepsilon = 1/2\Delta$, such an embedding is an isometry. Hence we can always extract a subset of V which is isometrically embeddable in ℓ_2 with cardinality

$$\Omega\left(\max\left\{\Delta, \frac{\log n}{\Delta \log \Delta}\right\}\right) = \Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right),$$

as claimed. □

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