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Matroidal Bijections between Graphs

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We study a hierarchy of five classes of bijections between the edge sets of two graphs: weak maps, strong maps, cyclic maps, orientable cyclic maps, and chromatic maps. Each of these classes contains the next one and is a natural class of mappings for some family of matroids. For example, f: $E(G) \to E(H)$ is cyclic if every cycle (eulerian subgraph) of G is mapped onto a cycle of H. This class of mappings is natural when graphs are considered as binary matroids. A chromatic map $E(G) \to E(H)$ is induced by a (vertex) homomorphism from G to H. For such maps, the notion of a vertex is meaningful so they are natural for graphic matroids. In the same way that chromatic maps lead to the definition of $\chi(G)$—the chromatic number—the other classes give rise to new interesting graph parameters. For example, $\phi(G)$ is the least order of H for which there exists a cyclic bijection f: $E(G) \to E(H)$. We establish some connection between ϕ and χ, e.g., $\chi(G) \geqslant \phi(G) > \chi(G)/2$. The exact relation between ϕ and χ depends on knowledge of the chromatic number of C_n^2, the square of the n-dimensional cube. Higher powers of C_n are considered, too, and tight bounds for their chromatic number are found, through some analysis of their eigenvalues.

NOTATION

Our terminology is mostly standard. We follow Berge [1] for graph theory, in particular: a *graph* is a loopless multigraph. When multiple edges are forbidden we talk about a *simple graph.* The star of a vertex x, st(x) is the set of all edges incident with x. G^k, *the* kth *power of the graph* $G = (V, E)$, is a simple graph on the same vertex set V where two vertices are adjacent iff their distance in G is k. We refer to Welsh [13] for matroids. The graphic (cycle) matroid of a graph G is denoted by $M(G)$. We use the term *cycle* for any element of the cycle space of a binary matroid, while a *circuit* is a minimal dependent set. In paticular if G is a graph, the cycles of $M(G)$ are the edge sets of eulerian subgraphs (we call a graph eulerian if every vertex has even degree). If M is a regular matroid

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which is oriented (see [13, Sect. 10.3]) and e is an element in the cycle c then $\operatorname{sgn}(e, c)$ is the (e, c)th entry in the oriented cycle matrix of M. Since the main topic of the paper are mappings between graphic matroids, whenever we mention a mapping $f\colon G \to H$ we refer to the *edge sets* of the graphs unless it is explicitly stated otherwise.

1. Introduction

Our starting point is the following observation: Let G be a graph and let $c\colon V(G) \to [n]$ be a proper n-coloring. This coloring gives rise to a graph H on $[n]$ with an edge with ends $c(x)$, $c(y)$ for each edge of G with ends x, y. The edges of G and H are in $1:1$ correspondence and it is easily verified that a cycle (i.e., eulerian subgraph) in G is mapped to one in H under this correspondence. A bijection between the edge sets of two graphs with this property is called a cyclic map and we see that for every graph G there exists H of order $\chi(G)$ with a cyclic map $G \to H$. What is the least order that such H may have? Such questions are studied for a few natural classes of edge bijections between graphs. These five classes form a hierarchy which we now describe.

Let G and H be graphs and $f\colon E(G) \to E(H)$ a bijection between their edge sets.

(1.1) f is a *weak bijection* if for every independent set $A \subset E(H)$, $f^{-1}(A)$ is independent in G.

(1.2) f is a *strong map* if for every closed set A of $M(H)$, $f^{-1}(A)$ is a closed set in $M(G)$. If G is not bipartite and all the edges of H are parallel, f is said to be a *trivial* strong map. Observe that any graph G can be mapped to a collection of $|E(G)|$ parallel edges and this is a trivial strong map (unless G is bipartite).

(1.3) f is called a *cyclic map* if the image of every cycle of G is a cycle of H.

(1.4) A cyclic map f is said to be *orientable cyclic* if $M(G)$ and $M(H)$ may be oriented so that if c is a cycle in G then $\operatorname{sgn}(e, c) = \operatorname{sgn}(f(e), f(c))$ for every $e \in c$. By a result of Bland and Las Vergnas [4, Corollary 6.2.8] such orientations always come from orientations of the underlying graph.

