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LEGAL COLORING OF GRAPHS

N. LINIAL

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The following computational problem was initiated by Manber and Tompa (22nd FOCS Conference, 1981): Given a graph $G=(V, E)$ and a real function $f\colon V\to\mathbf{R}$ which is a proposed vertex coloring. Decide whether f is a proper vertex coloring of G. The elementary steps are taken to be linear comparisons.

Lower bounds on the complexity of this problem are derived using the chromatic polynomial of G. It is shown how geometric parameters of a space partition associated with G influence the complexity of this problem.

Existing methods for analyzing such space partitions are suggested as a powerful tool for establishing lower bounds for a variety of computational problems.

1. Deciding the legality of a graph coloring

Let $G=(V,E)$ be a graph and let $f\colon V\to\mathbf{R}$ be a real function which is a proposed vertex coloring of G. We are to determine whether or not f is a proper coloring, namely, whether or not for every two adjacent vertices x, y we have $f(x)\neq f(y)$. The elementary step is one linear comparison.

The *element uniqueness problem* which was investigated by Dobkin and Lipton [1] is the following: Given n reals $f_1, \dots, f_n$ decide whether or not they are mutually distinct. The elementary steps are linear comparisons. This problem is the case of the legal coloring problem where the graph G is K_n, the complete graph on n vertices. The article [1] serves as the starting point for Manber and Tompa [5] who were the first to consider the legal coloring problem. Following [1] they transform the combinatorial problem into an equivalent geometric problem. The defining parameters of the geometric objects involved can be used to derive lower bounds for the complexity of the legal coloring problem. Letting $V=V(G)$ be denoted by $\{a_1, \dots, a_n\}$ we associate with G a subset of $\mathbf{R}^n$ as follows:

$$S(G) = \{(x_1, \dots, x_n)\in\mathbf{R}^n | x_i \neq x_j \quad \text{if} \quad [a_i, a_j]\in E(G)\}.$$

The geometric equivalent of the legal coloring problem is the question whether $\vec{f}=(f(a_1), \dots, f(a_n))$ belongs to $S(G)$ or not. To be able to apply the methods of [1]

AMS subject classification (1980): 68 C 25, 05 C 15 one needs to know the number of connected components of $S(G)$. The answer to this is implicit in [2] and related results were subsequently obtained by Zaslavsky [9], [10].

Theorem 1. [2], [5, Theorem 2]. *Let G be a graph with $a(G)$ acyclic orientations. There are $a(G)$ connected components in $S(G)$.*

The main tool from [1] is:

Lemma 2. [1]. *Let $S\subseteq \mathbf{R}^n$ be an open set with k connected components. Consider the problem of deciding for $x\in\mathbf{R}^n$ whether or not x belongs to S, using linear comparisons. This problem requires at least $\log_2 k$ steps.*

(All logarithms in this paper are base 2).

The conclusion that is derived is:

Theorem 3. [5]. *Let $G=(V, E)$ be a graph and consider the problem of deciding whether a given vertex coloring is legal, using linear comparisons. In worst case $\log a(G)$ steps are required.*

Most of the discussion in [5] concentrates around finding lower bounds on $a(G)$ given the numbei of vertices and the number of edges in G. In this article we find the exact minimum and we solve, in fact, a much more general problem.

We also point out that a large variety of computational problems can be restated as: "Decide for a point $x\in\mathbf{R}^n$, whether x belongs to S". The set S would typically be a space partition or a polyhedral set. The geometric parameters of S can be used to estimate the complexity of the computational problem. The methods developed by Zaslavsky ([9], [10]) to compute those parameters can be used to realize this general plan.

We start with a theorem of Stanley [8]: Let $G=(V, E)$ be a graph, let D be an acyclic orientation of G and let $f\colon V\to\{1, \dots, k\}$ be a mapping. We say that f is a k-partition *compatible* with the orientation D if the edge $[x, y]\in E$ being oriented from x to y in D implies that $f(x)\leqq f(y)$.

