# Internal Partitions of Regular Graphs

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**Abstract:** An *internal partition* of an $n$-vertex graph $G = (V, E)$ is a partition of $V$ such that every vertex has at least as many neighbors in its own part as in the other part. It has been conjectured that every $d$-regular graph with $n > N(d)$ vertices has an internal partition. Here we prove this for $d = 6$. The case $d = n - 4$ is of particular interest and leads to interesting new open problems on cubic graphs. We also provide new lower bounds on $N(d)$ and find new families of graphs with no internal partitions. Weighted versions of these problems are considered as well.  J. Graph Theory 00: 1–14, 2015

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## 1. INTRODUCTION

It is well known that every finite graph $G = (V, E)$ has an *external partition*, that is, a splitting of $V$ into two parts such that each vertex has at least half of its neighbors in

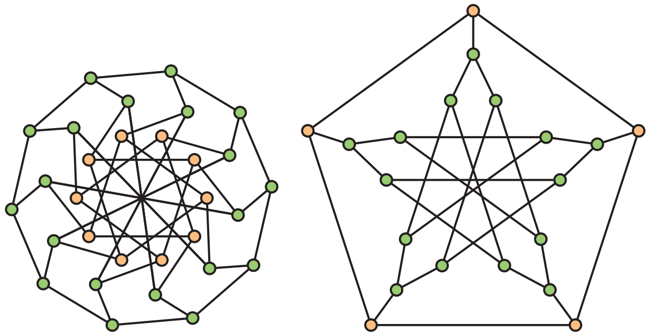

FIGURE 1. Examples of internal partitions.

the other part. This is, for example, true for $G$'s max-cut partition. Much less is known about the *internal partition* problem in which $V$ is split into two nonempty parts, such that each vertex has at least half of its neighbors in its own part. Not all graphs have an internal partition and their existence is proved only for certain classes of graphs. Several investigators (e.g., DeVos[5]) have raised the conjecture that for every $d$ there is an $n_0$ such that every $d$-regular graph with at least $n_0$ vertices has an internal partition. Here we prove the case $d = 6$ of this conjecture.

A related intriguing concept in this area is the notion of *external bisection*. This is an external partition in which the two parts have the same cardinality. We conjecture that the Petersen graph is the only connected bridgeless cubic graph with no external bisection. We take some steps in resolving this problem.

These concepts have emerged in several different areas and as a result there is an abundance of terminologies here. Thus Gerber and Kobler [7] used the term *satisfactory partition* for internal partitions. Internal/external partitions are called *friendly* and *unfriendly* partitions sometimes. Morris[9] studied social learning, and considered a more general problem. Now we want to partition $V = A\dot{\cup}B$ with $A, B \neq \emptyset$ such that every $x \in A$ (respectively, $y \in B$) has at least $qd(x)$ of its neighbors in $A$ (respectively, at least $(1 - q)d(y)$ neighbors in $B$). He refers to such sets as $(q/1 - q)$-*cohesive*. Here we use the term *q-internal partitions*. The complementary notion of *q-external partitions* is considered as well.

Figure 1 shows examples of internal partitions of cubic graphs.

Bazgan, Tuza, and Vanderpooten have written several papers [1, 2] on internal partitions. In [3] they give a survey of this area. Much of their work concerns the complexity of finding such partitions, a problem that we do not address here.

Our own interest in this subject arose in our studies of learning in social or geographical networks. Vertices in these graphs represent individuals and edges stand for social connection or geographical proximity. The individuals adopt one of two choices of a social attribute (e.g., PC or Mac user). Society evolves over time, with each individual adopting the choice of the majority of her neighbors. We asked whether a stable, diverse assignment of choices is possible in such a society. This amounts to finding an internal partition if the social choices are equally persuasive. It is also of interest to consider the problem when choices carry different persuasive power (say a neighbor who is a Mac user is more persuasive than a PC neighbor). If the merits are in proportion $q : 1 - q$, this leads to the problem of finding a *q-internal partition*.

Thomassen [12] showed that for every two integers $s, t > 0$ there is a $g = g(s, t)$ such that every graph $G = (V, E)$ of minimum degree at least $g$ has a partition $V = V_1 \dot\cup V_2$ so that the induced subgraphs $G(V_1)$, $G(V_2)$ have minimum degree at least $s$, $t$, respectively. He conjectured that the same holds with $g(s, t) = s + t + 1$, which would be tight for complete graphs. Stiebitz [11] proved this conjecture, and extended it as follows: For every $a, b : V \mapsto \mathbb{Z}_+$ such that $\forall v \in V, d_G(v) \geq a(v) + b(v) + 1$, there exists a partition of $V = A \dot\cup B$, such that $\forall v \in A, d_A(v) \geq a(v)$ and $\forall v \in B, d_B(v) \geq b(v)$. Kaneko [8] showed that in triangle-free graphs the same conclusion holds under the weaker assumption $d_G(v) \geq a(v) + b(v)$.

Stiebitz's result shows that, given $q \in (0, 1)$, every graph has a nontrivial partition, which is at most one edge (for each vertex) short of being a $q$-internal partition. Shafique and Dutton [10] showed the existence of internal partitions in all cubic graphs except $K_4$ and $K_{3,3}$ and in all 4-regular graphs except $K_5$. In this article, we settle the problem for 6-regular graphs.