(1.5) f is called a *chromatic map* if it is induced by a homomorphism, that is, if there exists a mapping $c\colon V(G) \to V(H)$, such that for every $e = (x, y) \in E(G)$, $f(e)$ has ends $c(x)$, $c(y)$.

Now we establish the fact that each class in this list contains the next one:

Chromatic maps easily seen to be orientable cyclic. To show that every cyclic map is strong, let us first remind that a set A in a matroid on E is closed iff for every circuit c, $|c-A| \neq 1$. Let f be cyclic and A a closed set in H. For every cycle c in G, $|c-f^{-1}(A)| = |f(c)-A| \neq 1$ (because $f(c)$ is an edge disjoint union of circuits of $M(H)$), hence $f^{-1}(A)$ is closed and f is strong.

Every strong bijection is known to be weak (see [13, Chap. 17, Sect. 4, Theorem 1, and Sect. 2, Exercise 2.3]). This last statement may be wrong if the mapping is not a bijection. Weak non-bijective maps are considered in the next section.

Each of these classes is closed under composition and they form the natural mappings for certain classes of matroids. Although we defined the mappings for graphs, the generalization for the appropriate class of matroids is obvious:

Weak and strong maps are based on independent sets and closed sets and thus are defined for general matroids (see [13, Chap. 17]). The definition of a cyclic map relies on the concept of a cycle and hence it reflects the properties of graphs as binary matroids. The notion of an orientation makes orientable cyclic maps natural for regular matroids. Finally, chromatic maps, based on vertex homomorphisms, are specific to graphic matroids.

Notice that our classes also dualize in the proper way: If f: $M \to N$ is strong (for general matroids), cyclic (for binary matroids), or orientable cyclic (for regular matroids), then the inverse map regarded as a mapping from N^* to M^* (the dual matroids) is also strong, cyclic, orientable cyclic, respectively. Regarding a strong bijection, see [13, Theorem 17.4.2]. For a cyclic or orientable cyclic map: Every cycle of N^* is orthogonal (mod 2 or over the rationals for a binary matroid or an orientation of a regular matroid, respectively) to all the cycles of N, including f images of all cycles of M. Thus its inverse image is orthogonal to all cycles of M, hence it is a cycle of M^*. A similar result holds for weak bijections if M and N are of the same rank [13, Exercise 17.4.2]. Since the family of graphic matroids is not closed under duality, chromatic maps do not fit here well as do the other classes.

In our framework we think about the chromatic number $\chi(G)$ as the least order of a graph H, for which there exists a chromatic map from G to H. We proceed with this approach as follows:

Let G be a graph. The parameters $\phi_o(G)$, $\phi(G)$, $\phi_s(G)$ are defined as the least order of a graph H, for which there exists a bijection f: $E(G) \to E(H)$, such that f is an orientable cyclic map, a cyclic map, or a non-trivial strong map, respectively. We postpone the discussion of a similar parameter for weak bijections to the next section.

Obviously $\chi(G) \geqslant \phi_o(G) \geqslant \phi(G) \geqslant \phi_s(G)$. It will be shown later that

$\phi_o(G)=\chi(G)$ (Theorem 3.2) and $\phi(G)\geqslant\frac{1}{2}\chi(G)$. More precise relations between ϕ and χ will also be derived (Theorem 3.1). Further understanding of this relation requires finding $\chi(C_n^2)$, the chromatic number of the square of the n-dimensional cube. In Section 5 we study in some detail the chromatic number of higher powers of C_n and obtain rather tight bounds.

It is then recognized that the notion of a cyclic map is very natural for binary matroids in general. One of us has recently studied maps between graphs [11], which in our terminology can be described as cyclic maps from cographic to graphic matroids. These are more general then duality between planar graphs, which in our terms is a bijection which is cyclic both as a map $M(G)\to M^*(H)$ and as a map $M^*(G)\to M(H)$. From this point of view $\phi(G)$ can be interpreted as the smallest number of cut sets required to cover every edge of G exactly twice.

