On the expansion rate of Margulis expanders

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Abstract

In this note we determine exactly the expansion rate of an infinite 4-regular expander graph which is a variant of an expander due to Margulis. The vertex set of this graph consists of all points in the plane. The point (x, y) is adjacent to the points $S(x, y), S^{-1}(x, y), T(x, y), T^{-1}(x, y)$ where $S(x, y) = (x, x+y)$ and $T(x, y) = (x+y, y)$. We show that the expansion rate of this 4-regular graph is 2. The main technical result asserts that for any compact planar set A of finite positive measure,

$$\frac{|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A|}{|A|} \geq 2,$$

where $|B|$ is the Lebesgue measure of B.
The proof is completely elementary and is based on *symmetrization* - a classical method in the area of isoperimetric problems.
We also use symmetrization to prove a similar result for a directed version of the same graph.

1 Introduction

The Greek *isoperimetric problem* asks for the largest possible area of a planar figure of a given circumference. It took about two millennia to prove that everyone's guess is true: the optimal figure is the disk. Modern proofs for this fact are pretty easy, but perhaps the most conceptual proofs known are based on the notion of *symmetrization*. The basic idea is this: Given any planar figure K, we seek a "more symmetric" figure K' of the same circumference, so that $|K'| \geq |K|$ (where $|X|$ is the Lebesgue measure of X). After the appropriate symmetries are identified, one shows that the optimum is attained for figures that are invariant under all relevant symmetries. In the planar case, the relevant symmetries can be taken to be reflections with respect to lines through the set's center of gravity. In this case the disk is the unique invariant set which is, therefore, also the unique optimal body. As the reader probably knows, we are telling here only part of the story that is relevant to us, and some additional argumentation is needed to complete the proof.

This classical problem is the starting point for a lot of modern mathematics. Specifically, it is often more challenging to answer similar questions for underlying geometries other than the Euclidean plane. Indeed, the theory of expander graphs can be viewed as a modern discrete version of this classical

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problem. Here we attempt to deal with the problem of presenting a family of expander graphs, and of proving their expansion properties by calculating their *exact* expansion rate. The expanders under consideration were the first to have been explicitly constructed, and are due to Margulis ([4], 1973). Technically, we determine the expansion rate of a variant of his infinite 4-regular graph.

In his work, Margulis relied on deep theorems from the theory of groups representations of the group $SL_2(Z_p)$. He used 5 transformations that generate the associated affine group (namely $(x, y) \to (x, y)$, $(x, y) \to (x+1, y)$, $(x, y) \to (x, y+1)$, $(x, y) \to (x, x+y)$, and $(x, y) \to (-y, x)$) and considered the induced graph on Z_p^2. Gabber and Galil ([1], 1979) used Fourier analysis to prove that a very similar construction yields a family of expander graphs. They were also able to provide, for the first time, a lower bound on the expansion rate. This bound seems, however, far from being tight. Based on a theorem of Selberg (1965), Lubotzky, Philips, and Sarnak ([2], 1986), showed that the Cayley graphs of $SL_2(Z_p)$ with respect to the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are expanders. This implies that the symmetric quotient graphs $Y_p = (V_p, E_p)$ defined by $V_p = Z_p \times Z_p \setminus \{(0,0)\}$, $E_p = \{((a,b); (a, (b \pm a) \ mod \ p))\} \cup \{((a,b); ((a \pm b) \ mod \ p, b))\}$ are expanders (see also ([3])). The celebrated LPS graphs are Ramanujan, i.e., they have the largest possible spectral gap. However, this fact yields only crude estimates for their expansion rate.

The main technical result in this paper is that for any planar compact set A of finite positive measure,

$$\frac{|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A|}{|A|} \geq 2,$$

where $S(x,y) = (x, x+y)$ and $T(x,y) = (x+y, y)$.

We follow the great tradition of solving isoperimetric problems by means of symmetrization arguments. The vertex set of the graph we consider is the whole plane. We show that for any planar set A, we can find another set A' of the same area, that is "more symmetric" and expands at most as much as A does. We then determine the set of a given area that is invariant under the relevant symmetries and show that it is optimal. The proof is elementary.

Later we prove a similar statement for a directed version of the above graph. Namely,

$$\frac{|S(A) \cup T(A) \cup A|}{|A|} \geq \frac{4}{3},$$

for any planar compact set A of finite positive measure.