Shafique and Dutton also conjectured that $K_{2k+1}$ is the only $d = 2k$-regular graph with no internal partition. We disprove this and present a number of counterexamples. Many of these exceptions are with $d \geq n - 4$. This range turns out to be of interest and we discuss it as well. As we show, there exist $d$-regular $n$-vertex graphs with no internal partitions with both $d$ and $n - d$ arbitrarily large. We conjecture that every $2k$-regular graph with $n \geq 4k$ has an internal partition. In the process, we consider external bisections of regular graphs, and especially cubic graphs. We note that all class 1 cubic graphs have an external bisection, and speculate that for bridgeless class 2 cubic graphs, only graphs that have the Petersen graph as a component do *not* have such a bisection.

Bollobás and Scott[4] made a related conjecture that any graph $G$ has a near-bisection $(A, B)$ such that

$$|N(a) \cap A| \leq |N(a) \cap B| + 1 \tag{1}$$

for every $a \in A$, and similarly for every $a \in B$. This entails that every $2d$-regular graph has an external bisection.

Finally, we conjecture that for every integer $d$ and real $1 > q > 0$, where $qd$ is an integer, there is an integer $\mu$ such that every $d$-regular graph of order at least $\mu$ has a $q$-internal partition. We also conjecture this for $q = 1/2$ and $d$ odd. As we show, for $d$ fixed and large $n$, every $n$-vertex $d$-regular graph has *many* $q$-internal partitions for *some* $q$. This lends some support to our conjecture.

## 2. TERMINOLOGY

We consider undirected graphs $G = (V, E)$ with $n$ vertices. For $S \subset V$, we denote by $G(S)$ the induced subgraph of $S$. The degree of $v \in V$ is denoted by $d(v) = d_G(v)$ and the number of neighbors that $v$ has in $S \subseteq V$ is called $d_S(v)$. The complement of $G$ is denoted by $\bar{G}$.

A partition $(A, B)$ is *trivial* if $|A| = 0$ or $|B| = 0$. A *bisection* of $V = A \dot\cup B$ is a partition with $|A| = |B|$. If $||A| - |B|| \leq 1$, then we call it a near-bisection. Corresponding to the partition $(A, B)$ of $V$ is the *cut* $E(A, B) = E_G(A, B) = \{xy \in E | x \in A, y \in B\}$. For $x \in A$ and $y \in B$, we call $d_A(x), d_B(y)$, respectively, the vertices' *indegrees*, and $d_B(x), d_A(y)$ the *outdegrees*. These terms usually refer to directed graphs, but we could not resist the convenience of using them in the present context.

We denote by $\delta(G)$, $\Delta(G)$, respectively, the minimum and maximum degree of $G$'s vertices. A subset $S \subseteq V$ is called *p-cohesive* if $\delta(G(S)) \geq p$. (Note that our notion of cohesion differs from that of Morris [9]). It is called $(p-1)$-*degenerate* if no $S' \subseteq S$ is $p$-cohesive.

A nontrivial partition $(A, B)$ *is q-internal* for $q \in (0, 1)$ if $\forall x \in A$, $d_A(x) \geq q d_G(x)$ and $\forall x \in B$, $d_B(x) \geq (1-q) d_G(x)$. A $\frac{1}{2}$-internal partition is simply *internal*.

If $\forall x \in A$, $d_B(x) \geq q d_G(x)$ and $\forall x \in B$, $d_A(x) \geq (1-q) d_G(x)$ we call a nontrivial partition *q-external*. A $\frac{1}{2}$-external partition is *external*.

A $q$-internal or a $q$-external partition is called *integral* if for every $v \in V$, $q d_G(v)$ is an integer.

A $q$-internal partition $(A, B)$ is *exact* if $|A| = qn$, and *near-exact* if $||A| - qn| < 1$. A $q$-external partition $(A, B)$ is *exact* if $|B| = qn$, and *near-exact* if $||B| - qn| < 1$. For $q = \frac{1}{2}$, exact partitions are *bisections*, and near-exact partitions are *near-bisections*.

## 3. INTERNAL PARTITIONS OF 6-REGULAR GRAPHS

**Lemma 1.** *Let $G = (V, E)$ be a graph with minimal degree $d$. For $0 < k < |V|$, let $(A, B)$ be a partition of $V$ that attains* $\min |E(A, B)|$ *over all partitions with $|A| = k$ or $|B| = k$. Then, either*

1. *$A$ is $l$-cohesive and $B$ is $m$-cohesive for some integers $l, m$ with $l + m = d$, or*
2. (a) *$A$ is $l$-cohesive and $B$ is $m$-cohesive for some integers $l, m$ with $l + m = d - 1$, and*
   (b) *The vertices in $A$ with indegree $l$ and the vertices in $B$ with indegree $m$ form a complete bipartite subgraph in $G$, and*
   (c) *For every $x \in A$ with indegree $l$, $B \cup \{x\}$ is $(m+1)$-cohesive. Similarly, $A \cup \{x\}$ is $(l+1)$-cohesive for every $x \in B$ with indegree $m$.*

***Proof.*** Let $x \in A$, $y \in B$. If $xy \notin E$ then

$$\begin{aligned} |E((A \backslash \{x\}) \cup \{y\}, (B \backslash \{y\}) \cup \{x\})| - |E(A, B)| \\ = d_A(x) - d_B(x) + d_B(y) - d_A(y) \\ \leq 2[d_A(x) + d_B(y) - d]. \end{aligned}$$

If $xy \in E$, then

$$\begin{aligned} |E(((A \backslash \{x\}) \cup \{y\}, (B \backslash \{y\}) \cup \{x\})| - |E(A, B)| \\ = d_A(x) - d_B(x) + (d_B(y) + 1) - (d_A(y) - 1) \\ \leq 2[d_A(x) + d_B(y) - (d-1)]. \end{aligned}$$

Since $E(A, B)$ is minimal, it follows that the sum of indegrees is at least $d - 1$ if $x, y$ are adjacent, and $d$ otherwise.