2. Weak and Strong Maps

These two classes of maps between matroids have been studied by a number of authors (for references see [13]). We are not aware, though, of a discussion specialized to graphic matroids. Our definitions (1.1)–(1.5) explicitly require that the mappings involved be bijections. As far as strong mappings are considered, this requirement is not essential: Every strong or weak mapping f: $E(G)\to E(H)$ might be considered as a bijection, simply by replacing every $e\in E(H)$ by a set of $|f^{-1}(e)|$ parallel edges. This procedure does not affect the order of H, which is our main concern. However, this approach fails for weak maps because for that kind of maps, graphs H with parallel edges might not be very interesting: Every graph can be weakly mapped onto a set of parallel edges on two vertices. Even if forbid this triviality we can still use H on three vertices with all edges but one being parallel. (The difference between strong and weak mapping with respect to adding parallel edges lies in the fact that a set of edges is closed or not regardless of containing parallel edges, while a set which contains parallel edges is never independent.) Still, as we just mentioned, the study of weak maps need not be restricted to bijections. We start by treating a class of weak maps, not necessarily bijections, where the target graph H does not allow parallel edges.

We define $\phi_w(G)$ to be the least order of a *simple graph* H (no parallel edges) for which there exists a weak mapping f: $E(G)\to E(H)$ (the inverse image of an independent set is independent). The following theorem relates $\phi_w(G)$ to $\chi(G)$:

Theorem 2.1. (i) *For every graph G, $\phi_w(G)\geqslant\chi(G)$.*

(ii) *There are graphs with $\chi(G)=2$ and $\phi_w(G)$ as large as we wish.*

Proof. (i) Let f: $G \to H$ be a weak map where H is simple and has order k. For every $x \in V(H), f^{-1}(\text{st}(x))$ is an independent set in $M(G)$. This yields a collection of k forests in G which cover every edge exactly twice. Therefore if S is a set of vertices in G it has at most $k(|S|-1)/2$ induced edges. Consequently the subgraph induced by S has a vertex of degree at most $k-1$. This is well known to imply that G is k-colorable (e.g., [5, p. 221]).

(ii) This is obtained by showing $\phi_{\text{w}}(K_{n,n}) \geqslant n+1$. Consider a weak map f: $K_{n,n} \to K_m$ (not necessarily a bijection). The average of $|f^{-1}(u)|$ over $u \in E(K_m)$ is $n^2/\binom{m}{2}$. Since every edge of a clique belongs to the same number of spanning trees, the average of $|f^{-1}(T)|$ over spanning trees T in K_m is $(m-1)n^2/\binom{m}{2}$. But this cannot exceed $2n-1$, which is the size of a spanning tree in $K_{n,n}$. Hence $m \geqslant n+1$. ∎

Before we pass to strong maps let us remind the reader that a closed subset in a graphic matroid is the union of vertex disjoint, induced subgraphs. First we show:

THEOREM 2.2. *$E(K_n)$ is not the proper disjoint union of fewer than $\sqrt{n}+1$ closed subgraphs. The bound is attained iff there is an affine plane of order $\sqrt{n}$.*

Proof. A closed subset of $M(K_n)$ is the union of edge sets of vertex disjoint cliques. Consider any decomposition of K_n into at least two closed sets and let r be the largest size of a clique in any of the closed sets involved. If x is a vertex outside this r-clique, then the r edges connecting x to this clique must all belong to different closed subsets. This means that at least $r+1$ closed subsets participate in the decomposition. On the other hand, if no clique with more than r vertices is used, then at least $(n-1)/(r-1)$ closed subsets will be needed to cover the $n-1$ edges incident with a vertex. These two bounds become equal when $r=\sqrt{n}$ so it follows that at least $\sqrt{n}+1$ closed subsets are needed.

When equality holds these cliques are all of size $\sqrt{n}$. They form then an $S(2, \sqrt{n}, n)$ design, that is, an affine plane of order $\sqrt{n}$ (e.g., [3, p. 28]).

On the other hand, if there is an affine plane of order $\sqrt{n}$, consider the vertices of K_n as its points. Every line corresponds to the clique on its points. Each parallel class of lines yields a closed subgraph and the $\sqrt{n}+1$ classes supply the required decomposition. ∎

The parameter ϕ_{s} seems to have less in common with known graphic parameters than do ϕ and ϕ_{w}. This is due to the existence of trivial strong maps.

