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Geodesic Geometry on Graphs

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Abstract

We investigate a graph theoretic analog of geodesic geometry. In a graph $G = (V, E)$ we consider a system of paths $\mathcal{P} = \{P_{u,v} : u, v \in V\}$ where $P_{u,v}$ connects vertices u and v. This system is *consistent* in that if vertices y, z are in $P_{u,v}$, then the subpath of $P_{u,v}$ between them coincides with $P_{y,z}$. A map $w : E \to (0, \infty)$ is said to *induce* $\mathcal{P}$ if for every $u, v \in V$ the path $P_{u,v}$ is w-geodesic. We say that G is *metrizable* if every consistent path system is induced by some such w. As we show, metrizable graphs are very rare, whereas there exist infinitely many 2-connected metrizable graphs.

Keywords Graph metrizability · Path systems · Shortest paths

Mathematics Subject Classification 05

1 Introduction

The idea of viewing graphs from a geometric perspective has been immensely fruitful. We refer the reader to Lovász' recent book [14] for a beautiful exposition of many of these success stories. The present paper adds a new connection between graph theory and differential geometry. Some themes of common interest between these two disciplines can be found in the literature. Thus, the study of graph eigenvalues has benefited considerably from the spectral theory of Laplacians on manifolds. It

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suffices to mention the connections between spectral gaps and expansion of graphs. Work in differential geometry and, in particular, Cheeger's work in the field, has been a major source of inspiration to graph theorists. Some of this story is surveyed in [12]. The study of graph *eigenvectors* should also be mentioned here, and while we know much less about them, many intriguing open questions remain, e.g., the study of *nodal domains* in graphs (e.g., [7]). The Laplacian is by no means the only differential operator of interest in graph theory. The heat equation and random walks on graphs constitute a major chapter in modern graph theory. There is also a growing body of work concerning *curvature* in graphs. An early and influential paper in this line is [15]. Here we study graphs from the perspective of *geodesic geometry*. Our main discovery is that for the vast majority of graphs the geodesic theory is way richer than the metric one. The relevant literature seems rather sparse. Ore [16] defines a graph to be *geodetic* if between any two vertices in it there is a unique shortest path. He sought to characterize geodetic graphs, and notwithstanding a considerable body of work on this problem (e.g., [1,3,4,17]) no definitive solution is in sight. Bodwin's beautiful recent work [2] is somewhat relevant, but as we explain below, his problems and results differ from ours.

In this work we introduce and study the notion of graph metrizability. A graph G is said to be *metrizable* if every consistent path system $\mathcal{P}$ in G is induced by a graph metric d, in that every path in $\mathcal{P}$ is d-geodetic. Graph metrics are defined in terms of positive edge weights. Here a path system is a collection of paths in G which contains a unique path $P_{u,v}$ between every pair of vertices u, v. Consistency means that if vertices y, z are in $P_{u,v}$, then the subpath of $P_{u,v}$ between them coincides with $P_{y,z}$. Strict metrizability means that d makes every $P_{u,v}$ the unique shortest u-v path.

Here are our main findings:

- Metrizability is rare: E.g., (i) every large 2-connected metrizable graph is planar, Theorem 6.2, (ii) no large 3-connected graph is metrizable, Corollary 6.10.
- However, arbitrarily large 2-connected metrizable graphs do exist: E.g., every outerplanar graph is metrizable, Corollary 7.5.
- We reveal some of the structural underpinnings of metrizability. The class of metrizable graphs is closed under the topological minor relation and is characterized by finitely many forbidden topological minors, Theorem 8.1.
- On the computational side, metrizability can be decided in polynomial time, Theorem 10.3.

Our main focus is on metrizability as a property of graphs. In contrast, Bodwin [2] investigates metrizability as a property of path systems. His main question is which *partial* path systems are strictly[1] induced by a metric. A partial path system is a collection of paths such that if the vertices u and v are in two of these paths, then their u-v subpaths must coincide. Bodwin has found an infinite family of intersection patterns such that a partial path system is strictly metrizable if and only if no such pattern occurs within the system. The difference between his work and ours goes deeper, since, as we show in Sect. 4, not every partial path system can be extended to

[1] In his terminology *strongly metrizable*.

a full path system. Moreover, it is possible for a graph to be strictly metrizable and yet contain a partial path system which is not strictly metrizable.

The Role of Computers in This Work

All of the results in this paper can be verified by hand, although this paper would not exist without our use of the computer. Although we were initially able to prove by hand that the Petersen graph (Fig. 1a) is non-metrizable, it quickly transpired that we needed a larger supply of such graphs. To this end we wrote a brute-force search program that found eleven such graphs (Fig. 22) and gave certificates that they are indeed non-metrizable. These certificates, see Appendix A, are easily verifiable by hand.

2 Definitions

Unless explicitly stated otherwise, paths that are mentioned throughout are *simple*. Let $G = (V, E)$ be a connected graph.

- A *path system* $\mathcal{P}$ in G is a collection of simple paths in G such that for every $u, v \in V$ there is exactly one member $P_{u,v} \in \mathcal{P}$ that connects between u and v.
- A *tree system* $\mathcal{T}$ in G is a collection of spanning trees in G such that for every $u \in V$ there is exactly one member $T_u \in \mathcal{T}$ which we think of as *rooted* at u.
- Let $\mathcal{P}$ be a path system in G. We say that it is *consistent* if for every $P \in \mathcal{P}$ and two vertices x, y in P, the x-y subpath of P coincides with $P_{x,y}$.
- Let $\mathcal{T}$ be a tree system in G. We say that it is *consistent* if for every two vertices $u, v \in V$ the u-v paths in T_u and in T_v are identical.

As we observe next, consistent path systems $\mathcal{P}$ and consistent tree systems $\mathcal{T}$ in the same graph G are in a simple one-to-one correspondence: Given $\mathcal{P}$, we define the tree T_u for every $u \in V$ via $E(T_u) := \bigcup_v E(P_{u,v})$. This is clearly a spanning subgraph of G and it is acyclic due to the consistency of $\mathcal{P}$. Given $\mathcal{T}$, we let $P_{u,v}$ be the v-u path in T_u, or, what is the same, the u-v path in T_v. This yields a consistent path system, because the path between any two vertices in a tree is unique. Therefore we can and will interchangeably talk of consistent path systems and consistent tree systems. Unless otherwise stated, all path systems and tree systems mentioned henceforth are assumed to be consistent. Given a weight function $w\colon E(G) \to \mathbb{R}$, we assign to every subgraph H of G the weight $w(H) = \sum_{e\in E(H)} w(e)$.

Definition 2.1 A path system $\mathcal{P}$ in $G = (V, E)$ is *induced* by $w\colon E(G) \to (0, \infty)$ if for each $u, v \in V$, $P_{u,v}$ is a u-v *geodesic*, i.e., $w(P_{u,v}) \le w(Q)$ for every u-v path Q. A path system that is induced by a positive weight function is said to be *metrizable*. A map $w\colon E(G) \to (0, \infty)$ *strictly induces* a path system $\mathcal{P}$ if for each $u, v \in V$, $P_{u,v}$ is the *unique* u-v geodesic, i.e., $w(P_{u,v}) < w(Q)$ for every u-v path $Q \ne P_{u,v}$. A path system that is strictly induced by some positive weight function is said to be *strictly metrizable*. A graph G is (*strictly*) *metrizable* if every path system in G is (strictly) metrizable.

Remark 2.2 It clearly suffices to consider connected graphs. In a disconnected graph, we deal with each connected component separately. In fact, it suffices to consider

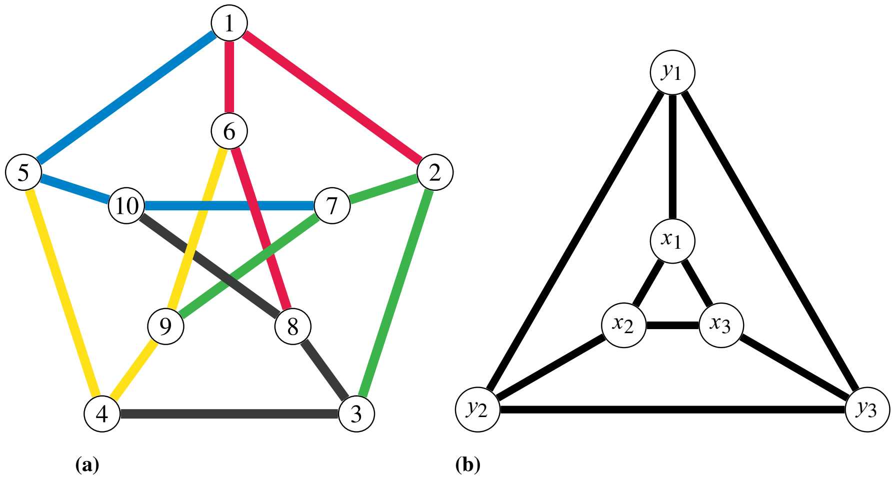


Fig. 1 **a** Non-metrizable path system in the Petersen graph. **b** A graph which has path system which is metrizable but not strictly metrizable

only 2-connected graphs. If vertex a separates a connected graph G, then $G \setminus \{a\}$ is the disjoint union of two (not necessarily connected) graphs H_1 and H_2, where $G_1 := H_1 \cup \{a\}$, $G_2 := H_2 \cup \{a\}$ are connected graphs, and any path system in G is uniquely defined by its restrictions to G_1, G_2. Indeed, if u and v belong to the same G_i, then any simple u-v path is contained in G_i, while if $u \in G_1$ and $v \in G_2$, then any u-v path is the concatenation of a u-a path and an a-v path.

3 Some Examples

Not all path systems are metrizable. Figure 1a exhibits a non-metrizable path system in the Petersen graph Π.

If $uv \in E(\Pi)$, then the path P_{uv} is comprised of the single edge uv. Between any two nonadjacent vertices $x, y \in V(\Pi)$ there is a unique path of length 2. For most such pairs this is taken to be $P_{x,y}$. There are five exceptional pairs of nonadjacent vertices, those which are connected by a colored path in Fig. 1a. For example, $P_{1,7} = 1, 5, 10, 7$. It is easily verified that this path system is consistent, and as we show next, this path system is nonmetrizable. If w is a weight function that induces it, then by considering the colored paths, the following inequalities must hold:

$$
\begin{aligned}
w_{1,2} + w_{1,6} + w_{6,8} &\leq w_{2,3} + w_{3,8},\\
w_{2,3} + w_{2,7} + w_{7,9} &\leq w_{3,4} + w_{4,9},\\
w_{3,4} + w_{3,8} + w_{8,10} &\leq w_{4,5} + w_{5,10},\\
w_{4,5} + w_{4,9} + w_{6,9} &\leq w_{1,5} + w_{1,6},\\
w_{1,5} + w_{5,10} + w_{7,10} &\leq w_{1,2} + w_{2,7},
\end{aligned}
$$

which implies

$$w_{6,8} + w_{7,9} + w_{8,10} + w_{6,9} + w_{7,10} \leq 0,$$

showing a weight function inducing these paths cannot be strictly positive.