Let us apply this for $x, y$ of minimum indegree. Then (1) follows if there is such a pair with $xy \notin E$. On the other hand, if $xy \in E$ for all such pairs, then (2a) and (2b) follow. We obtain (2c) directly from (2b). ■

**Corollary 1.** *Every $n$-vertex $d$-regular graph has a $\lceil \frac{d}{2} \rceil$-cohesive set of at most $\lceil \frac{n}{2} \rceil$ vertices (respectively, $\frac{n}{2} + 1$) for $d$ even (for $d$ odd).*

***Proof.*** Consider a near-bisection of $G$ that minimizes $|E(A, B)|$. By Lemma 1 if $d$ is even, at least one of $A$, $B$ is $\frac{d}{2}$-cohesive. If $d$ is odd, and if neither $A$ nor $B$ are $\lceil\frac{d}{2}\rceil$-cohesive, then by (2a) both are $\lfloor\frac{d}{2}\rfloor$-cohesive, and by (2c) each can be made $\lceil\frac{d}{2}\rceil$-cohesive by adding a vertex of the other. ■

**Theorem 1.** *Every* 6*-regular graph with at least* 12 *vertices has an internal partition. The bound is tight.*

***Proof.*** We argue by contradiction and consider an $n$-vertex 6-regular graph $G = (V, E)$ with no internal partition. Let $(A, B)$ be the near-bisection of $V$ that attains $\min |E(A, B)|$ over all near-bisections. By Lemma 1 either $A$ or $B$ must be 3-cohesive. We may assume $A$ is 3-cohesive while $B$ is not, for else $(A, B)$ is an internal partition.

We repeatedly carry out the following step: As long as there is some $y \in B$ with outdegree $d_A(y) > 3$ we move that vertex from $B$ to $A$. If $A$ is 3-cohesive then clearly so is $A \cup \{y\}$, while if $B$ is 2-degenerate, so is $B\backslash\{y\}$. By assumption no internal partition exists, so this process must terminate with a trivial partition, that is, $B$ must be 2-degenerate. The move of $y$ from $B$ to $A$ decreases $|E(A, B)|$ by $2d_A(y) - 6 \geq 2$. Every step of the process therefore decreases the cut by at least 2, while $|B|$ decreases by 1. Also in the last two moves $|E(A, B)|$ decreases by $\geq 4$, and 6 in this order, and at termination $E(A, B) = \emptyset$. We conclude that $|E(A, B)| \geq 2|B| + 6$.

On the other hand $|E(A, B)| \leq 2|A| + 4$: By Lemma 1 all vertices in $A$ have outdegree $\leq 2$, except for at most 4 (that are adjacent to a vertex in $B$ with outdegree $\leq 4$) vertices with outdegree 3. Therefore $2|A| + 4 \geq |E(A, B)| \geq 2|B| + 6$ so that $|A| \geq |B| + 1$. It follows that $|A| = |B| + 1$, $n$ is odd and $B$ is a "tight" 2-degenerate. Namely, exactly four vertices in $A$ have outdegree 3, and in all moves (except the last two) $|E(A, B)|$ is reduced by exactly 2. If $n \geq 9$ then $|B| \geq 4$, so the first two vertex moves are of outdegree 4. Let $y', y'' \in B$ be these first two vertices, let $(A', B') = (A \cup \{y'\}, B\backslash\{y'\})$ be the partition after the first move, and let $(A'', B'') = (A \cup \{y', y''\}, B\backslash\{y', y''\})$ be the partition after the second move. By the above $|E(A', B')| = |E(A, B)| - 2$ and $|E(A'', B'')| = |E(A, B)| - 4$.

By Lemma 1 (2c) all vertices in $A'$ have outdegree 2. Therefore, in $A''$, all vertices have outdegree 2 except 4 with outdegree 1. Suppose that some pair of these outdegree-2 vertices in $A''$, say $x', x''$ are adjacent. Then it would be possible to move both vertices to $B''$ while increasing the cut size by only 2. Namely, $|E(A''\backslash\{x', x''\}, B'' \cup \{x', x''\})| = |E(A'', B'')| + 2 < |E(A, B)|$. This yields a near-bisection that contradicts the minimality of $|E(A, B)|$. Alternatively, if the outdegree-2 vertices in $A''$ form an independent set, then all their neighbors in $A''$ must have outdegree 1 and indegree 5. It follows that there are at most five vertices in $A''$ of outdegree 2. Therefore $|A''| \leq 9$.

In the case $|A''| = 9$, we observe that by the above, $G(A'')$ is a complete bipartite graph $K_{5,4}$, and so $G(A)$ is a complete bipartite graph $K_{4,3}$. If in $A$, one indegree 3 vertex $a$ is moved to $B$, it becomes $K_{3,3}$, which is 3-cohesive. But by Lemma 1(2c), $B \cup \{a\}$ is 3-cohesive. Therefore the partition $(A \setminus \{a\}, B \cup \{a\})$ is an internal partition.

Therefore $|A''| \leq 8 \Rightarrow |A| \leq 6 \Rightarrow n \leq 11$.

We now comment on the range $n \leq 11$. Note that the proof covers all even $n$. The complete graph $K_7$ is an exception with $n = 7$.