THEOREM 2.3. *$\phi_{\text{s}}(K_n)=n$, for $n \neq 4$, and $\phi_{\text{s}}(K_4)=3$.*

Proof. Clearly $\phi_s(K_n)=n$ for $n \leqslant 3$. Let H be obtained by replacing each edge in K_3 by two parallel edges. A bijective map $K_4 \to H$ which maps every two non-incident edges of K_4 to parallel edges of H is strong. Hence $\phi_s(K_4)=3$. Next notice that there is no non-trivial strong map of K_n onto a graph of order 3 for $n>4$. The inverse of such a map induces a decomposition of $E(K_n)$ into two or three closed subsets, in contradiction with Theorem 2.2.

Let $f\colon K_n \to H$, $n>4$, be a strong map. It induces an equivalence relation on $E(K_n)$ where two edges in K_n are related iff their images under f are parallel. Let a, b be two incident edges in K_n. We claim that $f(a)$ and $f(b)$ are incident in H. If a and b are equivalent this is clear, so let us assume they are inequivalent. The equivalence classes $[a]$ and $[b]$ are closed, being inverse images of closed sets in H. If $f(a)$ and $f(b)$ are not incident, then for the same reason $[a] \cup [b]$ is closed. But the clique of $[a]$ containing a and that of $[b]$ containing b intersect in exactly one vertex. In a complete graph this forbids the union $[a] \cup [b]$ from being closed, a contradiction.

Since f preserves incidence, for every $x \in V(K_n)$, $\mathrm{st}(x)$ is mapped onto either a triangle or a star in H (possibly with multiple edges). Consider first the case where for some x in $V(K_n)$, $f(\mathrm{st}(x))$ contains the edges of a simple triangle (p, q, r) in H. Together with their parallels this is a closed set in H whose inverse image contains $\mathrm{st}(x)$. But this inverse image is closed and so it contains all edges in K_n. In other words, we have a non-trivial strong map $K_n \to K_3$ which we already showed to be impossible. In the same way, if for some x, $f(\mathrm{st}(x))$ is formed of parallel edges, then f is trivial, contradicting our assumption.

The only case that we still need to consider is when f maps stars to stars with at least two non-parallel edges. This induces a mapping $V(K_n) \to V(H)$ as follows: If y is the center of $f(\mathrm{st}(x))$, then map x to y. Assuming $V(H)<n$ it implies that two distinct stars $\mathrm{st}(v)$ and $\mathrm{st}(w)$ are mapped to stars of H with the same center c. These two stars of H will have just one edge in common, corresponding to the edge e_1 of K_n joining v and w. Let $e_2 \in \mathrm{st}(v)$, and let $e_3 \in \mathrm{st}(w)$ form a triangle with e_1 and e_2. Then $f(e_2)$, $f(e_3)$ are incident to c. Let F_{ij} $(1 \leqslant i<j \leqslant 3)$ be the closure in H of $\{f(e_i), f(e_j)\}$. Then $f^{-1}(F_{ij})$ is a closed set which contains e_i, e_j, and hence the whole triangle e_1, e_2, e_3. It follows that $\{f(e_1), f(e_2), f(e_3)\} \subseteq F_{ij}$ $(1 \leqslant i<j \leqslant 3)$. This is possible only if $f(e_1)$, $f(e_2)$, $f(e_3)$ are parallel. Then all edges of $f(\mathrm{st}(v))$ must be parallel, a contradiction as we have just seen. ∎

3. Cyclic Maps

We find the parameter ϕ, defined by cyclic maps, to be closely related to the chromatic number.

THEOREM 3.1. (i) $2^{\lceil \log_2 \phi(G) \rceil} \geq \chi(G) \geq \phi(G)$. *In particular* $2\phi(G) > \chi(G) \geq \phi(G)$ *and if* $\phi(G)$ *is a power of* 2 *then* $\phi(G) = \chi(G)$.

(ii) *If* $\chi(G) = 4$ *then* $\phi(G) = 3$.

(iii) *If* $\omega(G) \geq 5$, *then* $\phi(G) \geq \omega(G)$. ($\omega(G)$ *is the largest clique size in* G.)