Figure 1b shows a metrizable path system which is not strictly metrizable. Namely, every edge is the chosen path between its two vertices. For $i = 1, 2, 3$, let $P_{x_i, y_{i+1}} = x_i y_i y_{i+1}$ and $P_{y_i, x_{i+1}} = y_i x_i x_{i+1}$, with indices taken mod 3. It is easy to see that the constant weight function induces this path system. If a weight function w strictly induces this system, then for $i = 1, 2, 3$ the following inequalities must hold:

$$w(x_i y_i) + w(y_i y_{i+1}) < w(x_i x_{i+1}) + w(x_{i+1i} y_{i+1}) \quad \text{and}$$
$$w(y_i x_i) + w(x_i x_{i+1}) < w(y_i y_{i+1}) + w(y_{i+1i} x_{i+1}).$$

Summing the first inequality for $i = 1, 2, 3$ and canceling identical terms yields

$$\sum_{i=1}^{3} w(y_i y_{i+1}) < \sum_{i=1}^{3} w(x_i x_{i+1}).$$

Similarly, adding up the second inequality gives

$$\sum_{i=1}^{3} w(x_i x_{i+1}) < \sum_{i=1}^{3} w(y_i y_{i+1}),$$

a contradiction.

4 Basic Properties of Path and Tree Systems

Before getting into graph metrizability we establish some basic properties of path and tree systems. The notion of consistency makes sense also for a partial system of paths and we ask when a consistent system of paths can be extended to a full consistent path system. We give an example of a consistent partial collection of paths for which this is impossible. We then go on to prove some conditions under which the answer is positive. Next we establish certain lemmas which help us better elucidate the structure of path systems. Aside of the inherent interest in these lemmas, they help us to fully describe path systems in cycles. Path systems in cycles play a key role in the study of metrizability (Theorem 7.2).

We start with the following easy observation.

Proposition 4.1 *Let $G = (V, E)$ be a connected graph. Then every weight function $w\colon E \to (0, \infty)$ induces a consistent path system on G.*

Proof If the w-shortest u-v path is unique, then there is nothing to prove. What we need is a rule to break ties between u-v paths of the same w length. To this end fix

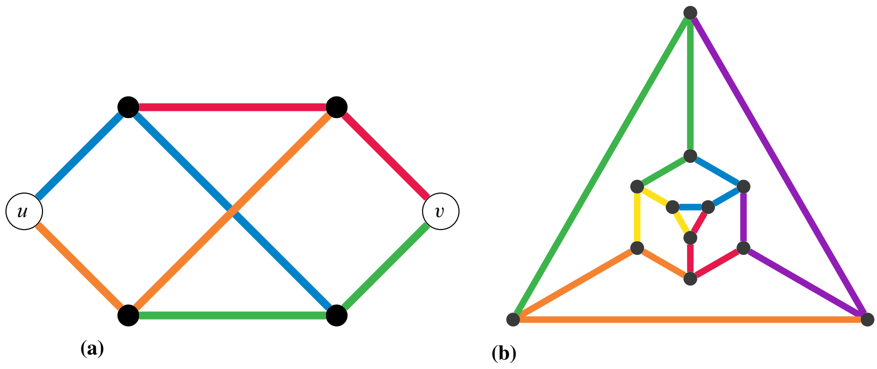


Fig. 2 **a** A partial path system which cannot be extended to a full system. **b** Another partial path system which cannot be extended to a full system

some ordering $e_1, e_2, \ldots$ on the edges of G and break ties between two such paths by comparing their edge sets lexicographically. It is easy to see that this guarantees consistency.

A *partial* path system Π in a connected graph $G = (V, E)$ is a collection of paths between *some* pairs $u, v \in V$. We say Π is a consistent partial path system, if for all paths $P, Q \in \Pi$ and vertices $u, v \in V(P) \cap V(Q)$, the u-v subpaths of P and Q coincide. As Fig. 2b shows, not every consistent partial path system can be extended to a full consistent path system. Here, the partial path system consists of the colored paths. It easy to see that the addition of any u-v path to this system makes it inconsistent. Another example of a non-extendable partial path system is given by Fig. 2b. Here again the colored paths form our partial path system. It is not difficult to see that any path which connects the two triangles in this graph is inconsistent with this partial system.

A partial path system Π is said to be *strictly metrizable* if there exists a weight function w such every path in Π is the unique geodesic w.r.t. w between its end vertices. We observe:

Observation 4.2 *A consistent partial path system* Π *that cannot be extended to a full consistent path system is not strictly metrizable.*

Proof Suppose that Π is strictly induced by some weight function w. By Proposition 4.1 w induces some full path system $\mathcal{P}$. Moreover, necessarily $\Pi \subseteq \mathcal{P}$ since every path in Π is assumed to be a unique shortest path w.r.t. w. It follows that Π can be extended to a full system.

We take this opportunity to explain how our work differs from Bodwin's. Although the basic concepts may seem similar, the difference is substantial. Thus, a path system in the language of [2], is what we call a partial path system. By Observation 4.2 the partial path system in Fig. 2a is not strictly metrizable. Therefore, from the point of view of [2] (e.g., his Fig. 6) Fig. 2a shows a path system which is not strictly metrizable. From our perspective, since we only consider full path systems, the graph in Fig. 2a is in fact an example of a strictly metrizable graph. (We use a computer to prove this claim. For more on this, see Sect. 10.1.) Bodwin lays out an infinite family of intersection patterns and proves that a consistent partial path system is strictly metrizable if and only if it does not contain any of these patterns. The above example illustrates that a graph may admit one of these intersection patterns, and still be strictly metrizable according to our definitions.

We discuss next some cases where an extension of a partial path system is possible. A path system is called *neighborly* if the path between any two adjacent vertices is the edge between them.

Proposition 4.3 *Let $\mathcal{P}'$ be a consistent neighborly path system in $G' = (V', E')$, an induced subgraph of $G = (V, E)$. Then $\mathcal{P}'$ can be extended to a consistent neighborly path system in G.*

Proof We may and will assume that G is connected. Also, by induction it suffices to consider the case where $G' = G \setminus \{v\}$ for some $v \in V$. Of course, we only need to specify the paths $P_{x,v}$ for all $x \neq v$.

Let us start with the case where G' is connected. We set $P_{v,u} := vu$ for every neighbor u of v. Now fix some $u_0 \in N_G(v)$ and set the paths $P_{x,v}$ for $x \notin N_G(v)$ according to the following intuitive rule: Seek a way from x to v via u_0, but if this path visits another neighbor of v, then hop to v as early as possible. More formally, $P_{x,v} := P'_{x,u_x} v$ where u_x is the first neighbor of v on the path P'_{x,u_0}. Since $\mathcal{P}'$ is consistent, it suffices to show that $P_{y,v}$ is a subpath of $P_{x,v}$ whenever $y \in P_{x,v}$. By construction $P_{x,v} = P'_{x,u_x} v$ and therefore $y \in P'_{x,u_x}$. But P'_{y,u_x} is a subpath of P'_{y,u_0}, since $\mathcal{P}'$ is consistent. It follows that $u_x = u_y$ is the first neighbor of v along P'_{y,u_0}. Therefore $P_{y,v} = P_{y,u_x} v$ is a subpath of $P_{x,v}$, as claimed.

If G' is disconnected, then v is a cut-vertex. We apply the above procedure to every component $C = (V_C, E_C)$ of G' with $V = V_C \cup \{v\}$ and $V' = V_C$, yielding a consistent system of paths $\mathcal{P}_C$ for the graph $G[V_C \cup \{v\}]$. For vertices u and w are in two distinct components C_1 and C_2, we let $P_{u,w} = P_{u,v} P_{v,w}$ with $P_{u,v} \in \mathcal{P}_{C_1}$, $P_{v,w} \in \mathcal{P}_{C_2}$ to determine a consistent system in G.

We turn our attention to tree systems and draw significant structural consequences from the assumption that two trees in the system coincide.

Lemma 4.4 *Let $\mathcal{T}$ be a tree system in a graph $G = (V, E)$, and $\mathcal{P}$ its corresponding path system. For $u, v \in V$, the following are equivalent:*

(1) $T_u = T_v$.
(2) *Every tree $T_w \in \mathcal{T}$ contains the path $P_{u,v}$.*
(3) $T_z = T_u$ *for every* $z \in P_{u,v}$.

Proof We recall the following standard fact: Every three (not necessarily distinct) vertices a, b, c in a tree T have a *median*. This is a vertex μ which is uniquely characterized by the property that every vertex other than μ is in at most one of the paths

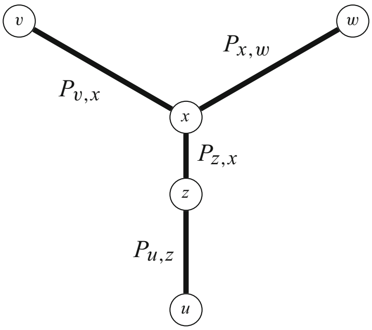


Fig. 3 The paths $P_{v,x}$, $P_{x,w}$, and $P_{u,x}$ intersect only at x, the median of v, u, and w in T_w

μ-a, μ-b, μ-c in T. Recall that for any two vertices $\gamma, \delta \in V$ the γ-δ path in T_γ is $P_{\gamma,\delta}$.

(3)$\Rightarrow$(1). This is obvious.

(1)$\Rightarrow$(2). $P_{u,v}$ is clearly contained in $T = T_u = T_v$. We wish to show that it is contained in T_w for every $w \neq u, v$. Let x be the median in T of u, v, w. Note that $P_{u,w} = P_{u,x}P_{x,w}$ and $P_{v,w} = P_{v,x}P_{x,w}$ are both paths in T_w. Together with the fact that $T_u = T_v$ this implies that the paths $P_{u,x}$, $P_{v,x}$ are contained in T_u, T_v and T_w. Therefore, $P_{u,v}$, the u-v path in $T_u = T_v$, is precisely the path $P_{u,x}P_{x,v}$. It follows $P_{u,v} = P_{u,x}P_{x,v}$ is contained in T_w as well.

(2)$\Rightarrow$(3). When are two trees on the same vertex set V (in our case T_z and T_u) identical? Fix some $\alpha \in V$. Then two trees on V are identical iff the α-β paths in both trees coincide for every $\beta \in V$. We use this criterion with $\alpha = z$ and show that for every $w \in V$ the z-w paths in T_z and T_u are identical. When $w = u$ this is clearly true as the z-u path in both trees is a u-z subpath of $P_{u,v}$. The situation is similar when $w = v$. For other vertices w, by assumption, the path $P_{u,v}$ is contained in T_w. Therefore, the w-u path in T_w is $P_{w,u} = P_{w,x}P_{x,u}$, where x is the median in T_w of the vertices u, v, w, see Fig. 3. Likewise the w-v path in T_w is $P_{w,v} = P_{w,x}P_{x,v}$. Since $x \in P_{u,v}$ it follows that $P_{u,v} = P_{u,x}P_{x,v}$. Also $z \in P_{u,v}$ so z must belong to either $P_{u,x}$ or $P_{x,v}$ (or both, when $z = x$). In either case it we have that the w-z path in T_z is $P_{w,z} = P_{w,x}P_{x,z}$. Notice that $P_{w,x}$, a subpath of $P_{w,u}$, and $P_{x,z}$, a subpath of $P_{u,v}$, are both contained in T_u. It follows that $P_{w,z} = P_{w,x}P_{x,z}$ is also the w-z path in T_u. □

An edge e is said to be $\mathcal{T}$-*persistent* if it belongs to every tree in the tree system $\mathcal{T}$. It is $\mathcal{P}$-persistent for a path system $\mathcal{P}$ if for every vertex u there is a vertex v with $e \in P_{uv}$. It is easily verified that for a corresponding pair $\mathcal{T}$ and $\mathcal{P}$, as described above, the two conditions are equivalent. We sometimes say such an edge is persistent if $\mathcal{P}$ or $\mathcal{T}$ is clear from context. We denote by G/e the graph obtained from G by contracting the edge e.