As mentioned above, we only determine the expansion rate of a variant of the infinite planar version of Margulis' construction. There is also a bounded version of his construction where the vertex set is the unit square and operations are carried out modulo 1. This bounded construction can be easily discretized to yield an infinite family of finite regular expander graphs. We conjecture that the statement analogous to our theorem holds also for the *mod* 1 version and hence for the finite expanders derived from it. Namely, we conjecture that there is a constant $c > 0$, such that if $A \subset [0,1]^2$ has measure $|A| \leq c$, then $|\tilde{S}(A) \cup \tilde{S}^{-1}(A) \cup \tilde{T}(A) \cup \tilde{T}^{-1}(A) \cup A| \geq 2|A|$. Here $\tilde{S}(x,y) = (x, (x+y) \ mod \ 1)$ and $\tilde{T}(x,y) = ((x+y) \ mod \ 1, y)$. A proof of this conjecture would yield the exact expansion rate for the graphs Y_p defined above.

2 Statement of results

We consider all planar compact measurable sets under the $L_1(\Re^2, |\ \ |)$ metric: the distance between two such sets is the measure of their symmetric difference. Moreover, two sets are considered equal if their symmetric difference has measure zero. Let $S, T : \Re^2 \to \Re^2$ be defined by $S(x,y) = (x, x+y)$, $T(x,y) = (x+y, y)$. All sets under consideration here are measurable and bounded. We prove

Theorem 2.1 *For any planar set A of finite positive measure,*

$$\frac{|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A|}{|A|} \geq 2.$$

The bound 2 is tight.

Since the expression we consider is homogeneous, we may restrict our attention to sets of measure 1 and normalize whenever necessary.

The following example shows that the bound in Theorem 2.1 is attained:

Example 2.2 *Let A be the square $\{(x,y); |x \pm y| \leq 1\}$. Then $S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A$ is the square $\{(x,y); |x|, |y| \leq 1\}$. In this case $|A| = 2$, and $|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A| = 4$.*

(a) (b)

Figure 1: (a) $A = \{(x,y); |x \pm y| \leq 1\}$; (b) $S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A = \{(x,y); |x|, |y| \leq 1\}$.

3 Notations and definitions

- $U(A) = S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A$.
- $-A = \{(-x,-y); (x,y) \in A\}$.
- $A^t = \{(y,x); (x,y) \in A\}$.
- $A^c = \{(-y,x); (x,y) \in A\}$.
- $\mu(A) = \frac{|U(A)|}{|A|}$.
- $\Im$ is the set of planar measurable compact sets of positive measure.
- Q_i is the i-th quadrant $(1 \leq i \leq 4)$.

4 Proof

Outline of proof: We seek symmetries that are related to the actions of T and S. We observe that a compact planar set that is invariant under three simple transformations (namely, reflection through the x axis, reflection through the y-axis, reflection through the line $x = y$), has expansion ≥ 2. Given a set $A \in \Im$, our plan is to find a set $\hat{A} \in \Im$ that is invariant under these three transformations, with $\mu(\hat{A}) \leq \mu(A)$. As mentioned, such sets have expansion rate ≥ 2, and so this completes the proof.

The following straightforward observations will be used later:

Lemma 4.1: *Let $p, q, r, s > 0$ if $\frac{p}{r}, \frac{q}{s} \geq \frac{p+q}{r+s}$, then $\frac{p}{r} = \frac{q}{s} = \frac{p+q}{r+s}$.* ▮

Lemma 4.2: *For any set A, $|A| = |S(A)| = |S^{-1}(A)| = |T(A)| = |T^{-1}(A)|$.* ▮

We begin with the following simple statement:

Lemma 4.3: *For any set $A \in \Im$, $\mu(A) > 1$.*

Proof: Let $A \in \Im$. Assume for contradiction that $\mu(A) = 1$. Namely, $|U(A)| = |A|$. But $T(A), S(A) \subset U(A)$, and $|S(A)| = |A| = |T(A)|$. Consequently, $A = S(A) = T(A)$, so $A = S(A) = S^2(A) = S^3(A) = ... = S^n(A) = T(A) = T^2(A) = T^3(A) = ... = T^n(A)$ for all $n \in Z$. Note that $S^n(x, y) = (x, nx + y)$, and $T^n(x, y) = (x + ny, y)$. Since $A \in \Im$, it has positive measure. Therefore there are positive-measure "chunks" of A at large distance from the origin, i.e., A is not bounded, a contradiction. ▮

The following symmetrization lemma proves central:

Lemma 4.4: *Let $A, \bar{A} \in \Im$ be two sets satisfying the following conditions:*

- $|A| = |\bar{A}|$.
- $|U(A)| = |U(\bar{A})|$.
- $|U(A \setminus \bar{A})| = |U(\bar{A} \setminus A)|$.
- $|U(A \cap \bar{A}) \cap U(A \setminus \bar{A})| = |U(A \cap \bar{A}) \cap U(\bar{A} \setminus A)|$.

