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Note

The Equivalence of Two Problems on the Cube

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Denote by Q_n the graph of the hypercube $C^n = \{+1, -1\}^n$. The following two seemingly unrelated questions are equivalent: 1. Let G be an induced subgraph of Q_n such that $|V(G)| \neq 2^{n-1}$. Denote $\Delta(G) = \max_{x \in V(G)} \deg_G(x)$ and $\Gamma(G) = \max(\Delta(G), \Delta(Q_n - G))$. Can $\Gamma(G)$ be bounded from below by a function of n?; 2. Let $f: C^n \to \{+1, -1\}$ be a boolean function. The *sensitivity* of f at x, denoted $s(f, x)$, is the number of neighbors y of x in Q_n such that $f(x) \neq f(y)$. The sensitivity of f is $s(f) = \max_{x \in C^n} s(f, x)$. Denote by $d(f)$ the degree of the unique representation of f as a real multilinear polynomial on C^n. Can $d(f)$ be bounded from above by a function of $s(f)$?

1. PRELIMINARIES

Denote by Q_n the graph on the n-dimensional cube $C^n = \{+1, -1\}^n$, where any two vertices are adjacent iff they differ in exactly one component. For an induced subgraph G of Q_n, denote the *maximal degree* of G by $\Delta(G)$, i.e.,

$$\Delta(G) = \max_{x \in V(G)} \deg_G(x).$$

In [1], it was shown that if G contains more than 2^{n-1} vertices, then

$$\Delta(G) > \tfrac{1}{2}(\log n - \log\log n + 1)$$

and there exists a G such that

$$\Delta(G) < \sqrt{n} + 1.$$

The value $\sqrt{n}$ is also conjectured to be the correct order of magnitude for a lower bound on $\Delta(G)$. Denote $\Gamma(G) = \max(\Delta(G), \Delta(Q_n - G))$.

Let $f: C^n \to \{+1, -1\}$ be a boolean function. The *sensitivity* of f at x, denoted by $s(f, x)$, is the number of neighbors y of x for which $f(x) \neq f(y)$. The sensitivity of f is

$$s(f) = \max_{x \in C^n} s(f, x)$$

The sensitivity of f is sometimes called the *critical complexity* of f.

In theoretical computer science, much effort has been expended in the definition of various measures of complexity of boolean functions. Some are derived from an underlying computational model, such as *decision tree depth*. Here the function is computed by repeatedly reading input bits, until the function can be determined from the bits accessed. The *cost* of an algorithm is the number of bits read on the worst case input, and the complexity of a function is the cost of the best algorithm for this function. A similar measure is the *certificate complexity*. A 1-certificate (0-certificate) for f is an assignment to some subset of the variables that forces the value of f to 1 (0). The certificate complexity of f on x, denoted $C(f, x)$, is the size of the smallest certificate that agrees with x. The certificate complexity of f is

$$C(f) = \max_{x \in C^n} C(f, x).$$

Other measures of complexity are of a combinatorial nature, e.g., sensitivity. A related measure is *block-sensitivity*, defined: Denote $[n] = \{1, ..., n\}$ and let $R \subset [n]$. If x is the vector $(x_1, ..., x_n)$, then $x^{(R)}$ is defined as the vector with coordinates:

$$x_i^{(R)} = \begin{cases} x_i, & i \notin R \\ -x_i, & i \in R. \end{cases}$$

The block sensitivity of f at x, denoted $bs(f, x)$, is the largest number t such that there exist t *disjoint* sets $R_1, ..., R_t$ such that for all $1 \leqslant i \leqslant t$, $R_i \subset [n]$, and $f(x) \neq f(x^{(R_i)})$. The block-sensitivity of f is

$$bs(f) = \max_{x \in C^n} bs(f, x).$$

A central activity in this field is determining the relation between various

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measures. The measures of complexity s_1 and s_2 are *equivalent* if they are polynomially related; i.e., there exist polynomials $p_1(x)$ and $p_2(x)$ such that

$$\forall f, \quad s_1(f) \leqslant p_2(s_2(f)), \quad s_2(f) \leqslant p_1(s_1(f)).$$

Nisan [3] showed that decision tree depth, certificate complexity, and block-sensitivity are equivalent. Nisan first considered the more natural measure of sensitivity (which is block-sensitivity restricted to singletons), but was unable to prove equivalence to decision tree depth and certificate complexity. However, only after introducing block-sensitivity was equivalence obtained.