Proposition 4.5 *Let $G = (V, E)$ be a connected graph, $\mathcal{T}$ be a consistent tree system in G and e a $\mathcal{T}$-persistent edge. Then $\mathcal{T}/e := \{T/e : T \in \mathcal{T}\}$ is a consistent tree system in G/e.*

Proof Trees are closed under edge contraction, and so $\mathcal{T}/e$ is a collection of trees. Write $e = uv$ and let z be the vertex obtained from contracting uv. By Lemma 4.4, $T_u = T_v$ so that $T_u/e = T_v/e$. For $w \in V \setminus \{u, v\}$ the tree rooted at w is then T_w/e and the tree rooted at z is $T_u/e = T_v/e$. The consistency of such a system follows from the consistency of $\mathcal{T}$.

Similarly, if $\mathcal{P}$ be a consistent path system in G and e is a $\mathcal{P}$-persistent edge then $\mathcal{P}/e := \{P/e : P \in \mathcal{P}\}$ is a consistent path system in G/e. We remark that the assumption in Proposition 4.5 that e is a persistent edge is crucial. In general, if e is not persistent then $\mathcal{P}/e$ is not a path system. In this case, there may be multiple paths in $\mathcal{P}/e$ connecting two vertices in G/e.

Lemma 4.6 *Let $\mathcal{P}$ be a path system in a connected graph $G = (V, E)$, and let $e \in E$ be a $\mathcal{P}$-persistent edge. If $\mathcal{P}/e$ is strictly metrizable then so is $\mathcal{P}$.*

Proof Let $\tilde{w}\colon E/e \to (0, \infty)$ be a weight function that strictly induces $\mathcal{P}/e$. Fix some small $\delta > 0$ and large $N > 0$. Define $w\colon E \to (0, \infty)$ by

$$w(e') := \begin{cases} \tilde{w}(e'/e) & e' \neq e \text{ participates in some paths of } \mathcal{P}, \\ N & e' \neq e \text{ does not participate in any path of } \mathcal{P}, \\ \delta & e' = e. \end{cases}$$

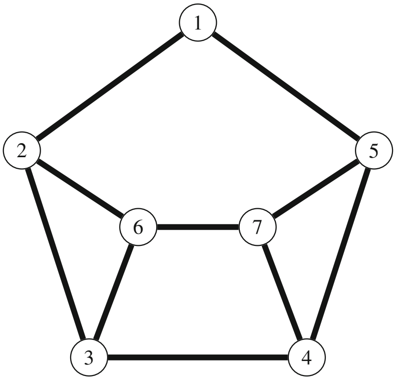
We argue that w strictly induces $\mathcal{P}$. Let $P_{u,v} \in \mathcal{P}$ and Q some other u-v path. We first consider the case where $P_{u,v}/e \neq Q/e$. By construction of $\mathcal{P}/e$, $P_{u,v}/e \in \mathcal{P}/e$, so that $\tilde{w}(Q/e) - \tilde{w}(P/e) > 0$. We take $\delta > 0$ to be small enough so that $w(Q) - w(P) \geq \tilde{w}(Q/e) - (\tilde{w}(P/e) + \delta) > 0$.

Now suppose $P_{u,v}/e = Q/e$. We argue that Q contains an edge not contained in $\mathcal{P}$. Indeed, first note that since $P_{u,v}$ and Q are distinct u-v paths the set $E(P_{u,v}) \triangle E(Q)$ contains a cycle. Moreover, all the edges in the set $E(P_{u,v}) \triangle E(Q)$ must be incident to the edge e. To see this note that if $e', e'' \in E$ are edges which are not incident to e then $e'/e = e''/e$ implies $e = e''$. Since $P_{u,v}/e = Q/e$, it follows that for any edge e' not incident with e, $e' \in P_{u,v}$ iff $e' \in Q$. Let $\mathcal{T}$ be the tree system corresponding to $\mathcal{P}$ and write $e = xy$. If $\mathcal{P}$ contains all the edges in $E(P_{u,v}) \triangle E(Q)$ then these edges are either in T_x or T_y. Since e is a persistent edge, $T_x = T_y$, implying T_x contains a cycle, a contradiction. Therefore Q contains an edge not in $\mathcal{P}$ and $w(Q) - w(P_{u,v}) \geq N - w(P_{u,v}) > 0$.

We note that the condition of strict metrizability cannot be removed from the statement of Proposition 4.6, see Fig. 4.

Let F be the set of $\mathcal{P}$-persistent edges. It is possible to contract the edges in F either sequentially or in parallel. It is easy to check that the resulting path system $\mathcal{P}/F$ is *reduced*, i.e., it is a path system with no persistent edges. If every edge in $\mathcal{P}$ is persistent then all trees in the corresponding tree system are identical, and $\mathcal{P}$ is the path system of a tree. We call such a path system *trivial*, and observe that this happens if and only if G/F is a single vertex.

Trees have a single trivial path system, and so the simplest non-trivial graphs for us are cycles. We turn to investigate path systems in cycles. A spanning tree T in the



(12), (123), (1234), (15), (126), (157),

(23), (234), (215), (26), (2157), (34), (345),

(36), (367), (45), (476), (47), (5126), (57), (67)

Fig. 4 This non-metrizable path system becomes metrizable by contracting the persistent edge 12

n-cycle C_n has the form $E(C_n)\setminus\{e\}$ for some edge $e \in E(C_n)$. Therefore a tree system in C_n is completely specified by a map $f: V(C_n) \to E(C_n)$ where $T_v = G \setminus f(v)$ for all $v \in V$. The following lemma says for which $f: V \to E$ the resulting path system is consistent:

Lemma 4.7 *Let $C_n = (V, E)$, $n \geq 3$. A mapping $f: V \to E$ defines a consistent path system in C_n if and only if for each $x, y \in C_n$ either $f(x) = f(y)$, or x and $f(x)$ separate y from $f(y)$.*

A function which satisfies the conditions in Lemma 4.7 is called a *crossing* function. A look at Fig. 5 shows why it is called the crossing condition. This is further elaborated in Sect. 9.

Proof Suppose that f defines a consistent system $\mathcal{P}, \mathcal{T}$ as above. If $f(v) = wz$, the the spanning tree T_v is comprised of two paths $P_{v,w}, P_{v,z}$ whose intersection is the vertex v. Thus any $x \neq v$ is in either $P_{v,z}$ or $P_{v,w}$. Suppose the latter and moreover $f(x) \neq wz$. We wish to show that $f(x) \notin P_{v,w}$ which implies that v and $f(v)$ separate x from $f(x)$. But $P_{v,w} = P_{v,x} \cup P_{x,w}$ and by the consistency of $\mathcal{P}$ both $P_{v,x}$ and $P_{x,w}$ are paths in T_x. However, if $f(x) \in P_{v,w}$ then $f(x) \in T_x$, contrary to the definition of f and we conclude that f is indeed crossing.

Now we show that when f is crossing the u-v path in T_v and the u-v path in T_u coincide for all $u, v \in C_n$. As before $T_v = C_n \setminus f(v)$ is the union of two paths P_1, P_2 that share v as an endpoint. The u-v path of T_v is contained in either P_1 or P_2, say $u \in P_1$. By the crossing condition $f(u) \notin P_1$ and we see that P_1 is a subpath of T_u implying the u-v paths of T_u and T_v coincide. □

For odd n the n-cycle C_n is a geodetic graph. We denote by $\mathcal{S}_n$ the geodesics in C_n w.r.t. unit weight edges and note that $\mathcal{S}_n$ is a consistent path system. We show that these are essentially the only consistent path systems of cycles.

Proposition 4.8 *Let $\mathcal{P}$ be a path system in the cycle C_n, $n \geq 3$, and let $F \subseteq E(C_n)$ be the set of all $\mathcal{P}$-persistent edges. Then either $\mathcal{P}$ is trivial or $\mathcal{P}/F = \mathcal{S}_m$, for some odd $3 \leq m \leq n$.*

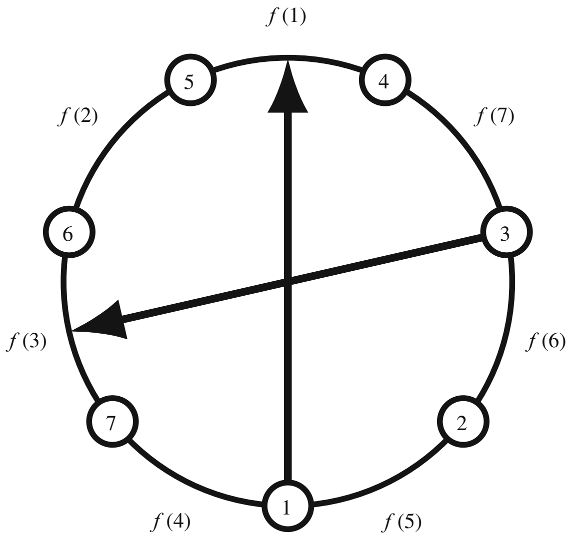


Fig. 5 A crossing function f in the 7-cycle

Proof Let $\mathcal{T}$ be the tree system corresponding to $\mathcal{P}$ and $f: V \to E$ its crossing function. By Lemma 4.4 an edge e is persistent iff it belongs to every tree in $\mathcal{T}$, i.e., the set of persistent edges of $\mathcal{T}$ is precisely $F = E(C_n) \setminus \operatorname{Im} f$. But $|V(C_n)| = |E(C_n)| = n$, so $\mathcal{P}$ is reduced if and only if f is a bijection. If $\mathcal{P}$ is not trivial, the graph C_n/F is a cycle and $\mathcal{P}/F$ is a reduced path system over C_n/F.

We now argue if $\mathcal{P}$ is a reduced system then $\mathcal{P} = \mathcal{S}_n$. As before, T_v is the union of the paths $P_{v,w}$ and $P_{v,z}$ with $V(P_{v,w}) \cap V(P_{v,z}) = \{v\}$, where $f(v) = wz$. Consider $x \in P_{v,w}$, $v \neq x$. As $\mathcal{P}$ is reduced, f is a bijection and therefore $f(x) \neq f(v)$, and by the crossing condition $f(x) \in P_{v,z}$. This gives an injection from $V(P_{v,w}) \setminus \{v\}$ into $E(P_{v,z})$. By symmetry there is also an injection from $V(P_{v,z}) \setminus \{v\}$ into $E(P_{v,w})$. This implies $|E(P_{v,w})| = |E(P_{v,z})| = k$ and $n = 2k + 1$. Moreover, for each $v \in C_n$ the edge $f(v)$ is the the edge antipodal to v in C_n. This is precisely the path system $\mathcal{S}_n$.