Theorem 4. [8]. *Let G be a graph on n vertices whose chromatic polynomial is $P(G, \lambda)$ and let k be a natural number. Then $(-1)^n P(G, -k)$ equals the number of pairs (D, f) where D is an acyclic orientation of G and f a compatible k-partition. In particular, $a(G)=|P(G, -1)|$.*

To state our result let us define for integers $k>l\geqq 0$ the graph $A_{k,l}$ which consists of a complete graph on k vertices and one more vertex which has l neighbors among the vertices of the complete k-graph. Notice that $P(A_{k,l}, \lambda)=(\lambda-l)\lambda(\lambda-1)\dots$ $\dots(\lambda-k+1)$.

Theorem 5. *Let m, n be integers with $\binom{n}{2}\geqq m\geqq 0$, and let k, l be the integers defined by $m=\binom{k}{2}+l$, $k>l\geqq 0$. Let $B=B_{m,n}$ be the graph obtained by adding $n-k-1$ isolated vertices to $A_{k,l}$. If G is a graph on n vertices with m edges, then for any integer λ, $|P(G, \lambda)|\geqq|P(B, \lambda)|$ holds.*

Proof. We need a lemma:

Lemma 5.1. *Let $G=(V,E)$ be a graph and let $x\in V$ have degree $d=d(x)$. For an integer λ with $\lambda\leqq 0$ or $\lambda\geqq d$, $|P(G,\lambda)|\geqq|(\lambda-d)\cdot P(G\setminus x,\lambda)|$ holds.*

Proof. Let us start with $\lambda\geqq d$: Any λ-coloring of $G\setminus x$ can be extended to a λ-coloring of G in at least $\lambda-d$ different ways, so $P(G,\lambda)\geqq(\lambda-d)\cdot P(G\setminus x,\lambda)\geqq 0$. For $\lambda=-t<0$ consider an acyclic orientation of G and a compatible t-partition φ. Suppose that x has d_i neighbors in $C_i=\varphi^{-1}(i)$, so $d=\sum_{i=1}^{t} d_i$. We want to show that the orientation and the partition of $G\setminus x$ can be extended to an orientation and a compatible t-partition of G in at least $(d+t)$ different ways. Let us extend φ so that $\varphi(x)=i$, for some $t\geqq i\geqq 1$. The compatibility condition dictates the orientation of all edges incident with x, except for those d_i whose other vertex is in C_i. We show d_i+1 different ways to extend the orientation so that it remaines acyclic. Notice that compatibility is already granted. Since the orientation of $G\setminus x$ is acyclic, we can order $y_1,\dots,y_{d_i}$, the neighbors of x in C_i in such a way that if there is a directed path from y_α to y_β in $G\setminus x$, then $\alpha>\beta$. Pick any $d_i\geqq s\geqq 0$ and consider the orientation where we orient (x,y_j) for $s\geqq j\geqq 1$ and (y_j,x) for $d_i\geqq j\geqq s+1$. For each value of s we get a different acyclic orientation, altogether d_i+1 orientations. Summing over $t\geqq i\geqq 1$ we get $\sum_{i=1}^{t}(d_i+1)=d+t$ extensions and the lemma follows. ∎

We can proceed now to complete the proof of theorem 5. For integers λ in the range $k\geqq\lambda\geqq 0$ we have $P(B,\lambda)=0$ and the claim is obvious. For λ outside this range we use induction on m. For small m the claim is easy to verify. We want to show next that the average degree of vertices in G must be strictly less than k. Since $m\geqq\binom{k}{2}$, n must be at least k. If $k=n$ than $G=K_n$ and the claim is true. If $n\geqq k+1$, use the fact that $nk/2\geqq(k+1)k/2>m$ to conclude that $2m/n$, the average degree in G is strictly less than k. Let us pick a vertex x with degree $d=d(x)\leqq k-1$. Denote $G\setminus x$ by G' and apply induction to G'. We use the facts that

$$P(A_{k,l},\lambda)=(\lambda-l)\lambda(\lambda-1)\dots(\lambda-k+1)$$

and

$$P(B_{m,n},\lambda)=(\lambda-l)\lambda^{n-k}(\lambda-1)\dots(\lambda-k+1).$$

We consider two cases:

Case 1. $d\leqq l$.