Yet another complexity measure is obtained from the unique representation of the boolean function f as a real multilinear polynomial over the cube:

$$f(x) = \sum_{I \subset [n]} \left[\alpha_I \prod_{i \in I} x_i \right].$$

The coefficient α_I (which satisfies $-1 \leqslant \alpha_I \leqslant 1$ for all $I \subset [n]$) is also called $\hat{f}(I)$, the *Fourier transform* of f at I. Denote by $d(f)$ the *degree* of this polynomial, i.e.,

$$d(f) = \max_{I \subset [n]} \{|I|: \alpha_I \neq 0\}.$$

Nisan and Szegedy [4] show that $d(f)$ is also equivalent to the three complexity measures mentioned above. As for the relation between sensitivity and degree, Szegedy [6] showed that

$$d(f) \geqslant \sqrt{s(f)}.$$

This can easily be shown to be tight. Whether $s(f)$ is also equivalent to all of the above is still unknown. In particular, an upper bound on $d(f)$ in terms of $s(f)$ is sought and is conjectured to be $s^2(f)$. Such a bound would mean that sensitivity is equivalent to all the previously mentioned quantities. In the next section we show that this upper bound is equivalent to a lower bound on $\Gamma(\ )$.

2. The Equivalence Theorem

Theorem 2.1. *The following are equivalent for any function $h: N \to R$:*

1. *For any induced subgraph G of Q_n such that $|V(G)| \neq 2^{n-1}$, $\Gamma(G) \geqslant h(n)$.*
2. *For any boolean function f, $d(f) < h^{-1}(s(f))$.*

Proof. We first transform 1 into a statement concerning boolean functions: Associate with the subgraph G a boolean function g such that $g(x) = 1$ iff $x \in V(G)$. Note that $\deg_G(x) = n - s(g, x)$ for $x \in V(G)$ and the same holds in $Q_n - G$ for $x \notin V(G)$. Denote by $\mathbf{E}(g)$ the average value of g on C^n. Now 1 and 2 are clearly equivalent to the following:

1′. For any boolean function g, $\mathbf{E}(g) \neq 0$ implies $\exists x: s(g, x) \leqslant n - h(n)$.

2′. For any boolean function f, $s(f) < h(n)$ implies $d(f) < n$.

To see the equivalence of 1′ and 2′, define

$$g(x) = f(x)\, p(x),$$

where $p(x)$ is the parity function of x: $p(x) = \prod_{i=1}^{n} x_i$. Note that for all $x \in C^n$, $s(g, x) = n - s(f, x)$ and for all $I \subset [n]$, $\hat{g}(I) = \hat{f}([n] - I)$, therefore $\mathbf{E}(g) = \hat{g}(\varnothing) = \hat{f}([n])$, where $\hat{f}([n])$ is the Fourier transform of f at $[n]$, i.e., the highest order coefficient in the representation of f as a polynomial.

$1' \to 2'$. Assume that $d(f) = n$, i.e., $\hat{f}([n]) \neq 0$. This is equivalent to $\mathbf{E}(g) \neq 0$. By 1′, $\exists x: s(g, x) \leqslant n - h(n)$; therefore $\exists x: s(f, x) \geqslant h(n)$, contradicting the premise.

$2' \to 1'$. Assume that $\forall x$, $s(g, x) > n - h(n)$. This implies that $s(f) < h(n)$. By 2′, $d(f) < n$, which is equivalent to $\hat{f}([n]) = \hat{g}(\varnothing) = \mathbf{E}(g) = 0$, contradicting the premise. ∎

3. Conclusion

Substituting $h(x) = \sqrt{x}$ in Theorem 2.1 shows that the two bounds $\Delta(G) \geqslant \sqrt{n}$ and $d(f) < s^2(f)$ are equivalent. The example from [1] which shows that there exists G such that $\Delta(G) \leqslant \sqrt{n} + 1$ can be used to show that the upper bound on $d(f)$ would be tight if it were true. All of this means that a proof of $\Delta(G) \geqslant \sqrt{n}$ would imply that boolean function sensitivity is equivalent to all other complexity measures mentioned in [3].

The sensitivity complexity measure $s(f)$ is especially important, since it also lower bounds $T(f)$—the time needed by a parallel RAM to compute f (a parallel RAM is a collection of synchronous parallel processors sharing a global memory with no write-conflicts allowed). Cook and Dwork [2] have shown that $T(f) \geqslant \log s(f)$. In fact, Nisan [3] later improved this to $T(f) \geqslant \log bs(f)$ (this is a stronger inequality, since for any f, $bs(f) \geqslant s(f)$). Simon [5] has also shown that a n-variable boolean function which depends on all its variables must have sensitivity at least $\Omega(\log n)$.