5 Metrizable Graphs

In this section we seek to improve our understanding of (strictly) metrizable graphs. Recall that a graph H is a topological minor of a graph G if G contains a subgraph which is a subdivision of H. For more on this, see Sects. 7 and 8.

Proposition 5.1 *A topological minor of a (strictly) metrizable graph $G = (V, E)$ is (strictly) metrizable, respectively.*

Proof We start with edge removals. Let G' be a graph obtained by removing an edge from G. By Remark 2.2, we can assume that G is 2-connected, whence G' is connected. Every path system in G' is also a path system of G, and since G is metrizable, so is G'. This applies verbatim to the strict case as well.

Next we consider vertex removals. Let G' be the graph obtained by removing the vertex v from G and let G'' be the graph obtained from G by removing all edges incident to v. By why what just shown, G'' is metrizable, and in particular, every connected component of G'' is metrizable. It follows that the graph G', which consists of all the components of G'' other than v, is also metrizable. The same argument applies to the strict case.

Consider next what happens when we suppress a vertex z of degree 2 in G. Let u, v be the two neighbors of z. We can assume w.l.o.g. that $uv \notin E$, or else we first delete this edge from G to obtain another metrizable graph. Let G' be the graph obtained from G by suppressing the vertex z. Given any path system $\mathcal{P}'$ in G', we construct the following path system $\mathcal{P}$ in G. For $x, y \in V(G) \setminus \{z\}$, if $P'_{x,y}$ does not contain the edge uv then $P_{x,y} := P'_{x,y}$. Otherwise, and if $P'_{x,y} = P'_{x,u}uvP'_{v,y}$, then $P_{x,y} := P'_{x,u}uzvP'_{v,y}$. Ditto when $P'_{x,y} = P'_{x,v}vuP'_{u,y}$. Next we need to define the paths $P_{z,w}$ for all $w \neq z$. If $P'_{u,w}$ does not contain the vertex v, set $P_{z,w} = zP'_{u,w}$. Otherwise, set $P_{z,w} := zP'_{v,w}$.

Note that $\mathcal{P}$ contains exactly one path between any two vertices, and we turn to show its consistency, i.e., that for every $P \in \mathcal{P}$ and every two vertices $x, y \in P$ the subpath of P between x and y is also a member of $\mathcal{P}$. Since $\mathcal{P}'$ is consistent, it suffices to prove this when $z \in P$, i.e., when $P = P_1 z P_2$ where P_1 is either empty or a path in $\mathcal{P}'$ that ends in u, and P_2 is either empty or a path in $\mathcal{P}'$ starting with v. If $x, y \in P_1$ or $x, y \in P_2$ then the consistency of $\mathcal{P}'$ implies that the x-y subpath of P is in $\mathcal{P}$. If $x \in P_1$, $y \in P_2$, then both P_1 and P_2 are non-empty and the path $P'_{x,y}$ contains the edge uv. By construction $P_{x,y} = P'_{x,u}zP'_{v,y}$ and $P_{x,y}$ is a subpath of P. If $x \in P_1$ then $P'_{u,x}$ is a subpath of P_1 not containing v so that $P_{z,x} = zP'_{u,x}$ is a subpath of P. Finally if $x \in P_2$ this means that the path $P'_{u,x} = P'_{u,v}P'_{v,x}$ contains v and therefore $P_{z,x} = zP'_{v,x}$ is again a subpath of P. Since G is metrizable, there exists a weight function $w\colon E(G) \to (0, \infty)$ inducing $\mathcal{P}$. The weight function $w'\colon E(G') \to (0, \infty)$ defined by $w'(uv) = w(uz) + w(zv)$ and $w'(e) = w(e)$, $e \neq uv$, induces $\mathcal{P}'$. Indeed, let $P' \in \mathcal{P}'$ be an x-y path and Q another x-y path. If P' contains the edge uv, then we can write $P' = R_1 uv R_2$, for some subpaths R_1 and R_2 of P', and we set $\tilde{P}' = R_1 uzv R_2$. If $uv \notin P'$ we set $\tilde{P}' = P'$. We define $\tilde{Q}$ similarly. Notice that by construction of $\mathcal{P}$, $\tilde{P}' \in \mathcal{P}$. Since w induces $\mathcal{P}$, by definition of w' it follows, $w'(P') = w(\tilde{P}') \leq w(\tilde{Q}) = w'(Q)$. Moreover, if the original path system is strictly metrizable then $w'(P') = w(\tilde{P}') < w(\tilde{Q}) = w'(Q)$. ◻

A subgraph is also a topological minor, so

Corollary 5.2 *If a graph G contains a subdivision of a non-metrizable graph then G is non-metrizable. In particular, if G has non-metrizable subgraph, then it is non-metrizable.*

While the property of graph metrizability is closed under topological minors it is not closed under minors. In general, the graph obtained by contracting an edge of a metrizable graph is not necessarily metrizable. An example of this is given by Fig. 6.

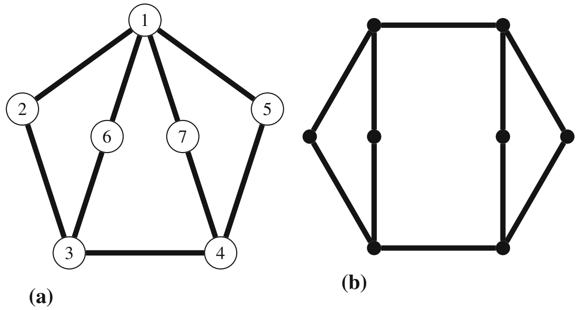


Fig. 6 Contracting an edge of the metrizable graph in **b** yields the non-metrizable graph in **a**

Here is a non-metrizable path system in the graph Fig. 6a. Paths $P_{u,v}$ with $u < v$ are listed in lexicographic order:

$$(12);\ (163);\ (174);\ (15);\ (16);\ (17);\ (23);\ (234);\ (2345);\ (216);\ (217);\ (34);\ (345);\ (36);\ (347);\ (45);\ (436);\ (47);\ (5436);\ (517);\ (6347).$$

A weight function w that induces this system must satisfy

$$w_{2,3} + w_{3,4} + w_{4,5} \le w_{1,2} + w_{1,5}, \qquad w_{1,2} + w_{1,6} \le w_{2,3} + w_{3,6},$$
$$w_{1,5} + w_{1,7} \le w_{4,5} + w_{4,7}, \qquad w_{3,6} + w_{3,4} + w_{4,7} \le w_{1,6} + w_{1,7}.$$

Adding these inequalities and canceling terms yields $w_{3,4} \le 0$. At present, we can only verify that Fig. 6b is metrizable by using a computer program to check that all of the possible path systems in this graph are metrizable. For more on this, see Sect. 10.1.

6 Metrizable Graphs are Rare

By Proposition 5.1 metrizability can be characterized by a set of forbidden topological minors. Of course it suffices to consider the minimal such forbidden graphs. As we will see (Theorem 8.1), there are only finitely many such graphs. With the help of a computer we found several such graphs, Fig. 22, but we suspect that this list is not exhaustive. For a discussion on how these graphs were found see Sect. 10.1. Proposition 5.1 and the graphs in Fig. 22 already imply that metrizability is a rare property of graphs.

First, we clarify on some notation used throughout this section. Given a graph $G = (V, E)$ and disjoint subsets $A, B \subseteq V$, an A-B path is a path which connects A and B but whose internal vertices are contained in $V \setminus (A \cup B)$. Similarly, if H and K are subgraphs of G then an H-K path is a path in G which connects H and K but whose internal vertices are contained in $V \setminus (V(H) \cup V(K))$.

Theorem 6.1 *Every* 4*-connected metrizable graph has at most six vertices.*

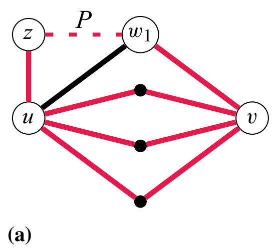


Fig. 7 **a** The red edges form a subdivision of Fig. 22a. **b** The graph $K_{4,4}$ contains a subdivision of Fig. 22c. **c** This graph is a copy of Fig. 22c

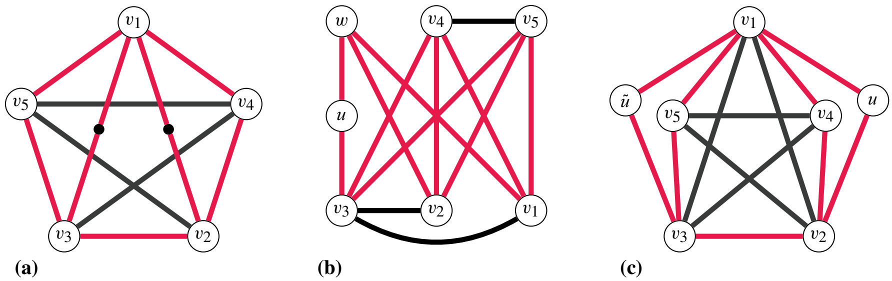


Fig. 8 **a** Subdividing two incident edges of K_5 yields a subdivision of Fig. 22b. **b** This graph is not metrizable since it contains a Fig. 22c subgraph. **c** The red edges form a copy of Fig. 22b

Proof Let G be a 4-connected graph with at least seven vertices. In view of Corollary 5.2 is suffices to show that G contains a subgraph or subdivision of a non-metrizable graph. Since there exist non-metrizable graphs on seven vertices, K_n is metrizable only if $n \leq 6$. Therefore, it suffices to consider the case where G contains a pair of non-adjacent vertices, say u and v. There exist at least four disjoint paths of length at least 2 connecting u and v. If any of these paths has length at least 3, then G is not metrizable, since it contains Fig. 22a as a topological minor. Therefore, u and v have at least four common neighbors, say $A = \{w_1, w_2, w_3, w_4\}$ is a set of four common neighbors. Suppose that u has a neighbor $z \notin A$. By the fan lemma there exist four internally disjoint paths from z to A. At least two of these paths contain neither u nor v. Say P is a z-w_1 path with this property, then the paths $uzPw_1v$, uw_2v, uw_3v, uw_4v are four disjoint u-v paths, Fig. 7a. Therefore G is not metrizable, since it contains Fig. 22a as a topological minor. There remains the case where $N(u) = A$. Every vertex $z \notin A \cup \{u, v\}$ has an A-fan. As argued above this fan must simply be comprised of four edges, i.e., $N(z) \supseteq A$. If G contains another vertex y then again $N(y) \supseteq A$ and G has a $K_{4,4}$ subgraph. But $K_{4,4}$ is not metrizable as it contains a subdivision of Fig. 22c, Fig. 7b. What if G has only seven vertices? Since G is 4-connected, the set $\{u, v, z\}$ cannot disconnect it, and there must be edges within A, say, $w_1 w_2 \in E$. But then G contains Fig. 22c as a subgraph, Fig. 7c, and is therefore not metrizable. □

Next we show most non-planar graphs are not metrizable.