For $n = 9$, there is a unique unpartitionable 6-regular graph (see Fig. 2). We prove this statement when we discuss the case $d = n - 3$ in the following section.

For $n = 11$, there exist 6-regular graphs with no internal partition. One such example, $Q_3$, is a member of a class of unpartitionable graphs that we construct in Section 6. ■

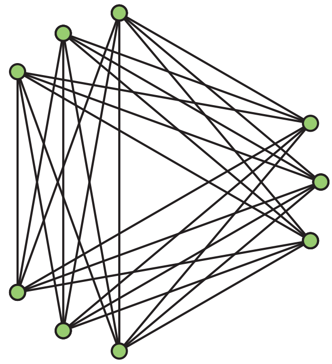

FIGURE 2. $K_{3,3,3}$: a 6-regular graph with no internal partition.

## 4. PARTITIONS OF COMPLEMENTARY GRAPHS

**Proposition 1.** *For every $q \in (0, 1)$, every graph $G$ has a $q$-external partition.*

***Proof.*** For a partition $(A, B)$ define

$$w(A, B) := |E(A, B)| - q \sum_{x \in A} d_G(x) - (1 - q) \sum_{x \in B} d_G(x) \tag{2}$$

The partition that maximizes $w(A, B)$ is nontrivial, since for every nonisolated vertex $x$ there holds $w(V \setminus \{x\}, \{x\}) > w(V, \emptyset)$ and $w(\{x\}, V \setminus \{x\}) > w(\emptyset, V)$. Furthermore $w(A, B) - w(A \setminus \{x\}, B \cup \{x\}) = d_B(x) - d_A(x) - qd_G(x) + (1 - q)d_G(x) = 2d_B(x) - 2qd_G(x)$ and $w(A, B) - w(A \cup \{x\}, B \setminus \{x\}) = d_A(x) - d_B(x) + qd_G(x) - (1 - q)d_G(x) = 2d_A(x) - 2(1 - q)d_G(x)$, so the maximality of $(A, B)$ implies that it is $q$-external. ■

**Proposition 2.** *For $q \in (0, 1)$ every exact $q$-internal partition of $G = (V, E)$ is an exact $(1 - q)$-external partition of $\bar{G}$.*

***Proof.*** Let $|V| = n$ and let $(A, B)$ be an exact $q$-internal partition of $G$. Namely, $|A| = qn$, $|B| = (1 - q)n$ and $\forall x \in A$, $d_A(x) \geq qd_G(x)$ and $\forall x \in B$, $d_B(x) \geq (1 - q)d_G(x)$. To indicate that we work in $\bar{G}$, we denote by $\bar{A}$, $\bar{B}$ the subgraphs of $\bar{G}$ induced by $A$, $B$. Then

$$\begin{aligned}
&\forall x \in V,\ d_{\bar{G}}(x) = n - d_G(x) - 1 \\
&\forall x \in A,\ d_{\bar{B}}(x) = |B| - d_B(x) = (1 - q)n - (d_G(x) - d_A(x)) \\
&\qquad \geq (1 - q)(n - d_G(x)) > (1 - q)d_{\bar{G}}(x) \\
&\forall x \in B,\ d_{\bar{A}}(x) = |A| - d_A(x) = qn - (d_G(x) - d_B(x)) \\
&\qquad \geq q(n - d_G(x)) > qd_{\bar{G}}(x)
\end{aligned}$$

So $(A, B)$ is a $(1 - q)$-external partition. ■

**Proposition 3.** *For $q \in (0, 1)$ every exact $(1 - q)$-external partition of $G = (V, E)$ is an exact $q$-internal partition of $\bar{G}$, provided the partition of $\bar{G}$ is integral.*

***Proof.*** Maintaining the notation of Proposition 2, consider an exact $(1 - q)$-external partition $(A, B)$ of $G$. Namely $|A| = qn$, $|B| = (1 - q)n$ and $\forall x \in B$, $d_A(x) \geq qd_G(x)$ and

$\forall x \in A, d_B(x) \geq (1-q)d_G(x)$. Then

$$\begin{aligned}
&\forall x \in V,\ d_{\bar{G}}(x) = n - d_G(x) - 1 \\
&\forall x \in A,\ d_{\bar{A}}(x) = |A| - d_A(x) - 1 = qn - (d_G(x) - d_B(x)) - 1 \\
&\qquad \geq q(n - d_G(x)) - 1 = qd_{\bar{G}}(x) - (1-q).
\end{aligned}$$

By rounding up we conclude that $d_{\bar{A}}(x) \geq qd_{\bar{G}}(x)$. (Note that $d_{\bar{A}}(x)$ and $qd_{\bar{G}}(x)$ are integers and $1 > q > 0$.)

$$\begin{aligned}
&\forall x \in B,\ d_{\bar{B}}(x) = |B| - d_B(x) - 1 = (1-q)n - (d_G(x) - d_A(x)) - 1 \geq \\
&\qquad \geq (1-q)(n - d_G(x)) - 1 = (1-q)d_{\bar{G}}(x) - q.
\end{aligned}$$

By a similar argument $d_{\bar{B}}(x) \geq (1-q)d_{\bar{G}}(x)$, so $(A, B)$ is a $q$-internal partition. ■

**Corollary 2.** *If G has an internal bisection, then $\bar{G}$ has an external bisection.*

**Corollary 3.** *If all degrees in G are even and $\bar{G}$ has an external bisection, then G has an internal bisection.*