Proof. (i) Let f: $E(G) \to E(H)$ be a cyclic map where $|V(H)| = \phi = \phi(G)$. We label the vertices of H with distinct binary vectors of length $k = \lceil \log_2 \phi \rceil$. Let $u \in E(G)$ and let $f(u) = (x, y) \in E(H)$, where x and y are two k-vectors. Define $g(u)$ to be $x + y$, so g maps $E(G)$ to the k-dimensional vector space over $GF(2)$. Now we describe h, a proper vertex coloring of G with binary k-vectors: Pick any vertex x and set $h(x) = 0$, the zero vector. If for two adjacent vertices y, z, $h(y)$ is already defined while $h(z)$ is not, then we set $h(z) = h(y) + g((y, z))$. The vertices of H have distinct labels thus $g((y, z)) \neq 0$ and $h(y) \neq h(z)$ as required. To show that h is well defined, independent of the choice of the specific pair y, z, it is enough to show that for every cycle c in G we have $\sum g(u) = 0$, where the sum is taken over all $u \in c$. This is, however, clear, by definition of g and the fact that the image of c is a cycle in H (see [12]).

(ii) First we notice that $\phi(K_4) = 3$. The strong mapping from K_4 to K_3 with every edge doubled (see the proof of Theorem 2.3) is cyclic. This implies that $\phi(G)$ never equals 4, since any cyclic mapping of G to H of order 4 can be composed with the previous mapping (parallel edges in H pose no difficulty), to yield a cyclic mapping of G onto a graph of order 3. The conclusion follows now easily from (i).

(iii) If f: $E(G) \to E(H)$ is cyclic, then f restricted to a subgraph of G is cyclic too. Thus, to prove our statement it suffices to show that for $n \geq 5$, $\phi(K_n) = n$. The identity mapping shows $\phi(K_n) \leq n$ and $\phi(K_n) \geq n$ is a direct consequence of Theorem 2.3. ▮

THEOREM 3.2. (i) *For every graph* G, $\phi_o(G) = \chi(G)$ (*thus*, ϕ_o *is not really a new parameter*).

(ii) *There exists orientable cyclic maps which are not chromatic maps.*

Proof. (i) Let f: $E(G) \to E(H)$ be an orientable cyclic map where $|V(H)| = n = \phi_o(G)$ Say that the vertices of H are labeled $0, 1, ..., n-1$. Now if an edge u is oriented from i to j, then we associate with

$f^{-1}(u) \in E(G)$ the number $j-i \pmod n$. This defines a mapping δ: $E(G) \to Z_n - \{0\}$, such that the signed sum along each cycle is 0 (mod n). Such a mapping induces an n-coloring g in a well-known way (e.g., [12]): Select a vertex $x \in V(G)$ and color it $g(x)=0$. Continue inductively as follows: If p is an already colored vertex and q is a not yet colored neighbor of p, then set $g(q)=g(p)+\operatorname{sgn}(u)\,\delta(u)$, where u is an edge between p and q and $\operatorname{sgn}(u)=1$ if it is oriented from p to q, and -1 otherwise. g is well defined because δ sums to zero on cycles, and it is a proper coloring since $\delta(u)$ is never zero.

(ii) Let G and H be two circuits of the same length. Any bijection between their edge sets is orientable cyclic, but it is not induced by a homomorphism in the case where two consecutive edges are mapped onto a non-consecutive pair. ∎

4. Square of the Unit Cube

How good are the bounds of Theorem 3.1? As we already know, $\chi(K_n) = \phi(K_n)$ for $n \geqslant 5$. But how large can $\chi(G)$ get, given $\phi(G)$? For $\phi(G)=2$, $\chi(G)=2$. For $\phi(G)=3$, $3 \leqslant \chi(G) \leqslant 4$ and K_4 shows that the upper bound can be attained. $\phi(G)=4$ is impossible as previously noted. For $\phi(G) \geqslant 5$ it turns out that there is a definite extreme case. C_n is the graph of the n-dimensional unit cube and C_n^2 is its square, having the binary n-vectors as vertices, with two vectors adjacent iff their vectors differ in exactly two places. Note that our definition for a power of a graph may not be standard.

THEOREM 4.1. (i) *$\phi(C_n^2)=n$ and among all graphs G with $\phi(G)=n$, C_n^2 has the largest chromatic number.*

(ii) $2^{\lceil \log_2 n \rceil} \geqslant \chi(C_n^2) \geqslant n$. *The upper bound holds with equality for n of the form 2^t, 2^t-1, 2^t-2, and 2^t-3.*

Proof. (i) As usual e_i denotes the ith unit vector. The set $\{e_i \mid 1 \leqslant i \leqslant n\}$ forms a clique in C_n^2 and so $\phi(C_n^2) \geqslant \omega(C_n^2) \geqslant n$. On the other hand, if $u=(x, y)$ is an edge in C_n^2, where x, y are considered as binary n-vectors, then $x+y=e_i+e_j$ for some $1 \leqslant i < j \leqslant n$. Let the mapping f be defined as follows: f maps u to (i, j) in a graph H on $V(H)=\{1, 2, ..., n\}$. It is straightforward to verify that f is indeed cyclic.