Theorem 6.2 *Every metrizable* 2*-connected non-planar graph has at most seven vertices.*

Proof Let G be a 2-connected non-planar graph with at least eight vertices. By Kuratowski's theorem G contains a subgraph H that is a subdivision of either K_5 or $K_{3,3}$. Consider first the case where H is a subdivision of K_5. Let v_1, v_2, v_3, v_4, v_5 be the vertices of H with degree 4 and P_{ij} the v_i-v_j path in H that is the subdivided edge $v_i v_j$.

Comment Below we arbitrarily and without further notice break symmetries between $v_1, \ldots, v_5$.

We may assume that every P_{ij} has length 1 or 2, for if, say, P_{12}, has length at least 3, then the union of paths $P_{12}, P_{13}P_{32}, P_{14}P_{42}, P_{15}P_{52}$ is a subdivision of Fig. 22a. If, say, P_{12} and P_{13} have length 2, then the union of the paths $P_{12}, P_{13}, P_{14}P_{42}, P_{15}P_{53}$, P_{23} is a subdivision of Fig. 22b, see Fig. 8a. Henceforth we can and will assume that distinct paths P_{ij} of length 2 are disjoint. This, in particular means that H has at most seven vertices and there is some vertex, say $u \in V(G) \setminus V(H)$. Next we show that some specific conditions imply that G is not metrizable.

(1) There is an edge $uw \in E(G)$, where w is a degree 2 vertex in H.

Proof W.l.o.g. w is the middle vertex of P_{12}. Since G is 2-connected there is another path from u to H which doesn't contain w. It suffices to consider the case where this other path is also an edge $e = u\tilde{w}$. If $d_H(\tilde{w}) = 2$, say it is the middle vertex of P_{34}, then $\tilde{w}v_4v_2v_5v_1w_1u\tilde{w}$ is a 7-cycle in G. Along with the edges $v_3\tilde{w}$, v_3v_1 and v_3v_2 this gives a copy of Fig. 22f, see Fig. 9a. Now suppose that $e = uv_i$. If $i = 1, 2$, say $i = 1$, then we redefine P_{12} to be the path v_1uwv_2, yielding a subdivision of K_5 where P_{12} has length 3. It follows that G is not metrizable. So we can assume $i = 3, 4, 5$, say $i = 3$. In this case, the union of the paths $wuv_3, wv_1, wv_2, P_{14}, P_{24}, P_{34}, P_{15}, P_{25}, P_{35}$ forms a subdivision of Fig. 22c, see Fig. 8b. □

(2) $uv_1, uv_2 \in E(G)$ and some path P_{1k} has length 2.

Proof If $k = 2$ then the paths $v_1uv_2, P_{12}, P_{13}P_{32}, P_{14}P_{45}P_{52}$ form a subdivision of Fig. 22a. Otherwise, say $k = 3$. The paths $v_1uv_3, P_{12}P_{23}, P_{14}P_{43}, P_{15}P_{53}$ form a subdivision of Fig. 22a. □

(3) $u \neq \tilde{u} \in V(G) \setminus V(H)$ and $uv_1, uv_2, \tilde{u}v_1, \tilde{u}v_k \in E(G)$, and $k \neq 1$.

Proof If $k = 2$, then the paths $v_1uv_2, v_1\tilde{u}v_2, P_{13}P_{23}, P_{14}P_{45}P_{25}$ form a subdivision of Fig. 22a. Otherwise, say $k = 3$ and the paths $v_1uv_2, v_1\tilde{u}v_3, P_{14}P_{24}, P_{15}P_{35}, P_{2,3}$ form a subdivision of Fig. 22b, see Fig. 8c. □

(4) There is an edge $e = u_1u_2$ in G that is disjoint from H.

Proof By Menger's theorem there are disjoint u_1-H, u_2-H paths Q_1, Q_2. By (1), we can assume that Q_1, Q_2 end at $v_i \neq v_j$ respectively. Replacing the path P_{ij} with $v_iQ_1u_1u_2Q_2v_j$ we get another subdivision of K_5 with P_{ij} having length 3. □

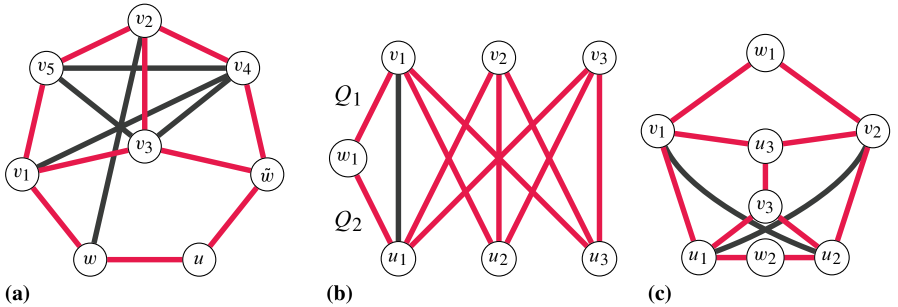


Fig. 9 **a** The colored edges form a Fig. 22f subgraph. **b** A proper subdivision of $K_{3,3}$ is a subdivision of Fig. 22c. **c** This graph contains a Fig. 22e subgraph and is therefore not metrizable.

Our analysis of the general case is based on the above conditions and is parametrized by $|V(H)|$. Since G is 2-connected for every $u \in V(G) \setminus V(H)$ there are two disjoint u-H paths in G. By condition (4) we can assume that these paths are in fact edges.

We have previously dealt with the case $|V(H)| \geq 8$ and move now to assume $|V(H)| = 7$. This means that two paths, say P_{12} and P_{34}, have length 2 and there is a vertex $u \in V(G) \setminus V(H)$ with two neighbors in H. We may assume that these two neighbors are v_i and v_j for some $i \neq j$. For if u has a neighbor in H whose degree in H is 2, then case (1) applies. Therefore the neighbors of u must be v_i and v_j, $i \neq j$. Since $\{i, j\} \cap \{1, 2, 3, 4\} \neq \emptyset$, G is not metrizable by condition (2).

We next assume $|V(H)| = 6$ in which case $P_{12} = v_1 w v_2$ is a path of length 2 and there are two vertices $u_1, u_2 \notin V(H)$. Again by (1) we can assume that w is neither a neighbor of u_1 nor u_2. If u_1 or u_2 share a neighbor in H then condition (3) holds. Otherwise the neighbors of u_1 and u_2 in H are all distinct. This implies at least one of these neighbors is either v_1 or v_2 and G is not metrizable by (2).

The last case to check is $|V(H)| = 5$. Then there exists three vertices $u_1, u_2, u_3 \in V(G) \setminus V(H)$, each having at least 2 neighbors in H. But $|V(H)| = 5$, so these neighbors cannot all be distinct and condition (3) applies.

Now we consider the case where G contains a subdivision H of $K_{3,3}$. If H contains any vertices of degree 2 then G contains a subdivision of Fig. 22c. Therefore we can assume that $H = K_{3,3}$ with vertex bipartition $A = \{v_1, v_2, v_3\}$, $B = \{u_1, u_2, u_3\}$ and $w_1, w_2 \in V(G) \setminus V(H)$. By the fan lemma there exists two disjoint w_1-H paths Q_1 and Q_2. If Q_1 is a w_1-A path and Q_2 a w_1-B path then G again contains a subdivision of Fig. 22c, see Fig. 9b. So we can assume that Q_1 and Q_2 are both u_1-A paths and have endpoints v_1 and v_2, respectively. If either Q_1 or Q_2 have length greater than 1 then the paths $v_1 Q_1 w_1 Q_2 v_2$, $v_1 u_1 v_2$, $v_1 u_2 v_2$, $v_1 u_3 v_2$ form a subdivision of Fig. 22a. Therefore $Q_1 = w_1 v_1$ and $Q_2 = w_1 v_2$. Likewise w_2 has two neighbors in H which both belong to either A or B. Say both are in A. If the neighbors of w_1 and w_2 coincide then the paths $v_1 w_1 v_2$, $v_1 w_2 v_2$, $v_1 u_1 v_2$, $v_1 u_2 v_3 u_3 v_2$ form a subdivision of Fig. 22a. Otherwise, w_2 shares only one neighbor with w_1, say v_2. Again, the paths $v_1 w_1 v_2 w_2 v_3$, $v_1 u_1 v_3$, $v_1 u_2 v_3$, $v_1 u_3 v_3$ form a subdivision of Fig. 22a. Now suppose the neighbors of w_2 are $u_1, u_2 \in B$. Then the graph $H' = (H - \{v_1 u_2, v_2 u_1\}) \cup \{w_1 v_1, w_1 v_2, w_2 u_1, w_2 u_2\}$ is a subdivision of Fig. 22e, see Fig. 9c. □

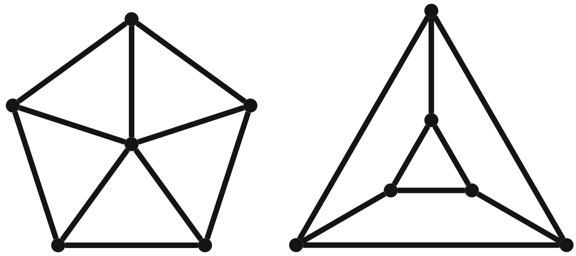


Fig. 10 Minimally 3-connected planar graphs on six vertices

Next we consider the planar case, starting with some technical lemmas. Recall the n-wheel W_n, which is the graph obtained by joining an n-cycle with a single vertex called its *pivot*.

Lemma 6.3 *The only minimally* 3*-connected graph on five vertices is* W_4. *The minimally* 3*-connected planar graphs on six vertices are precisely* W_5 *and the* 3*-prism* Y_3*, Fig.* 10. *The minimally* 3*-connected planar graphs on seven vertices are precisely those in Fig.* 11.

Proof This can be shown using the procedure outlined in [6]. Explicitly, if $G \neq K_4$ is a minimally 3-connected graph then there exists a minimally 3-connected graph G', of order strictly less than G, such that G can be obtained from G' by applying one of the following operations:

1. Take a vertex u and a non-incident edge e and subdivide e to include a new vertex z. Then add the new edge uz.
2. Subdivide two edges e_1 and e_2 with new vertices u_1 and u_2, respectively, and add the new edge u_1u_2.
3. Introduce a new vertex v and make it a neighbor of three distinct vertices x, y, and z.

Dawes [6] gives a sufficient and necessary condition for when one the above operations, applied to a minimally 3-connected, yields another minimally 3-connected graph. Moreover, any of the above operations yields a planar graph iff all the involved vertices and/or edges lie in the same face. Therefore, starting with K_4 one can generate all the 3-minimally connected planar graphs. In particular, using this procedure, one can easily find an exhaustive list of 3-minimally connected planar graphs on six and seven vertices. □

Lemma 6.4 *A metrizable* 2*-connected graph* G *which contains a subdivision of* W_5 *has at most seven vertices.*

Proof Let H be a subdivision of W_5 that is a subgraph of G, where u is H's pivot, and v_1, v_2, v_3, v_4, v_5 its degree 3 vertices in cyclic order. Let P_i be the path that is the possibly subdivided edge v_iv_{i+1} and Q_i the u-v_i path which is the possibly subdivided edge uv_i. We first prove that G is not metrizable under certain conditions.