**Proposition 4.** *For even n, every $(n-2)$-regular graph has an internal bisection.*

***Proof.*** The complement of an $(n-2)$-regular graph is a perfect matching. Split each matched pair between sides of a partition to obtain an external bisection. The proposition follows from Corollary 3. ■

**Proposition 5.** *An $(n-3)$-regular graph G has an internal partition if and only if its complementary graph $\bar{G}$ has at most one odd cycle. Furthermore this partition is a near-bisection.*

***Proof.*** Clearly $\bar{G}$ is 2-regular, that is, it is composed of vertex disjoint cycles. For every cycle, place the vertices alternately in $A$ and in $B$. If at most one cycle is odd, then $||A| - |B|| \leq 1$, so the partition is a near-bisection. It is also an internal partition of $G$, since the smaller side, say $B$, is a clique. Also, $A$ spans a clique if $|A| = |B|$ , or a clique minus one edge if $|A| = |B| + 1$, so its minimum indegree is also $|B| - 1$. As $|B| - 1 \geq (n-3)/2$, the partition is internal.

Let $G$ have an internal partition $(A, B)$. If $n$ is even, every vertex must have indegree $\geq n/2 - 1$. Therefore $|A| = |B| = n/2$ and the complementary graph $\bar{G}$ is bipartite so has no odd cycles. If $n$ is odd, assume $|A| > |B|$. $B$'s minimum indegree is $(n-3)/2$ so $|B| = (n-1)/2$, $|A| = (n+1)/2$ and the partition is a near-bisection. In $\bar{G}$, $|E(A, B)| = 2|B| = n - 1$ so $E(A) = (2|A| - |E(A, B)|)/2 = 1$. Therefore $(A, B)$ is bipartite in $\bar{G}$ except for a single edge internal to $A$. Therefore $\bar{G}$ has only one odd cycle. ■

We can now confirm that $K_{3,3,3}$, the graph in Figure 2, has no internal partition, as it is the complement of three disjoint triangles. Furthermore, as there is no other way for a 2-regular 9-vertex graph to have more than one odd cycle, this is the only $n = 9$, $d = 6$ graph with this property.

## 5. THE CASE $d = n - 4$ AND CUBIC GRAPHS

Let $G$ be a $d$-regular graph on $n$ vertices with $d = n - 4$. Clearly $n$ must be even, and its complement $\bar{G}$ is a cubic graph.

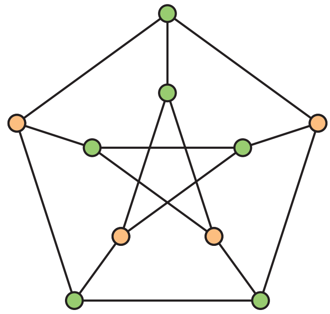

FIGURE 3. External partition of the Petersen graph.

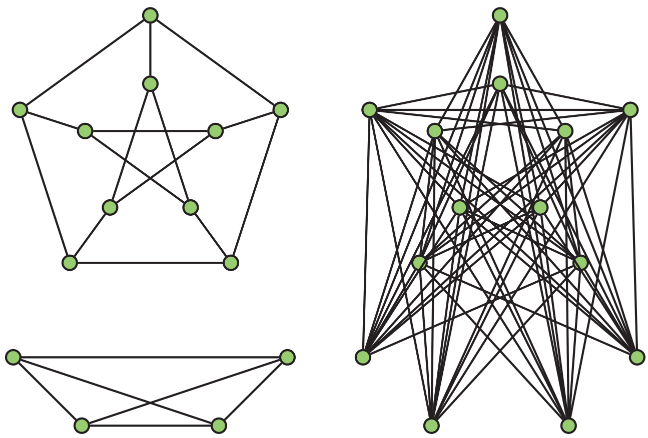

FIGURE 4. Smallest $d$=$n$−4 regular graph with no internal partition (right) is complement of cubic graph on left.

**Proposition 6.** *If an $(n-4)$-regular graph G has an internal partition then either*

- *$\bar{G}$ has an external bisection, or*
- *$\bar{G}$ has an independent set of size at least $n/2-1$.*

***Proof.*** To be internal, partition $(A, B)$ must have minimum degree $n/2-2$ in either part, so each part must have size $\geq n/2-1$. Then, either $(A, B)$ is an internal bisection, implying (by Corollary 2) that $\bar{G}$ has an external bisection, or $|A| = |B| + 2$, where $B$ is a clique in $G$ and thus an independent set in $\bar{G}$. ■

The Petersen graph (see Fig. 3) has no external bisection, but it has an independent set of size 4. Its complement is 6-regular, and in fact has an internal partition (but not a bisection), as already proven in Theorem 1.

The requirement of an independent set of size $n/2-1$ means that it is possible to remove at most three edges to make the graph bipartite. Clearly this is a rare phenomenon among cubic graphs, so our quest for graphs with internal partitions boils down to asking which cubic graphs have an external bisection.

We show next the following.

**Proposition 7.** *Every class 1, 3- or 4-regular graph G has an external bisection.*

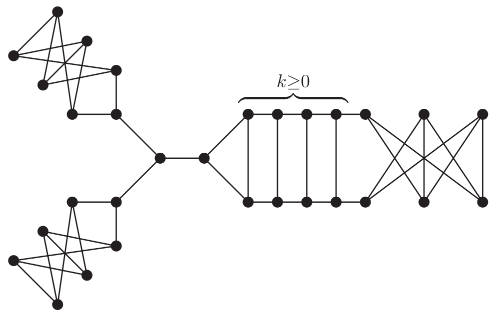


FIGURE 5. Family of cubic graphs that have no external bisection. These graphs were found by L. Esperet and G. Mazzuoccolo (private communication).