To complete the proof of (i) we show that if $\phi(G)=n$ there is a homomorphism h: $V(G) \to V(C_n^2)$ which maps adjacent vertices to adjacent vertices (which induces a chromatic bijection if we allow parallel edges in C_n^2 to make the mapping $1:1$). This immediately implies $\chi(C_n^2) \geqslant \chi(G)$. Here is how this homomorphism h is constructed: Let f: $E(G) \to E(H)$ be cyclic, where $V(H)=\{1, 2, ..., n\}$. Select any vertex v in G and set $h(v)=0$, the zero vertor. If for some edge $(x, y) \in E(G)$, $h(x)$ is defined but $h(y)$ is not and if $f(x, y) = (i, j) \in E(H)$, then we set $h(y) = h(x) + e_i + e_j$. The fact that h is a homomorphism is easy to verify.

(ii) The first statement is just a repetition of Theorem 3.1(i) for $G = C_n^2$. As for the second statement, let us remark that C_n^2 has two components: the subgraph on vertices of even weight and the one for odd weight. These two subgraphs are isomorphic and so we consider the even part. An independent set of vertices in this subgraph is thus a binary code of length n with all weights even and all distances $\geqslant 4$. If the last coordinate is omitted from each of the vectors, a code of length $n-1$ and distance $\geqslant 3$ is obtained. It is common to denote by $A(m, d)$ the largest size of binary codes of length m and least distance d. This quantity received a good deal of attention (see [10, Chap. 17; 2] for a thorough discussion). It follows that the largest size of an independent set, $\alpha(C_n^2)$, is equal to $2A(n-1, 3)$. For $n = 2^t - k$, $k = 0, 1, 2, 3$ it is known that $A(n-1, 3)$ is realized by the k times shortened Hamming code. Therefore for these values of n, $\alpha(C_n^2) = 2^{n-t}$ and

$$\chi(C_n^2) \geqslant \frac{2^n}{\alpha(C_n^2)} = 2^t = 2^{\lceil \log_2 n \rceil}$$

as claimed.

It was conjectured (see, for instance, [2]) that k times shortened Hamming codes remain optimal for $k \leqslant 2^{t-2}$ (they are known to be inferior to other codes for larger k). If this conjecture is true, then

$$\chi(C_n^2) = 2^{\lceil \log_2 n \rceil}$$

holds at least "half of the time." In fact we know of no counterexample to the last equation but we have very little evidence to support it beyond what was already mentioned. It should also be remarked that bounds, and sometimes exact values, are known for $A(n, 3)$ for many values of n. From these, non-trivial bounds on $\chi(C_n^2)$ may be deduced (see the tables of [10]). ∎

5. Coloring Higher Powers of the Cube

The discussion in Section 4 naturally extends to the more general question of finding the chromatic number of C_n^t, the tth power of the n-dimensional cube. This graph has all binary n vectors as vertices, two vertices being adjacent iff their Hamming distance is t. If t is odd, then C_n^t is bipartite and so only the case of even t is interesting.

Theorem 5.1. *For every integer d there exist $k_1 > k_2 > 0$ such that if $n > n_0(d)$, then*

$$k_1 n^d > \chi(C_n^{2d}) > k_2 n^d.$$

Proof. We start with the lower bound: By a theorem of Hoffman [8], if G is a graph of order p and $\lambda_1 \geqslant \cdots \geqslant \lambda_p$ are the eigenvalues of its adjacency matrix, then

$$\chi(G) \geqslant 1 - \lambda_1/\lambda_p.$$

Now in the case of $G = C_n^{2d}$, the spectrum may be determined by means of the theory of association schemes (see, e.g., [10, Chap. 21]). Let us recall that Krawtchouk polynomials are defined as

$$P_k(x, n) = \sum_{j=0}^{k} (-1)^j \binom{n-x}{k-j}\binom{x}{j}. \quad \blacksquare$$

THEOREM. *The set of eigenvalues of C_n^t is given by*

$$\{P_t(i, n) \mid i = 0, ..., n\}.$$

For proof see, e.g., [10, p. 657].