In this case G' has $n-1$ vertices and $m-d$ edges and so for all integral λ: $|P(G',\lambda)|\geqq|P(B',\lambda)|$ holds, where B' is the graph obtained by adjoining $n-k-2$ isolated vertices to $A_{k,l-d}$. Now by Lemma 5.1, $|P(G,\lambda)|\geqq|(\lambda-d)\cdot P(G',\lambda)|$, and all we have to verify is that $|(\lambda-d)\cdot P(B',\lambda)|\geqq|P(B,\lambda)|$ where $B=B_{m,n}$ so that $P(B,\lambda)=(\lambda-l)\cdot\lambda^{n-k}(\lambda-1)\dots(\lambda-k+1)$ and $P(B',\lambda)=(\lambda-l+d)\cdot\lambda^{n-k-1}(\lambda-1)\dots(\lambda-k+1)$. This reduces to $|(\lambda-d)(\lambda-l+d)|\geqq|\lambda(\lambda-l)|$ where $\lambda>k>l\geqq d\geqq 0$ or $l\geqq d\geqq 0>\lambda$. This is easily verified.

Case 2. $d>l$.

Here B' is obtained by adding $n-k-1$ isolated vertices to $A_{k-1,k+l-d-1}$. Therefore

$$P(B',\lambda)=(\lambda+d+1-k-l)\lambda^{n-k}(\lambda-1)\dots(\lambda-k+2).$$

The calculations are as above and reduce to showing $|(\lambda-d)P(B',\lambda)|\geqq|P(B,\lambda)|$ that is, $|(\lambda-d)(\lambda+d+1-k-l)|\geqq(\lambda-l)(\lambda-k+1)|$, where $\lambda>k>d>l\geqq 0$ or $k>d>l\geqq 0>\lambda$, which is easily verified. This completes the proof of theorem 5. Let us only remark that by carefully analyzing the cases of equality one can show that $B_{m,n}$ is the unique graph for which the minimum is attained. ∎

Theorem 3 and 5 can be combined to deduce

Theorem 6. *Let G be a graph with m edges and consider the problem of deciding the legality of a vertex coloring of G. It takes at least*

$$\sqrt{\frac{m}{2}}\log m+O(\sqrt{m})$$

linear comparisons in the worst case. ∎

2. Further lower bounds on the complexity of legal coloring

The basic Lemma 2 has been generalized by several authors [6], [7]. The more general lower bounds can be applied to our problem and we show how: A subset S of $\mathbf{R}^n$ is said to be *polyhedral* if each of its connected components in the intersection of finitely many halfspaces. The number of t-dimensional faces of S is denoted by $f_t(S)$. We refer the reader to [3] for background on polyhedra.

Lemma 7. *Let $S\subseteq\mathbf{R}^n$ be an open polyhedral set and consider the problem of deciding for $x\in\mathbf{R}^n$ whether x is in S using linear comparisons. If* op *is the least number of steps in the worst case, then*

$$2^{\mathrm{op}}\binom{\mathrm{op}}{n-t}\geqq f_t(S).\quad ∎$$

Notice that Lemma 2 is the case $t=n$ of Lemma 7. To be able to apply Lemma 7 to derive lower bounds on the complexity of the legal coloring problem we need to relate $f_t(S(G))$ to some graphic parameters of G. This has been done by Zaslavsky in [10] and we describe his solution: For a set of edges $T\subseteq E$ in G we denote by G_T the pseudo-graph which results by contracting all edges in T. Notice that G_T may have loops in which case it has no acyclic orientations and so $a(G_T)=0$.