***Proof.*** Pick some $d$-edge coloring of $G$, and choose any two of the colors. The corresponding alternating cycles form a 2-factor in $G$ of even cycles. Number the vertices of each of these cycles sequentially along the cycle path. Alternately assign the vertices in the cycles to the two sides of a partition, which is clearly a bisection. For $d \leq 4$, this partition is external, since every vertex has at least two neighbors at the opposite part. ■

While all class 1 cubic graphs have an external bisection, the same question for class 2 cubic graphs remains open, though below we present a partial result. As noted, the Petersen graph, the smallest *snark* (see, e.g., [13]), has no external bisection. We checked a substantial number of larger snarks and found external bisections in all of them. We make the conjecture.

**Conjecture 1.** *The Petersen graph is the only 2-edge connected cubic graph with no external bisection.*

We actually believe that the same is true for connected cubic graphs with a single bridge. Below we detail an attempt at proof, which still falls short.

Connected cubic graphs with more than one bridge that have no external bisection exist. A family of such graphs, discovered by Esperet and Mazzuoccolo [6], is shown in Figure 5.

Disconnected cubic graphs with no external bisection are easy to come by. For example, a graph that has an odd number of components that are Petersen graphs and any number of $K_4$ components.

As mentioned above, the complement of the Petersen graph has an internal partition, since Petersen has an anticlique of size $4 = n/2 - 1$ (as required by Proposition 6). But the above-mentioned disconnected cubic graphs do not meet that requirement and so their complements have no internal partition. The smallest of these is a 10-regular graph of order 14, whose complement is a Petersen graph plus a $K_4$ component (see Fig. 4). This is the smallest of an infinite class of $(n-4)$-regular graphs with no internal partition. We conjecture that all $d = n - 4$ exceptions are constructed this way, as stated in the following.

**Conjecture 2.** *If $G$ is $(n-4)$-regular and has no internal partition, then $\bar{G}$ is a disconnected cubic graph that has an odd number of components each of which has no external bisection. All other components of $\bar{G}$ have the property that all their external partitions are bisections.*

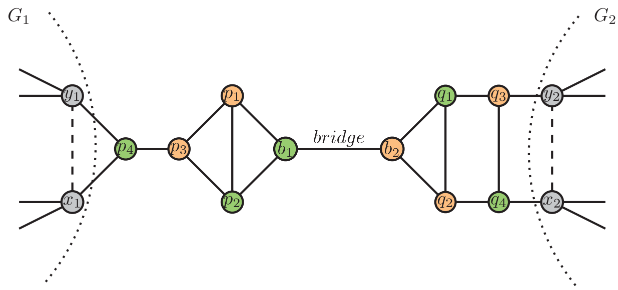


FIGURE 6. Cubic graph bridge decomposition.

An equivalent statement, based on Proposition 6 is this: All connected cubic graphs with no external bisection have independent sets of size $n/2 - 1$.

We also make the following conjecture.

**Conjecture 3.** *Every cubic graph has an external partition* $(A, B)$ *with* $||A| - |B|| \leq 2$.

The Petersen graph shows that if true, this statement is tight.

There exist graphs other than $K_4$ all of whose external partitions are bisections. Every cubic graph of order 6 or 8 has this property, since an uneven external partition has at most a 3:2 proportion of the sides. There are arbitrarily large connected cubic graphs with no external bisection, as shown by the construction in Figure 5.

As mentioned, we believe that every cubic graph with a single bridge has an external bisection. We are presently unable to establish that, but following is a partial result in that direction. This is a procedure that takes a cubic graph with a bridge $G = (V, E)$ and outputs two smaller cubic graphs $G_1, G_2$. We denote this by $Split(G) = (G_1, G_2)$. The reader may find it useful to follow Figure 6 where this procedure is illustrated.

**Procedure 1.** *Start by deleting the two vertices of the bridge* $(b_1, b_2)$*. In each of the two components all vertices then have degree* 3*, except for two vertices of degree* 2*. The following is repeated in a loop for each component until a cubic graph remains:*

- *If the two degree-2 vertices are not adjacent, add an edge between them. This yields a cubic graph, and the procedure is terminated. Otherwise remove them both. The continuation depends on whether the two vertices share a neighbor.*
- *If the removed degree-2 vertices had a common neighbor (such as* $p_1, p_2$ *and their common neighbor* $p_3$*), delete that neighbor and its remaining neighbor (in the example:* $p_4$*). There remain exactly two vertices of degree 2* $(x_1, y_1)$*, and the loop is repeated.*
- *Otherwise (as in* $q_1, q_2$*) their additional neighbors* $(q_3, q_4)$ *are distinct. Again, exactly two vertices with degree 2 remain, and the loop is repeated.*

*The terminal components* $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ *are nonempty and cubic, since during the run of the procedure the component always has two vertices of degree 2. They each contain a single edge that is not in E, namely* $x_1y_1 \in E_1, x_2y_2 \in E_2$*.*

**Proposition 8.** *Let G be a cubic graph with a bridge, and let* $Split(G) = (G_1, G_2)$ *be its decomposition described in Procedure 1. If* $G_1$ *and* $G_2$ *are class-1 cubic graphs, then G has an external bisection.*