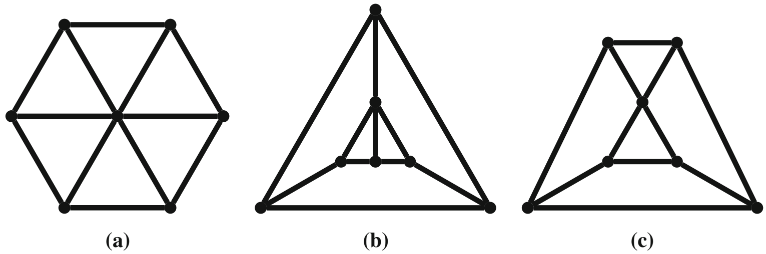


Fig. 11 Minimally 3-connected planar graphs on seven vertices

(1) H has exactly two vertices of degree 2.

Proof We consider all the possible cases up to symmetries.

- Q_1 has length 3. The graph $H - \{uv_5, u_3\}$ is a copy of Fig. 22h.
- Q_1 and Q_2 have length 2. The graph $H - \{uv_3, uv_5\}$ is a copy of Fig. 22i.
- Q_1 and Q_3 have length 2. The graph $H - \{uv_4, uv_5\}$ is a copy of Fig. 22f.
- P_1 has length 3. The graph $H - \{uv_3, uv_5\}$ is a copy of Fig. 22f.
- P_1 and P_2 have length 2. The graph $H - \{uv_4, uv_5\}$ is a copy of Fig. 22f.
- P_1 and P_3 have length 2. The graph $H - \{uv_3, uv_5\}$ is a copy of Fig. 22f.
- P_1 and Q_1 have length 2. The graph $H - \{uv_3, uv_5\}$ is a copy of Fig. 22g.
- P_1 and Q_2 have length 2. The graph $H - \{uv_3, uv_5\}$ is a copy of Fig. 22g.
- P_1 and Q_3 have length 2. The graph $H - \{uv_2, uv_4\}$ is a copy of Fig. 22h. □

(2) H has exactly one vertex of degree 2, w, and $wz \in E(G)$ for some $z \in V(G) \setminus V(H)$

Proof Since G is 2-connected there exists another z-H path not containing w. It suffices to consider the case where this path is a single edge. We go through all the possibilities up to symmetries.

- w in on the path P_1
  - $v_1 \in N(z)$. Redefine P_1 to be v_1zwv_2 a path of length 3, and return to case (1) above.
  - $v_5 \in N(z)$. The paths v_5zw, v_5v_1w, v_5uv_2, $v_5v_4v_3v_2$, v_2w form a subdivision of Fig. 22b, Fig. 12a.
  - $v_4 \in N(z)$. The paths v_4zw, $v_4v_5v_1w$, v_4uv_2, $v_4v_3v_2$, wv_2 form a subdivision of Fig. 22b.
  - $u \in N(z)$. The cycle $v_1wv_2v_3v_4v_5v_1$ along with the paths uzw, uv_3, uv_5 form a copy of Fig. 22g, Fig. 12b.
- w in on the path Q_1
  - $v_1 \in N(z)$. Redefine Q_1 to be the path of length 3, $uwzv_1$, and return to case (1) above.
  - $u \in N(z)$. Again redefine Q_1 to be the path of length 3, $uzwv_1$.
  - $v_2 \in N(z)$. The cycle $v_1v_2v_3v_4v_5v_1$, along with the paths $uwzv_2$, uv_4, uv_5 form a copy of Fig. 22h, Fig. 12c.

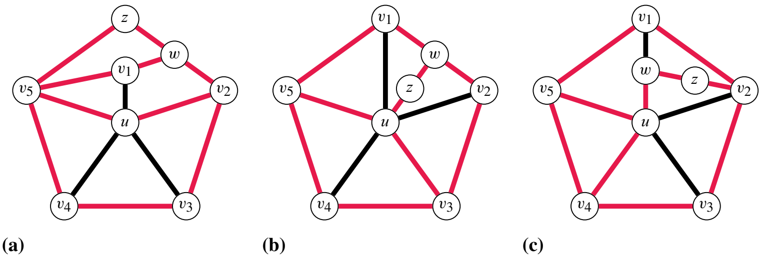


Fig. 12 **a** This graph contains the non-metrizable Fig. 22b. **b** The graph W_7 with a subdivided spoke contains Fig. 22b. **c** A graph which contains a copy of Fig. 22h

- $v_3 \in N(z)$. Then the graph $(H - \{uv_2, uv_3\}) \cup \{zw, zv_3\}$ is a subdivision of Fig. 22e, Fig. 13a. □

(3) $z \in V(G) \setminus V(H)$ and $zv_i, zv_j \in E(G)$ for some non-adjacent v_i, v_j, say $i = 2$ and $j = 5$.

Proof The paths v_5zv_2, P_5P_1, Q_5Q_3, P_4P_3, P_2 form a subdivision of Fig. 22b, Fig. 13b. □

(4) P_1 has length 2 and v_1, v_2 are both neighbors of some $z \in V(G) \setminus V(H)$.

Proof The paths P_1, v_1zv_2, Q_1Q_2, $P_5P_4P_3P_2$ form a subdivision of Fig. 22a. □

(5) Q_1 has length 2 and v_1, u are both neighbors of some $z \in V(G) \setminus V(H)$.

Proof The paths Q_1, uzv_1, P_5Q_5, $P_1P_2P_3P_4Q_4$ form a subdivision of Fig. 22a. □

We show G is not metrizable using the above conditions. Our analysis is split to cases according to the number of vertices in H. We show first that G is not metrizable when $|V(H)| = 8$, and derive the same conclusion whenever $|V(H)| \geq 8$, using Corollary 5.2. When $|V(H)| = 8$ there are exactly two vertices of degree 2 in H and this is dealt with in case (1).

Next assume $|V(H)| = 7$. Then there exists $w \in V(H)$, $\deg_H(w) = 2$, and $z \in V(G) \setminus V(H)$. Since G is 2-connected there exist two openly disjoint z-H paths. It is enough to assume that these paths are edges in G. If one of these edges is zw then by condition (2) G is not metrizable. Suppose these edges are zv_i, zv_j. If v_i and v_j are not adjacent then G is not metrizable by case (3). If v_i and v_j are adjacent with $j = i + 1$, then either P_i has length 2 and case (4) applies or we can redefine P_i to be v_izv_{i+1} and case (1) applies. Suppose these edges are zv_i, zu. If Q_i has length 2 then G is not metrizable by case (5). Otherwise we can redefine Q_i to be uzv_i and condition (1) applies.

Lastly assume $|V(H)| = 6$. Then there exists $z \in V(G) \setminus V(H)$. Since G is 2-connected there exists two openly disjoint z-H paths, R_1 and R_2. If the ends of R_1 and R_2 are not adjacent then G is not metrizable by condition (3). So we can assume that ends of these paths are adjacent and we can replace one of the edges in H by the path R_1R_2 so that $|V(H)| \geq 7$. In this case G was already shown to be not metrizable. □

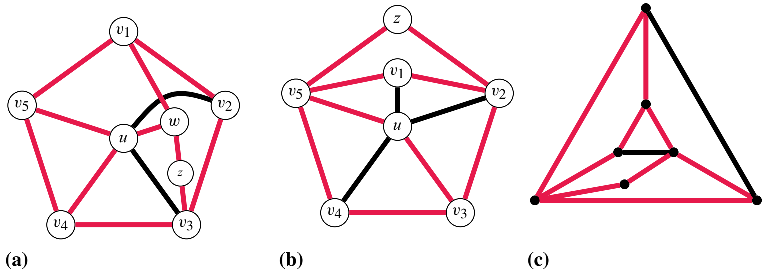


Fig. 13 **a** A graph which contains a subdivision of Fig. 22e. **b** The red edges form a subgraph isomorphic to Fig. 22b. **c** A path which connects two non-adjacent vertices in Y_3 creates a subdivision of Fig. 22b

Lemma 6.5 *A* 2*-connected graph which contains a subdivision of* Y_3 *and has at least seven vertices is not metrizable.*

Proof Let G be a graph with at least seven vertices containing a subdivision of Y_3. Subdividing any edge in Y_3 we either get a copy of Fig. 22d or Fig. 22e. So we need only consider the case where there is path of length 2 between two non-adjacent vertices of Y_3. In this case G contains a subdivision of Fig. 22b, see Fig. 13c.

We use the following lemma from [8]:

Lemma 6.6 *Every* 3*-connected graph* $G \neq K_4$ *contains an edge e such that the graph* $G - e$ *is* 3*-connected after suppressing any vertices of degree* 2.

An immediate consequence of Lemma 6.6 is

Lemma 6.7 *Every* 3*-connected graph* G *on* $n > 4$ *vertices contains a subdivision of a* 3*-connected graph* H *on either* $n - 1$ *or* $n - 2$ *vertices.*

Lemma 6.8 *Let* G *be* 3*-connected (planar) graph on* $n > 4$ *vertices. Then* G *contains a subdivision of a graph* H*, where* H *is a* 3*-connected* (*planar*) *graph which has either* $n - 1$ *or* $n - 2$ *vertices, respectively.*

We can now prove the following:

Theorem 6.9 *If* G *is a* 2*-connected graph of order at least* 8 *which contains a subdivision of a* 3*-connected graph other than* K_4*,* W_4*, and* $K_5 - e$*, then* G *is not metrizable.*

Proof By Theorem 6.2 it suffices to prove the planar case. By Lemma 6.8 if the theorem holds for 3-connected planar graphs on six and seven vertices then it holds for all such graphs with at least six vertices. Moreover, it suffices to consider minimally 3-connected graphs as a subdivision of a 3-connected graph always contains a subdivision of a minimally 3-connected graph. It follows from Lemmas 6.3, 6.4, and 6.5 that the theorem holds for 3-connected planar graphs on six vertices. The theorem is also true for minimally 3-connected planar graph on seven vertices as it is easy to see from Lemma 6.3 that such graphs contain a subdivision of W_5 or Y_3. Lastly we note that the only 3-connected planar graphs on less than six vertices are K_4, W_4, and $K_5 - e$. □

Corollary 6.10 *A* 3*-connected graph with at least eight vertices is not metrizable.*

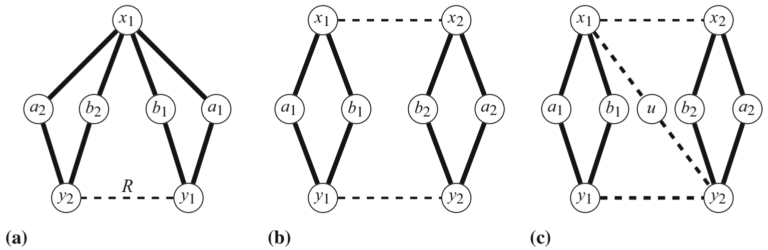


Fig. 14 **a** A subdivision of the graph Fig. 22b. **b** Disjoint paths between x_1 and x_2 and y_1 and y_2. **c** Removing the vertex b_2 gives a subdivision of Fig. 22b

We now state and prove more results regarding the non-metrizability of certain graphs that will be useful to us later. A vertex in a graph is called *essential* if its degree is greater than 2.

