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Combinatorics, Probability and Computing (1993) 2, 491–503

Local-Global Phenomena in Graphs

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Received 26 March 1993

For Paul Erdős on his 80th birthday

This is a survey of a number of recent papers dealing with graphs from a geometric perspective. The main theme of these studies is the relationship between graph properties that are local in nature, and global graph parameters. Connections with the theory of distributed computing are pointed out and many open problems are presented.

1. Introduction

How well can global properties of a graph be inferred from observations that are purely local? This general question gives rise to numerous interesting problems that we want to discuss here. Such a *local-global* approach is often taken in geometry, where it has a long and successful history, but a systematic study of graphs from this perspective has not begun until recently. Nevertheless, a number of older results in graph theory do fit very nicely into this framework, as we later point out. Most of the specific problems fall in two categories. In the first, local structural information on the graph is collected and then used to derive certain consequences for the graph as a whole. The other class of problems concerns *consistency* of local data. Namely, one asks to characterize those sets of local data that may come from some graphs.

As the reader will soon see, the local-global paradigm leads to many questions in which graphs are viewed as geometric objects, a point of view that we believe can greatly benefit graph theory. Besides the geometric connection, ties also exist with the theory of combinatorial algorithms. We suggest a specific test case for the heuristic notion that polynomial-time algorithms are capable of examining only local phenomena. In distributed computing, locality of computation is an already recognized and studied notion, and some connections with this discipline are pointed out as well.

2. Packing and covering with spheres and local-global averaging

Let $W \subseteq V(G)$ be a set of vertices in a graph G. If the vertices in W form a majority in every ball of radius between 1 and r in G, does this imply that W has a large cardinality?

As an illustration, consider the following example with $r = 1$. In this graph, W is a clique of $\sqrt{n}$ vertices. Each vertex in W has a set of $\sqrt{n} - 1$ neighbors not in W, each of which has degree 1. It is a routine matter to check that this graph satisfies the assumption for $r = 1$. It is also not hard to modify the construction for any fixed $\alpha < 1$ so that W occupies a fraction $\geq \alpha$ of any 1-neighborhood, while $|W| = O(\sqrt{n})$ (here α was 1/2).

Let us introduce some notation. The ball† of radius k centered at x, denoted $B_k(x)$ consists of all vertices y whose distance from x does not exceed k, and its cardinality $|B_k(x)|$ is denoted $\beta_k(x)$. Our question is how small $|W|$ may be in terms of r and n, the order of G.

If we represent W by its characteristic function, we are led to consider a more general problem. Namely, let f be a nonnegative function defined on the vertices of an n-vertex graph G. Suppose that we have a lower bound on the average of f on every ball in G of radius between 1 and r. What can we conclude for the overall average of f?

This subject has been recently taken up by Linial, Peleg, Rabinovich and Saks [23] who show the following.

Theorem 2.1. (Local Averages) *Let f be a nonnegative function defined on the vertices of an n-vertex graph G. Suppose that the average of f over every ball of radius $r \geq t \geq 1$ in G is at least μ. Then, the average of f over all of V is at least $\mu \cdot n^{-O(1/\log r)}$. The bound is tight.*

Consequently, if we let r be n^c for some positive constant c, local averages do reflect the true global behavior of f. Examples are given in [23] showing that smaller r's will not do. It is natural to ask at this point what happens if we only know a lower bound for the average of f over balls of radius r (and not for every $r \geq t \geq 1$). Examples are given showing that only very weak conclusions can be drawn about the overall average of f, however big r may be. Namely, it may be that the average of f is only $O(n^{-1/3})$. It is also worthwhile noting that the conclusions of the theorems remain unchanged even if we make the assumption only for balls whose radius $r \geq t \geq 1$ is *a power of* 2.

The result for local averages is proved as a consequence of tight theorems about sphere packing and about covering by spheres in general graphs. Either 0–1 or fractional packing and covering results will do for this purpose.

Theorem 2.2. (Covering by Spheres) *For integers $n > r$, the vertices of an n-vertex graph can be covered by a collection of balls with radii in the range $[1, \ldots, r]$, that cover no vertex more than $n^{O(1/\log r)}$ times. The bound is tight.*

Theorem 2.3. (Sphere Packing) *In any n-vertex graph, there is a collection of disjoint balls whose radii are in the range $[1, \ldots, r]$, which together cover at least $n^{1-O(1/\log r)}$ vertices. The bound is tight.*

† The words ball and sphere are used interchangeably here.