Theorem 8. [10]: *For a graph $G=(V,E)$ on n vertices*

$$f_t(S(G))=\sum_{\substack{T\subseteq E\\|T|=n-t}}a(G_T)\quad(n\geqq t\geqq 0).\quad ∎$$

Combining the last two results we have:

Theorem 9. *For a graph $G=(V,E)$ if* op *is the least number of linear comparisons required in the worst case to decide the legal coloring problem for G, then*

$$2^{\mathrm{op}}\binom{\mathrm{op}}{t}\geqq\sum_{T\subseteq E,|T|=t}a(G_t)\quad(n\geqq t\geqq 1).\quad ∎$$

3. Directions for further research

1. The quality of our lower bounds: The lower bound derived in [1] for the element uniqueness problem is quite sharp—by sorting the vector $\vec{f}=(f_1, \ldots, f_n)$ in $n \cdot \log n$ steps the problem can be decided. It would be very interesting to find such algorithms for the general cases of the legal coloring problem.

2. Just as legal coloring is the general case of the element uniqness problem one can ask questions on directec acyclic graphs which when specialized to the complete graphs are classical search—sort problems. Here are two examples:

a) The general case of the sorting problem. Given a graph G there is an unknown acyclic orientation on G that we are supposed to discover by successively asking the orientations of specific edges in G. How many steps are needed in worst case?

b) The general case of the maximum problem. Same set up as in a. Find a source of the unknown acyclic orientation of G.

3. Maximizing $a(G)$: An interesting question raised in [5] is to find the maximum of $a(G)$ given n and m, the number of vertices and edges in G. More generally one can ask for max $|P(G, \lambda)|$ as we did in this article. Our knowledge about this problem is very fragmentary and we mention some of our results without proof.

a) We have $a(G) \leqq \prod \max\big(2, d(x)\big)$ where the product is over all vertices x in G. This upper bound is better then the best bound given by [5] but is still very poor for graphs with many edges.

b) The graph which maximizes $a(G)$ given m and n is either a forest or a connected graph. (This is easy).

c) At first look it would seem that the graph G which maximizes $P(G, \lambda)$ given m, n, and λ should depend on λ. It seems however that for positive integral λ it does not that is there is a unique $G_{m,n}$ which attains the maximum for all natural λ. It would be extremely interesting to know if this is the case.

4. The complexity of deciding what is in a polyhedron: Let $S \subseteqq \mathbf{R}^d$ by a polyhedron that is defined by n essential linear inequalities. Consider the problem of deciding for $x \in \mathbf{R}^d$ whether x is in S using linear comparisons. From [3 ch. 10] we know that $f_t(S) \leqq n^{d/2}$ $(d \geqq t \geqq 0)$. The best lower bounds one can derive from Lemma 7 for the complexity of this problem are thus of the form op $\geqq O(d \log n)$. The question is whether this lower bound can be achieved. For $d=2$ this is clearly the case—if S is an open polygon with n sides, then deciding for $x \in \mathbf{R}^2$ whether or not x is in S can be decided by $O(\log n)$ linear comparisons. The result for $d=3$ may also be within reach, but for general dimension we know nothing.

5. Zaslavsky's method as a general tool for deriving lower bounds: In a series of papers ([9], [10], and more) T. Zaslavsky has shown how to compute $f_t(S)$ where S is $\mathbf{R}^n$ minus a finite collection of hyperplanes. This is a class of polyhedral sets which come up in many computational problems. The evaluation of the integers $f_t(S)$ combined with Lemma 7 can be a method of deriving interesting lower bounds for a number of interesting problems. The paper [4] can serve as an example how hard it is to derive such bounds and in fact this calls for redoing [4] systematically using Zaslavsky's methods.

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Nathan Linial

Institute of Mathematics and Computer Science
Hebrew University, Jerusalem, 91 904, Israel