Lemma 6.11 *Let $G = (V, E)$ be a 2-connected graph which contains internally disjoint paths P_1, Q_1, P_2, and Q_2, each of length at least 2. Suppose that for each $i = 1, 2$, P_i and Q_i share endpoints $x_i, y_i \in V$ such that x_i and y_i separate the vertices of P_i and Q_i from the rest of G. If G contains an essential vertex not contained in P_1, Q_1, P_2, or Q_2 then G is not metrizable.*

Proof It suffices to prove this when all these paths have length 2. If some path is longer, we can suppress a vertex of degree 2 in it and refer to Corollary 5.2. We write $P_i = x_i a_i y_i$ and $Q_i = x_i b_i y_i$, and let $A = \{x_1, a_1, b_1, y_1, x_2, a_2, b_2, y_2\}$, and take $u \notin A$ a vertex of degree at least 3. There are several cases to consider.

- $x_1 = x_2$ and $y_1 = y_2$. By the fan lemma there exist u-A paths R_1 and R_2 which intersect only at u. As x_1 and y_1 separate A from the rest of G these paths must terminate at x_1 and y_1. If either R_1 or R_2 has length 2 or more, then the graph comprised of the x_1-y_1 paths P_1, Q_1, P_2 and $R_1 R_2$ is a subdivision of Fig. 22a. So we can assume these paths are the edges ux_1 and uy_1. Since u has degree at least 3 it has another neighbor v. We now apply to v the same argument previously applied to u and conclude that v is a neighbor of both x_1 and y_1. But then the x_1-y_1 paths P_1, Q_1, P_2, and $x_1 u v y_2$ form a subdivision of Fig. 22a.
- $x_1 = x_2$ and $y_1 \neq y_2$. Let R be a y_1-y_2 path not containing x_1. Note that R contains none of the vertices in $\{a_1, b_1, a_2, b_2\}$. This is because x_1 and y_1 separate $\{a_1, b_1\}$ from y_2 while x_1 and y_2 separate $\{a_2, b_2\}$ from y_1. Therefore the paths P_1, Q_1, P_2 , Q_2 and R form a subdivision of Fig. 22b, see Fig. 14a.
- $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$. By Menger's theorem there are disjoint paths between $\{x_1, y_1\}$ and $\{x_2, y_2\}$, say, from x_1 to x_2 and y_1 to y_2, see Fig. 14b. Again these paths contain none of the vertices in $\{a_1, b_1, a_2, b_2\}$. Therefore, if either the x_1-x_2 path or the y_1-y_2 path have length greater than 1 we obtain a subdivision of Fig. 22j by taking the union of these two paths along with P_1, P_2, Q_1, Q_2. So we can assume these paths are the edges $x_1 x_2$ and $y_1 y_2$. Again, by the fan lemma there are two u-A paths R_1, R_2 which only intersect at u. Since the vertices $\{x_1, y_1, x_2, y_2\}$ separate A from the rest of G the endpoints of these paths must be in the set $\{x_1, y_1, x_2, y_2\}$.

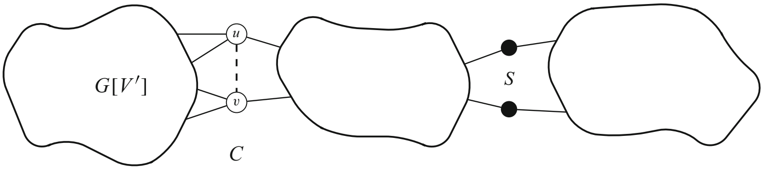


Fig. 15 An illustration of a graph satisfying the conditions of Lemma 6.12

If the endpoints are x_1 and x_2 then replacing the edge x_1x_2 with the path R_1R_2 we get a subdivision of Fig. 22j, as above. If the endpoints are x_1 and y_1 then the four x_1-y_1 paths P_1, Q_1, R_1R_2, and $x_1x_2P_2y_2y_1$ form a subdivision of Fig. 22a. If the endpoints are x_1 and y_2 then the paths P_1, Q_1, R_1R_2, P_2, and y_1y_2 form a subdivision of Fig. 22b, see Fig. 14c. Every other case is identical to one of the previous ones. □

As shown by Dirac [9], a 2-connected graph with minimum degree 3 contains a subdivision of K_4. We need the following variation of this result (Fig. 15).

Lemma 6.12 *Let $G = (V, E)$ be a graph of connectivity 2 and $S \subseteq V$, $|S| = 2$, a separating set. If C is a component of $G \setminus S$ such that each vertex in C is essential in G then there is a set $V' \subseteq V(C)$ and two vertices $u, v \in (V(C) \cup S) \setminus V'$ such that the graph $G[V' \cup \{u, v\}] + uv$ is 3-connected and $G[V']$ is a component of the graph $G \setminus \{u, v\}$.*

Proof We choose $u, v \in V(C) \cup S$ so as to minimize the size of the smallest component, C', of $G \setminus \{u, v\}$ such that $V(C') \subseteq V(C)$. Such pairs of vertices exist, e.g., the two vertices of S. Set $V' = V(C')$ and $\tilde{V} = V' \cup \{u, v\}$. We argue that the graph $G[\tilde{V}] + uv$ is 3-connected. First we show that $|\tilde{V}| \geq 4$. Indeed, both u and v have at least two neighbors in C'. To see this we note that since G is 2-connected both u and v must have at least one neighbor in C' for otherwise the removal of only one vertex disconnects G. Suppose that u has only one neighbor in C', say w. Since $d_G(w) \geq 3$, it has at least one neighbor $z \in C'$ so that $C' - \{w\}$ is not empty. Then removing w and v from G separates $C' - \{w\}$ from u. Moreover, $C' - \{w\}$ has a component that is strictly contained in C', contradicting our choice of u and v. Now suppose $H = G[\tilde{V}] \cup \{uv\}$ is not 3-connected. Then there exists x and y whose removal disconnects H. If $\{x, y\} = \{u, v\}$ then $H \setminus \{x, y\} = C'$, which is a connected graph. If $\{x, y\} \cap \{u, v\} = \emptyset$ then both u and v are in the same connected component since they are connected by an edge. Let C'' be the component $H \setminus \{x, y\}$ which does not contain u and v. Then any path from u or v to C'' must contain either x or y. Since any path connecting a vertex in $V \setminus \tilde{V}$ to C' must contain either u or v, it follows that x and y disconnects C'' from the rest of G. This implies C'' is a component of $G \setminus \{x, y\}$ which is strictly contained in C', a contradiction. Lastly suppose that w.l.o.g. $x = u$ and $y \neq v$. Then again we have that removing x and y disconnects G. Indeed, if C'' is a component of $H \setminus \{x, y\}$ which does not contain v then removing x and y from G we find there is no path connecting C'' to $V \setminus \tilde{V}$. Therefore C'' is a component of $G \setminus \{x, y\}$ which is strictly contained in C', a contradiction. □

Using Lemma 6.12 we prove this next metrizability lemma.

Lemma 6.13 *Let $G = (V, E)$ be a graph of connectivity 2 containing two internally disjoint paths P_1 and P_2, each of length at least 2, which share endpoints x and y such that x and y separate P_1 and P_2 from the rest of G. If $|V \setminus (V(P_1) \cup V(P_2))| \geq 6$ and every vertex in $V \setminus (V(P_1) \cup V(P_2))$ is essential in G then G is not metrizable.*

Proof This graph conforms with the assumptions of Lemma 6.12. Indeed, the set $\{x, y\}$ separates G, and every vertex in every connected component C of $G \setminus (V(P_1) \cup V(P_2))$ has degree at least 3 in G. Using Lemma 6.12 we find some $V' \subseteq V(C)$ and vertices $u, v \in V(C) \cup \{x, y\}$ such that $H := G[V' \cup \{u, v\}] + uv$ is 3-connected and $G[V']$ is a component of $G \setminus \{u, v\}$. If $H \notin \{K_4, W_4, K_5 - e\}$ then G is non-metrizable by Theorem 6.9. Otherwise $H \in \{K_4, W_4, K_5 - e\}$ and H has at most five vertices. As $|V \setminus (V(P_1) \cup V(P_2))| \geq 6$, there exists at least one more vertex $u \notin P_1 \cup P_2 \cup H$ of degree at least 3. Since H is 3-connected, there exists two disjoint u-v paths Q_1 and Q_2 of length at least 2. These paths are also paths in G since they do not contain the edge uv. Thus P_1, P_2, Q_1, and Q_2 are internally disjoint paths of length at least 2, where P_1, P_2 share endpoints, Q_1, Q_2 share endpoints, and some vertex not in any of these paths has degree at least 3. By Lemma 6.11 G is not metrizable. □

Theorem 6.14 *A metrizable 2-connected graph with all vertex degrees at least 3 has at most twelve vertices.*

With some more work the upper bound on the number of vertices can be reduced from 12 to 8.

Proof Let $G = (V, E)$ be a 2-connected graph with $\delta(G) \geq 3$ and at least thirteen vertices. Corollary 6.10 allows us to assume that that G has connectivity 2. If $\{x, y\}$ is a cut set of G and C is the smallest connected component of $G \setminus \{x, y\}$, then $|V \setminus (\{x, y\} \cup V(C))| \geq 6$. Let $V' \subseteq V(C)$ and $u, v \in (V(C) \cup \{x, y\}) \setminus V'$ be as in Lemma 6.12. Namely, $G[V' \cup \{u, v\}] + uv$ is 3-connected and $G[V']$ is a component of $G \setminus \{u, v\}$. Since $G[V' \cup \{u, v\}] + uv$ is 3-connected there are two disjoint u-v paths of length at least 2. Fix two such paths P_1 and P_2. Let E' denote all the edges in $G[V' \cup \{u, v\}] + uv$ which do not appear in P_1 or P_2. After deleting every edge in E' from G we get a subgraph G' which satisfies all the requirements of Lemma 6.13. Therefore G' and hence G is not metrizable. □

7 Establishing Metrizability

Up until this point we have shown that metrizable graphs are rare. Can we find a large class of graphs which *are* metrizable? Trivially, trees are strictly metrizable. As we show next:

Proposition 7.1 *Cycles are strictly metrizable.*

Proof Let $\mathcal{P}$ be path system in C_n for some $n \geq 3$. If $\mathcal{P}$ is trivial in the sense of Sect. 4, it is strictly metrizable. As Proposition 4.8 shows $\mathcal{P}/F = \mathcal{S}_m$ for some odd $m \geq 3$, where F is the set of persistent edges of $\mathcal{P}$. But $\mathcal{S}_m$ is strictly metrizable, and by Proposition 4.6 so is $\mathcal{P}$. □

Our next goal is to show, moreover that *all outerplanar graphs are strictly metrizable*. Recall that a graph is outerplanar if it can be drawn in the plane with all vertices residing in the outer face. Equivalently, a graph is outerplanar iff it contains no subdivision of $K_{2,3}$ or K_4 [5]. We start with some preparations before we can embark on the proof.

A path in a graph G is said to be *suspended* if all the vertices, except possibly the endpoints, have degree 2 in G.

Theorem 7.2 *If a metrizable graph G has a suspended path with endpoints x and y, then $G+xy$ is also metrizable. Similarly, if G is strictly metrizable then so is $G+xy$.*

We start with some easy lemmas.