Then either $\mu(A \cup \bar{A}) \leq \mu(A)$ or $\mu(A \cap \bar{A}) \leq \mu(A)$. (If $A \cap \bar{A} = \emptyset$ then necessarily $\mu(A \cup \bar{A}) \leq \mu(A)$.)

Proof: Let $B = A \cap \bar{A}$.

- If $|B| = 0$, then $\mu(A \cup \bar{A}) = \frac{|U(A \cup \bar{A})|}{|A \cup \bar{A}|} \leq \frac{|U(A)| + |U(\bar{A})|}{|A| + |\bar{A}|} = \mu(A)$.
- Otherwise ($|B| \neq 0$), define $B_1 = A \setminus B$, and $B_2 = \bar{A} \setminus B$. Then $|B_1| = |B_2|$, $|U(B_1)| = |U(B_2)|$, and
$$\mu(A) = \frac{|U(A)|}{|A|} = \frac{|U(B) \cup U(B_1)|}{|B| + |B_1|} = \frac{|U(B)| + |U(B_1)| - |U(B) \cap U(B_1)|}{|B| + |B_1|}.$$
Let $a = |U(B)| + |U(B_1)| - |U(B) \cap U(B_1)|$, $\alpha = |B| + |B_1|$, $b = |U(B)|$, and $\beta = |B|$. Then $\mu(A) = \frac{a}{\alpha}$ and $\mu(B) = \frac{b}{\beta}$.
Now
$$\mu(A \cup \bar{A}) = \mu(B \cup B_1 \cup B_2) = \frac{|U(B) \cup U(B_1) \cup U(B_2)|}{|B| + |B_1| + |B_2|} \leq$$

$$\leq \frac{|U(B)|+|U(B_1)|+|U(B_2)|-|U(B)\cap U(B_1)|-|U(B)\cap U(B_2)|}{|B|+|B_1|+|B_2|} =$$

$$= \frac{|U(B)|+2|U(B_1)|-2|U(B)\cap U(B_1)|}{|B|+2|B_1|} = \frac{2a-b}{2\alpha-\beta}.$$

If $\mu(A) < \mu(B), \mu(A\cup \bar{A})$ then $\frac{a}{\alpha} < \frac{b}{\beta}, \frac{2a-b}{2\alpha-\beta}$, resulting in a contradiction (by 4.1).∎

We now apply this observation with three different choices of $\bar{A}$, one for each of the above-mentioned symmetries:

- either $\mu(A\cup -A) \leq \mu(A)$ or $\mu(A\cap -A) \leq \mu(A)$. (If $A\cap -A = \emptyset$ then $\mu(A\cup -A)\leq \mu(A)$.)
- either $\mu(A\cup A^t) \leq \mu(A)$ or $\mu(A\cap A^t) \leq \mu(A)$. (If $A\cap A^t = \emptyset$ then $\mu(A\cup A^t)\leq \mu(A)$.)
- either $\mu(A\cup A^c) \leq \mu(A)$ or $\mu(A\cap A^c) \leq \mu(A)$. (If $A\cap A^c = \emptyset$ then $\mu(A\cup A^c)\leq \mu(A)$.)

Furthermore, we may combine all three properties (i.e., apply them one after the other), and from now on consider only sets that are invariant under all three symmetries.

Proof of Theorem 2.1: In view of the previous observations, it is enough to consider a set A with $A = -A = A^t = A^c$. In particular, $|A_1| = |A_2| = |A_3| = |A_4|$, where $A_i = A\cap Q_i$. Let $|A| = 4$. Then $|A_i| = 1$ for $1\leq i\leq 4$. Observe the following:

- $S(A_1), T(A_1)\subset Q_1$, $S(A_1)\cap T(A_1) = \emptyset$.
- $S^{-1}(A_2), T^{-1}(A_2)\subset Q_2$, $S^{-1}(A_2)\cap T^{-1}(A_2) = \emptyset$.
- $S(A_3), T(A_3)\subset Q_3$, $S(A_3)\cap T(A_3) = \emptyset$.
- $S^{-1}(A_4), T^{-1}(A_4)\subset Q_4$, $S^{-1}(A_4)\cap T^{-1}(A_4) = \emptyset$.

This results in $|U(A)| \geq |S(A_1)\cup T(A_1)\cup S^{-1}(A_2)\cup T^{-1}(A_2)\cup S(A_3)\cup T(A_3)\cup S^{-1}(A_4)\cup T^{-1}(A_4)| = |S(A_1)|+|T(A_1)|+|S^{-1}(A_2)|+|T^{-1}(A_2)|+|S(A_3)|+|T(A_3)|+|S^{-1}(A_4)|+|T^{-1}(A_4)| = 1+1+1+1+1+1+1+1 = 8$, and $\mu(A)\geq \frac{8}{4} = 2$.∎

5 The directed case

Next we consider a non-symmetric variant of the above construction. It turns out that once the appropriate symmetries are identified, the previous arguments work smoothly.