Thus to apply Hoffman's theorem to C_n^{2d} we have to find the maximum and minimum of $P_{2d}(i, n)$ over $i = 0, ..., n$. To this end we notice that $P_{2d}(x, n)$ is an even polynomial in $y = x - n/2$. We claim that

$$P_k(x, n) = \sum_{j=0}^{\lceil k/2 \rceil} (-1)^{k+j} y^{k-2j} n^j (A_{k,j} + O(n^{-1})), \tag{5.1}$$

where $A_{k,j} > 0$ and the coefficients in the O term depend only on k, j. This holds for $k = 0, 1$, where $P_0(x, n) = 1$ and $P_1(x, n) = n - 2x = -2y$. The proof of (5.1) follows from the following recursion that Krawtchouk polynomials satisfy:

$$(k+1)\, P_{k+1}(x, n) = (n - 2x)\, P_k(x, n) - (n - k + 1)\, P_{k-1}(x, n).$$

Using (5.1) for the r.h.s. we get

$$\begin{aligned}
&(n-2x)\, P_k(x, n) - (n-k+1)\, P_{k-1}(x, n) \\
&= -2y \sum_{j=0}^{\lceil k/2 \rceil} (-1)^{k+j} y^{k-2j} n^j (A_{k,j} + O(n^{-1})) - (n-k+1) \\
&\quad \times \sum_{j=0}^{\lceil (k-1)/2 \rceil} (-1)^{k+j-1} y^{k-2j-1} n^j (A_{k-1,j} + O(n^{-1})) \\
&= \sum_{j=0}^{\lceil k/2 \rceil} (-1)^{k+j+1} y^{k-2j+1} n^j (2A_{k,j} + O(n^{-1})) \\
&\quad + \sum_{j=0}^{\lceil (k+1)/2 \rceil} (-1)^{k+j+1} y^{k-2j+1} n^j (A_{k-1,j-1} + O(n^{-1})).
\end{aligned}$$

The last step involved writing $n-k+1$ as $n(1+O(n^{-1}))$ and a change in the index. Equating coefficients we get

$$P_{k+1}(x, n)=\sum_{j=0}^{\lceil (k+1)/2\rceil} (-1)^{k+j+1}\, y^{k+1-2j} n^j (A_{k+1,j}+O(n^{-1})),$$

where

$$A_{k+1,j}=\frac{1}{k+1}\,(A_{k-1,j-1}+2A_{k,j})$$

for $j=1, ..., \lceil k/2\rceil$.

$$A_{k+1,0}=\frac{2}{k+1}\,A_{k,0}$$

and if k is odd, $A_{k+1,(k+1)/2}=(1/(k+1))\,A_{k-1,(k-1)/2}$. For $k=2d$ we rewrite (5.1) as

$$P_{2d}(x, n)=\sum_{j=0}^{d} (-1)^j\, y^{2(d-j)} n^j (A_{2d,j}+O(n^{-1})).$$

It follows that for $n>n_0(d)$ and $|y|\geqslant \alpha_d\sqrt{n}$, we have

$$P_{2d}(x, n)>0.$$

(The terms alternate in sign, the y degree decreases by two, and the n degree goes up by one.) From general properties of orthogonal polynomials it is known that $P_k(x, n)$ has all its k roots in the interval $n\geqslant x\geqslant 0$. We have just found that in fact all roots of $P_{2d}(x, n)$ are concentrated in the interval $\alpha_d\sqrt{n}\geqslant |x-n/2|$. It follows that all the local extrema belong to that interval as well. Now

$$\max_{n\geqslant x\geqslant 0} P_{2d}(x, n)\geqslant P_{2d}(0, n)=\binom{n}{2d}$$

(in fact equality holds). Also

$$P_{2d}(x, n)\geqslant -\sum_{j=0}^{d} y^{2(d-j)} n^j (A_{2d,j}+O(n^{-1}))$$

and since the minimum is obtained for some $|y|\leqslant \alpha_d\sqrt{n}$,

$$P_{2d}(x, n)\geqslant -\sum_{j=0}^{d} \alpha_d^{2(d-j)}\, n^d (A_{2d,j}+O(n^{-1}))\geqslant -\beta_d n^d.$$