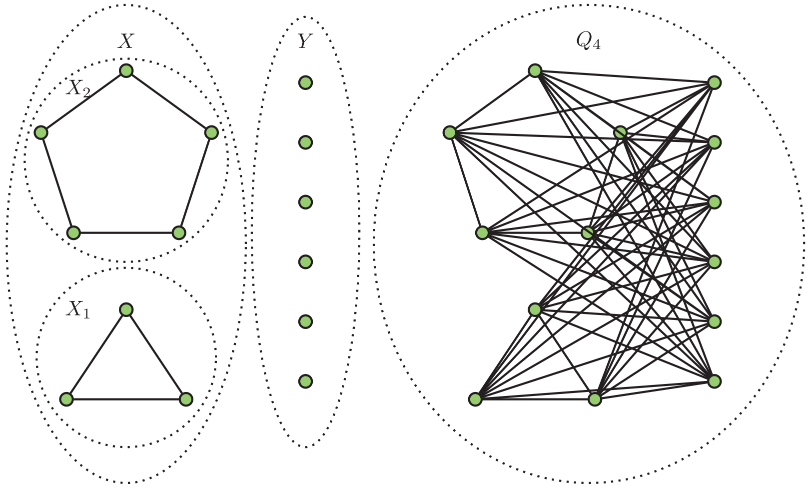


FIGURE 7. $Q_4$: 8-regular graph with no internal partition (right) is complete bipartite graph of subgraphs (left).

***Proof.*** $G$ has an external bisection, constructed as follows: Bisect the vertices in $V_1$ as in the proof of Proposition 7, taking care to choose the two colors other than $x_1y_1$'s color. This creates an external bisection of $G_1$ in which $x_1y_1$ may be removed and replaced by other edges without disturbing the fact that the partition is external. Similarly derive an external bisection of $G_2$, using two colors other than $x_2y_2$'s color. Finally assign the bridge vertices to different sides of the partition, and do the same with any nonbridge vertex pair that was deleted to obtain $G_1$ and $G_2$. The result is an external bisection of $G$. ■

This construction does not work if either $G_1$ or $G_2$ are class 2. For single-bridge cubic graphs, the case where either $G_1$ or $G_2$ are snarks therefore remains to be settled.

## 6. THE GENERAL CASE

The existence of internal partitions for $d$-regular graphs with $d = 5$ and with $7 \le d \le n - 5$ remains unsettled, as is the existence of $q$-internal partitions for $q \ne \frac{1}{2}$.

We construct a class of regular graphs $Q_m$, $m = 3, 4, \ldots$ without an internal partition, in which both $d$ and $n - d$ are unbounded. $Q_m$'s vertex set is composed of three disjoint sets $X_1, X_2$, and $Y$ (see Fig. 7), where $|X_1| = m - 1$, $|X_2| = m + 1$, $|Y| = m + 2$. The edge set is composed of (i) a clique on $X_1$, (ii) an arbitrary $(m - 2)$-regular graph on $X_2$, (iii) a complete bipartite graph between $X := X_1 \cup X_2$ and $Y$.

$Q_m$ is $2m$-regular with $3m + 2$ vertices. The first few such graphs are $Q_3(n = 11, d = 6)$, $Q_4(n = 14, d = 8)$, $Q_5(n = 17, d = 10), \ldots$

**Proposition 9.** *$Q_m$ has no internal partition.*

***Proof.*** Suppose to the contrary that $(A, B)$ is an internal partition of $Q_m$ with $|A| = a$ and $|B| = b$. In the complementary graph $\bar{Q}_m$, the set $Y$ spans a connected component that is a clique $K_{m+2}$. In the partition $(A, B)$ of $\bar{Q}_m$, each vertex in $A$ (respectively, $B$) has outdegree at least $b - m$ (respectively, $a - m$). The only way to partition $K_{m+2}$ to meet these requirements is to have $a - m$ of its vertices in $A$ and the other $b - m$ vertices in $B$.

Therefore $|V(X) \cap A| = |V(X) \cap B| = m$. Also for any $x \in (V(X) \cap A)$, $d_{V(X)\cap A}(x) \geq m - (a - m) = 2m - a$, and for any $x \in (V(X) \cap B)$, $d_{V(X)\cap B}(x) \geq m - (b - m) = 2m - b$. Therefore $(V(X) \cap A, V(X) \cap B)$ is either a trivial partition or a $q$-internal partition of $X$ for $q = \frac{2m-a}{m-2}$. Now $X_1$, being complete, has no $q$-internal partition for any $q$. Therefore its vertices are either all in $A$ or all in $B$. Say in $A$. Then $|V(X_2) \cap A| = m - |V(X_1)| = 1$, so there is a single $A$-vertex in the $X_2$ component, but a partition of a connected graph into a single vertex and its complement is not $q$-internal for any $q$. Neither is it trivial. A contradiction. ■

The reader will note that for all known examples $G$ of even-degree regular graphs with no internal partition, the complement $\bar{G}$ is disconnected. We raise the question of whether this is true in general. We observe that if true, this implies $2d > n$.[1] We venture a conjecture for this weaker statement.

**Conjecture 4.** *If the integer $d$ is even, then every $d$-regular graph with at least $2d$ vertices has an internal partition.*

This is clearly not true for $d$ odd, as shown by the complete bipartite graph $K_{d,d}$. Also, unpartitionable regular graphs with connected complements do exist, for example, with $d = 5, n = 18$.