Theorem 5.1 *For any planar set A of finite positive measure,*

$$\frac{|S(A)\cup T(A)\cup A|}{|A|} \geq \frac{4}{3},$$

where $S(x,y) = (x, x+y)$, $T(x,y) = (x+y, y)$. The bound $\frac{4}{3}$ is tight.

Here is an example showing that the bound $\frac{4}{3}$ is best possible.

Example 5.2 *Let A be the hexagon $\{(x,y); |x|, |y|, |x+y|\leq 1\}$. Then $S(A)\cup T(A)\cup A$ is the square $\{(x,y); |x|, |y|\leq 1\}$. In this case $|A| = 3$, and $|S(A)\cup T(A)\cup A| = 4$.*

(a) (b)

Figure 2: (a) The set $A = \{(x,y); |x|, |y|, |x+y| \le 1\}$; (b) $S(A) \cup T(A) \cup A = \{(x,y); |x|, |y| \le 1\}$.

Sketch of proof: As in the nondirected case, one should symmetrize via $A = -A = A^t$. However, here there is a "new" symmetrization that should be identified: $A = T^{-1}S(A)$. A compact planar set that is invariant under the linear transformation $T^{-1}S$ has expansion $\ge 4/3$. Therefore, given a set $A \in \Im$, we find a set $\hat{A} \in \Im$ satisfying $\hat{A} = -\hat{A} = \hat{A}^t$, $S(\hat{A}) = T(\hat{A})$, with $\mu(\hat{A}) \le \mu(A)$ (where now $\mu(X) = \frac{|S(X) \cup T(X) \cup X|}{|X|}$). Since such sets have expansion rate $\ge \frac{4}{3}$, the proof is complete.
We omit most of the technical details of the proofs and focus on the main lemmas.

Lemma 5.3: $(S^{-1}T)^3 = (T^{-1}S)^3 = -I$. ▮

Therefore, $(T^{-1}S)^6 = I$. What $T^{-1}S$ actually does is to decompose the plane to 6 regions that are rotated anti-clockwise as described in the following figure:

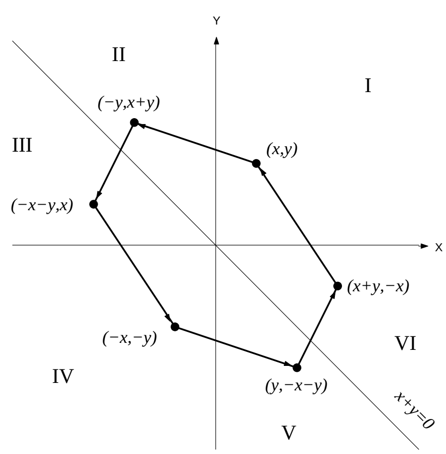

Figure 3: $T^{-1}S$ maps the regions $I \to II \to III \to IV \to V \to VI \to I$.

Lemma 5.4: *Given a set* $A \in \Im$, $\mu(A) \le \frac{4}{3}$, *there exists a set* $\hat{A} \in \Im$, *for which* $S(\hat{A}) = T(\hat{A})$ *with* $\mu(\hat{A}) \le \mu(A)$.▮

Proof of Theorem 5.1: Consider a set A with $T(A) = S(A)$, and assume $|A| = 6$. Since $T^{-1}S(A) = A$, and $(T^{-1}S)^6 = I$, A splits evenly between 6 regions in the plane (as in Figure 3): First quadrant/ Second

quadrant above the line $x+y=0$/ Second quadrant below the line $x+y=0$/ Third quadrant/ Fourth quadrant below the line $x+y=0$/ Fourth quadrant above the line $x+y=0$.
Let A_i be the intersection of A with the i-th region. Then $|A_i|=1$ for $1 \leq i \leq 6$. Furthermore, A's measure splits unevenly between the different quadrants: 1 unit in the first and in the third quadrants, 2 units in the second and the fourth quadrants. An easy calculation shows that if $S(A)=T(A)$, then $S(A)=T(A)=\{(-x,y)|(x,y) \in A\}$, i.e., $S(A)$ is obtained from A by reflection through the y-axis. $S(A)$'s measure (which is 6), splits unevenly between the quadrants, but this time there are 2 units in each of the first and third quadrants, and 1 unit in each of the second and fourth quadrants. Therefore: $|S(A) \cup T(A) \cup A| \geq |S(A) \cup A| = \sum_{i=1,..,4} |(S(A) \cup A) \cap Q_i| \geq 2+2+2+2 = 8$, resulting in $\mu(A) \geq \frac{8}{6} = \frac{4}{3}$.∎

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