The lower bound on $\chi(C_n^{2d})$ now follows

$$\chi(C_n^{2d}) \geqslant 1 + \frac{\binom{n}{2d}}{\beta_d n^d} > k_2 n^d.$$

The coloring we use for proving the upper bound was introduced by Graham and Sloane [7]. First, let us denote for $n \geqslant w \geqslant 0$ by $C_n^d(w)$ the subgraph induced by vertices of C_n^{2d} with weight w. Note that if $|w - w'| > 2d$ then vertices of $C_n^{2d}(w)$ and $C_n^{2d}(w')$ have no edge between them. Therefore

$$\chi(C_n^{2d}) \leqslant 2d \max_{0 \leqslant w \leqslant n} \chi(C_n^{2d}(w)).$$

Since we are not concerned with factors depending on d only it suffices to concentrate on $\max_w \chi(C_n^{2d}(w))$. Graham and Sloane [7] presented a mapping which properly colors $C_n^{2d}(w)$ with few colors: Let $q \geqslant n$ be a prime power and let $x_1, ..., x_n$ be distinct elements of $GF(q)$. Map $(a_1, ..., a_n) \in V(C_n^{2d})$ to

$$\left(\sum_{i=1}^{n} a_i x_i, \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant n} a_{i_1} a_{i_2} x_{i_1} x_{i_2}, ..., \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant n} a_{i_1} \cdots a_{i_d} x_{i_1} \cdots x_{i_d}\right),$$

which belongs to the d-dimensional vector space over $GF(q)$. It is shown in [7] and quite easily verified that this map is a proper coloring of $C_n^{2d}(w)$ for every $n \geqslant w \geqslant 0$. The number of colors used is at most $q^d < \gamma_d n^d$ if q is the next prime power following n. Altogether we have a coloring of C_n^{2d} by $k_1(d) \cdot n^d$ colors.

After this work was completed we learned about a recent paper of Frankl and Furedi [6] from which the lower bound follows. Their methods are completely different from ours.

6. Other Matroid Maps between Graphs

It is quite clear how cyclic maps can be considered in the more general setting of binary matroids. Let M, N be binary matroids with ground sets S, T, respectively. A mapping $f: S \to T$ is cyclic if the image of a cycle in M is a cycle in N.

Using this terminology, planar duality between the graphs G and H may be described as a bijection $E(G) \to E(H)$ which is cyclic both from $M(G)$ to $M^*(H)$ and from $M^*(G)$ to $M(H)$.

Relaxation of the notion of matroid duality turns out to be very interesting. One of us has recently observed [11] that the double cycle cover conjecture (see [9] for a survey) is equivalent to a conjecture on the existence of cyclic maps from the cographic matroid of every bridgeless graph to a loopless graphic matroid. More specifically, the order of the image (as a graph) of this cyclic map provides the number of cycles in the double cover. If we ask about a double cocycle cover of a graph then we should consider cyclic maps between graphs. Every graph has an obvious double cocycle cover consisting of the cocycles defined by all vertices. This cover is induced by the identity cyclic map of the graph on itself. However, we still may ask for the smallest number of cocycles in such a cover. It turns out that this number is exactly ϕ. This is a very simple observation and proving it is left to the reader.

What about cyclic maps of the form $M(G) \to M^*(H)$? We have very little to say here. We mention without proof the following two easy facts:

(i) If G is either planar or bipartite there exists a bridgeless graph H and a cyclic map $M(G) \to M^*(H)$. However, there are graphs G which are neither planar nor bipartite for which the same holds.

(ii) The above property does not hold for $G = K_5$.

Let us mention in closing two questions which arose in the course of this study:

(1) Find more precise estimates for $\chi(C_n^2)$. In particular, is it true that for every n

$$\chi(C_n^2) = 2^{\lceil \log_2 n \rceil}?$$

We do not known whether this holds even for $n = 9$.

(2) In Theorem 3.2(ii) we exihibit orientable cyclic maps which are not chromatic. However, the only examples we know of come from graphs of connectivity two or less. Do such maps exist for 3-connected graphs?

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