We return to the problem of the existence of a $q$-internal partition for arbitrary regular graphs. There is a distinction between integral and nonintegral partitions. Nonintegral partitions are rarer than integral partitions, since every $q$-internal partition of a $d$-regular graph $G$ is also an integral $q'$-internal partition of $G$ for $q' = \lfloor qd \rfloor / d$ as well as for $q' = \lceil qd \rceil / d$. We make the following conjecture.

**Conjecture 5.** *For every integer $d$ and $1 > q > 0$ such that either (i) $q = \frac{1}{2}$ or (ii) $qd$ is an integer, there is an integer $\mu$ such that every $d$-regular graph of order $\geq \mu$ has a $q$-internal partition.*

As already noted, $\mu = 8$ for $d = 3, q = \frac{1}{2}$. Numerical experiments suggest that for $q = \frac{1}{2}$ and $d = 5, 7$ there holds $\mu = 18, 26$, respectively.

In fact, the following stronger statement appears to be true: There exists an integer $\mu'$ that depends only on $\delta(G)$, $\Delta(G)$ and on $q$ such that every graph $G = (V, E)$ with order at least $\mu'$ has a $q$-internal partition if (i) $q = \frac{1}{2}$ or (ii) $qd_G(v)$ is a positive integer for all $v \in V$.

We do not expect that there is a simple-to-state theorem covering the case where $q \neq \frac{1}{2}$ and $qd$ is nonintegral. For example, a connected $d$-regular graph has no $q$-internal partition for $0 < q < \frac{1}{d}$. On the other hand, for $\frac{1}{d} < q < \frac{2}{d}$, a shortest cycle and its complement often yield a $q$-internal partition (e.g., when the girth is $\geq 5$).

Although the above conjecture remains open, the following theorem shows that every incomplete graph has an integral $q$-internal partition for *some* $q$. Moreover, for $d$ fixed and growing $n$ the number of such distinct partitions is unbounded.

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[1] $2d = n$ can be discounted because it holds only for $K_{d,d}$, which, for even $d$, is partitionable.

**Proposition 10.** *Let $G$ be a $d$-regular graph of order $n > d + 1$. If $(A, B)$ is a non-integral $(1 - r)$-external partition of $\bar{G}$, then it is also a $q$-internal partition of $G$ for $q = (|A| - \lceil r(n - d - 1)\rceil)/d$.*

***Proof.*** $\bar{G}$ is $(n - d - 1)$-regular and $(A, B)$ is $(1 - r)$-external in $\bar{G}$. Therefore, in $G$, its indegrees satisfy the inequalities:

$$\forall x \in A, d_A(x) > |A| - 1 - r(n - d - 1) \tag{3}$$

$$\forall x \in B, d_B(x) > |B| - 1 - (1 - r)(n - d - 1). \tag{4}$$

Since the partition is not integral, $r(n - d - 1)$ and $(1 - r)(n - d - 1)$ are not integers, and the inequalities are strict. Consequently,

$$\forall x \in A, d_A(x) \geq |A| - 1 - \lfloor r(n - d - 1)\rfloor = |A| - \lceil r(n - d - 1)\rceil \tag{5}$$

$$\forall x \in B, d_B(x) \geq |B| - 1 - \lfloor (1 - r)(n - d - 1)\rfloor = |B| - \lceil (1 - r)(n - d - 1)\rceil. \tag{6}$$

Set $q = (|A| - \lceil r(n - d - 1)\rceil)/d$. By (5):

$$\forall x \in A, d_A(x) \geq qd. \tag{7}$$

Noting that $\lfloor (1 - r)(n - d - 1)\rfloor + \lceil r(n - d - 1)\rceil = n - d - 1$,

$$\begin{aligned} |B| - 1 - \lfloor (1 - r)(n - d - 1)\rfloor &= n - |A| - 1 - (n - d - 1) + \lceil r(n - d - 1)\rceil \\ &= (1 - q)d \end{aligned} \tag{8}$$

and by (6),

$$\forall x \in B, d_B(x) \geq (1 - q)d. \tag{9}$$

By (7), (9) $(A, B)$ is a $q$-internal partition of $G$. ■

**Theorem 2.** *A $d$-regular graph $G$ of order $n > d + 1$ has a $q$-internal partition $(A, B)$ for some $q \in (0, 1)$ with $qd$ an integer. There are at least $\frac{n-d-1}{d}$ such distinct partitions.*

***Proof.*** $\bar{G}$ is $(n - d - 1)$-regular. Select $r \in (0, 1)$ such that $r(n - d - 1)$ is not an integer (recall that $n - d - 1 \neq 0$). By Proposition 1, $\bar{G}$ has an $(1 - r)$-external partition $(A, B)$, and so by Proposition 10, $G$ has a $q$-internal partition for $q = (|A| - \lceil r(n - d - 1)\rceil)/d$.

Furthermore since $0 \leq q \leq 1$,

$$\lceil r(n - d - 1)\rceil \leq |A| \leq \lceil r(n - d - 1)\rceil + d. \tag{10}$$

So for any given $r$, $|A|$ has a range of at most $d$. Since $\lceil r(n - d - 1)\rceil$ can take on $n - d - 1$ values, $|A|$ takes on at least $\frac{n-d-1}{d}$ different values. Consequently, the number of distinct $q$-internal partitions is at least $\frac{n-d-1}{d}$. ■

For $d$ fixed there are just $d - 1$ values of $q \in (0, 1)$ for which $qd$ is integral. By Theorem 2 every $d$-regular graph has $\Omega(n)$ distinct integral $q$-internal partitions. While this does not prove the existence of a $q$-internal partition for any *specific* $q$, it suggests that this becomes more likely as $n$ grows.

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