Lemma 7.3 *Let $\mathcal{P}$ be a path system in an n-vertex graph $G=(V,E)$. If $\mathcal{P}$ is strictly induced by a weight function $w\colon E(G)\to[0,\infty)$, then there exists $\varepsilon>0$ such that for any function $\delta\colon E(G)\to[0,\varepsilon]$, $w+\delta$ also strictly induces $\mathcal{P}$.*

Proof Since w strictly induces $\mathcal{P}$ there is some $\varepsilon'>0$ such that $w(Q)-w(P_{u,v})>\varepsilon'$ for every $u,v\in V$ and every u-v path $Q\neq P_{u,v}$. The claim clearly holds with $\varepsilon=\varepsilon'/(2n)$, since for $w'=w+\delta$, $w(P)\le w'(P)<w(P)+n\varepsilon=w(P)+\varepsilon'/2$ for any path P in G. Consequently, $w'(Q)-w'(P_{u,v})>w(Q)-w(P_{u,v})-\varepsilon'/2\ge\varepsilon'/2>0$ for every two vertices u,v and every u-v path $Q\neq P_{u,v}$. □

Lemma 7.4 *Let $f\colon V\to E$ be a crossing function corresponding to a path system $\mathcal{P}$ in the cycle $C_n=(V,E)$, $n\ge3$, as in Lemma 4.7. Then $w\colon E\to[0,\infty)$ strictly induces $\mathcal{P}$ if and only if for every $x\in V$, $|w(P_{x,a_x})-w(P_{x,b_x})|<w(a_xb_x)$, where $f(x)=a_xb_x$.*

Proof Suppose that w strictly induces $\mathcal{P}$. Then for any $x\in V$,

$$0<w(P_{x,b_x}b_xa_x)-w(P_{x,a_x})=w(P_{x,b_x})-w(P_{x,a_x})+w(a_xb_x)\quad\text{and}$$
$$0<w(P_{x,a_x}a_xb_x)-w(P_{x,b_x})=w(P_{x,a_x})-w(P_{x,b_x})+w(a_xb_x),$$

implying

$$|w(P_{x,a_x})-w(P_{x,b_x})|<w(a_xb_x).$$

Now suppose that $|w(P_{x,a_x})-w(P_{x,b_x})|<w(a_xb_x)$ for all $x\in V$. We need to show that $w(Q)-w(P_{u,v})>0$, where $P_{u,v}\in\mathcal{P}$ and Q is the other u-v path in C_n. W.l.o.g. $v\in P_{u,a_u}$, where $f(u)=a_ub_u$. By definition of f, it must be that $P_{u,v}$ is a subpath of P_{u,a_u} and that $P_{u,b_u}b_ua_u$ is a subpath of Q. As $|w(P_{u,a_u})-w(P_{u,b_u})|<w(a_ub_u)$, this implies

$$\begin{aligned}w(Q)-w(P_{u,v})&\ge w(P_{u,b_u}b_ua_u)-w(P_{u,a_u})\\&=w(P_{u,b_u})-w(P_{u,a_u})+w(a_ub_u)>0.\end{aligned}$$ □

Let $\mathcal{P}$ a path system in a graph G and let H be a subgraph of G. We say $\mathcal{P}$ *restricts* to H if for every two vertices $u,v\in H$ the path $P_{u,v}$ is contained in H. We now prove Theorem 7.2.

Proof of Theorem 7.2 We only deal with the metrizable case, since essentially the same arguments apply to the strictly metrizable case as well. We can assume $xy \notin G$, otherwise there is nothing to show. Let Q be a suspended path between x and y and let $C = Q + xy$ be the cycle formed by Q and the edge xy. Let $H := G \setminus E(Q) + xy$ be the graph obtained by removing Q's edges from G and adding the edge xy. By Proposition 5.1, H is metrizable, being a topological minor of G.

We need to show that every path system $\mathcal{P}$ in $G + xy$ is metrizable. We first reduce the problem to the case where $\mathcal{P}$ includes every edge in C. If $xy \notin \mathcal{P}$ then $\mathcal{P}$ is just a path system of G and is metrizable by assumption. On the other hand, if $e' \notin \mathcal{P}$ for some edge $e' \in Q$, then $\mathcal{P}$ is a path system of a graph whose biconnected components include those of H and the rest of the edges of Q. The metrizability of G then follows from Remark 2.2.

Next we claim that if $\mathcal{P}$ contains every edge in C, then it restricts to both H and C. Indeed, if the path $P_{u,v}$ is not contained in H for some $u, v \in H$, then it must contain vertices from $V(Q) \setminus \{x, y\}$. But Q is a suspended path, and is separated from H by x and y. Therefore any path between two vertices in H which meets $V(Q) \setminus \{x, y\}$ must contain all of Q. Since $\mathcal{P}$ is consistent, this implies $P_{x,y} = Q$, contradicting that $xy \in \mathcal{P}$. In the same way $\mathcal{P}$ restricts to a path system in C.

Let the restrictions of $\mathcal{P}$ to H and C be called path systems $\mathcal{P}_H$ and $\mathcal{P}_C$, respectively. By Lemma 4.7, there corresponds to the path system $\mathcal{P}_C$ a crossing function $f\colon V(C) \to E(C)$. Namely, for each $z \in C$, $f(z) = uv$ is the unique edge such that $V(P_{z,u}) \cap V(P_{z,v}) = \{z\}$.

We take a closer look at the path system $\mathcal{P}_C$. In Proposition 4.8 we saw that by contracting each set $f^{-1}(e)$, $e \in \operatorname{Im} f$, to a vertex we obtain an odd cycle equipped with its canonical path system. It follows that for some $n \geq 1$, $|\operatorname{Im} f| = 2n+1$ and that there exists an ordering of the edges $\operatorname{Im} f = \{e_1, \dots, e_{2n+1}\}$ so that $U_i = C[f^{-1}(e_i)]$ satisfy the following, for $1 \leq i \leq 2n + 1$:

- U_i is a subpath of C;
- U_i and U_{i+1} are connected by the edge e_{n+i+1}, with indices taken mod $2n + 1$, and $e_{n+1} = xy$, with $y \in U_1$ and $x \in U_{2n+1}$;
- for $1 \leq i \leq 2n + 1$, $f(u) = e_i$ for all $u \in U_i$.

This means we can express C as $C = yU_1U_2 \cdots U_{2n}U_{2n+1}xy$, Fig. 16. Let α and β denote the end vertices of the path U_{n+1} so that e_1 is incident with β and e_{2n+1} is incident with α. Also, we set $U := U_{n+1}$.

Let $\tilde{w}\colon E(G+xy) \to [0, \infty)$ be any non-negative weight function. If a u-v geodesic contains a vertex z, then clearly if R_1 is a u-z geodesic and R_2 is a z-v geodesic then the path R_1R_2 is also a u-v geodesic. We make the following simple observation:

> Suppose that $P_{u,x}$, $P_{u,y}$, $P_{x,v}$, and $P_{y,v}$ are $\tilde{w}$-geodesics, where $u \in C$, $v \in H$. If, w.r.t. $\tilde{w}$, some u-v geodesic contains x, resp. y, then $P_{u,x}P_{x,v}$, resp. $P_{u,y}P_{y,v}$, is a $\tilde{w}$-geodesic. Consequently, either $P_{u,x}P_{x,v}$ or $P_{u,y}P_{y,v}$ is a u-v geodesic w.r.t. $\tilde{w}$. $(*_1)$

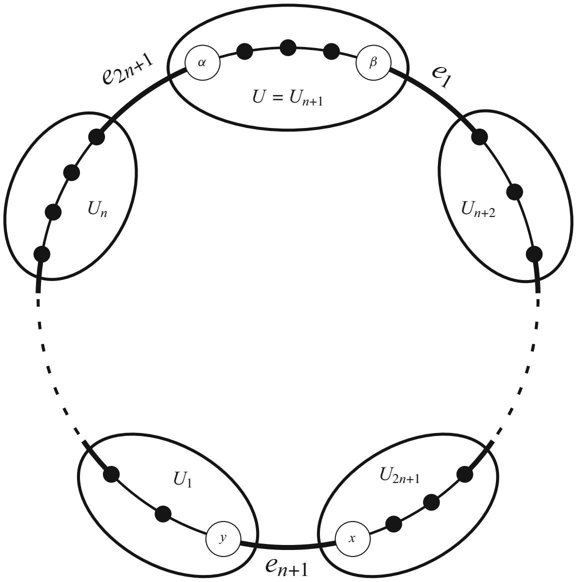


Fig. 16 The path system $\mathcal{P}_C$ partitions C

The last part follows from the fact that x and y separate C from H so that every u-v path contains either x or y. Next we prove an important claim which roughly says that a weight function which induces both $\mathcal{P}_C$ and $\mathcal{P}_H$ *almost* induces $\mathcal{P}$.

Suppose that $\tilde{w}$ induces both $\mathcal{P}_C$ and $\mathcal{P}_H$. Then every path in $\mathcal{P}$ is a $\tilde{w}$-geodesic, with the possible exception of those u-v paths where $v \in H$ and $u \in U$ (whence $f(u) = xy$). $(*_2)$

Paths in $\mathcal{P}$ with both endpoints in C or both endpoints in H are clearly $\tilde{w}$-geodetic, since $\tilde{w}$ induces both $\mathcal{P}_C$ and $\mathcal{P}_H$. An exceptional path must connect some $u \in C$ to some $v \in H$. Let $P_{u,v}$ be such an exceptional path. Since the paths $P_{u,x}$ and $P_{u,y}$ are in $\mathcal{P}_C$ these paths are $\tilde{w}$-geodesics. Similarly, $P_{x,v}$ and $P_{y,v}$ are path in $\mathcal{P}_H$ and therefore also $\tilde{w}$-geodesics. As x and y separate C from H we can assume w.l.o.g. $y \in P_{u,v}$. By consistency we get $P_{u,v} = P_{u,y}P_{y,v}$. Since $P_{u,v} = P_{u,y}P_{y,v}$ is not $\tilde{w}$-geodetic, by $(*_1)$ it follows that $P_{u,x}P_{x,v}$ is a $\tilde{w}$-geodesic. Note that this implies $x \notin P_{u,v}$, for otherwise by consistency $P_{u,v} = P_{u,x}P_{x,v}$, which contradicts that $P_{u,v}$ is not a geodesic. Similarly, $y \notin P_{u,x}P_{x,v}$ or else $P_{u,y}P_{y,v} = P_{u,v}$ is a $\tilde{w}$-geodesic by $(*_1)$. It follows that $y \notin P_{u,x}$ and $x \notin P_{u,y}$. Since these are two paths in C it must be that $V(P_{u,y}) \cap V(P_{u,x}) = \{u\}$, which, by definition of f, yields $f(u) = xy$.

Since both C and H are metrizable, there exist weight functions $w_C\colon E(C) \to (0, \infty)$ and $w_H\colon E(H) \to (0, \infty)$ which induce $\mathcal{P}_C$ and $\mathcal{P}_H$, respectively. Clearly $E(H) \cap E(C) = \{xy\}$, and we rescale these weight functions if necessary to guarantee that $w_C(xy) = w_H(xy)$. With this normalization there is a uniquely defined $\tilde{w}\colon E(G+xy) \to (0, \infty)$ such that $\tilde{w}|_{E(C)} = w_C$ and $\tilde{w}|_{E(H)} = w_H$. If $f^{-1}(xy) = \emptyset$

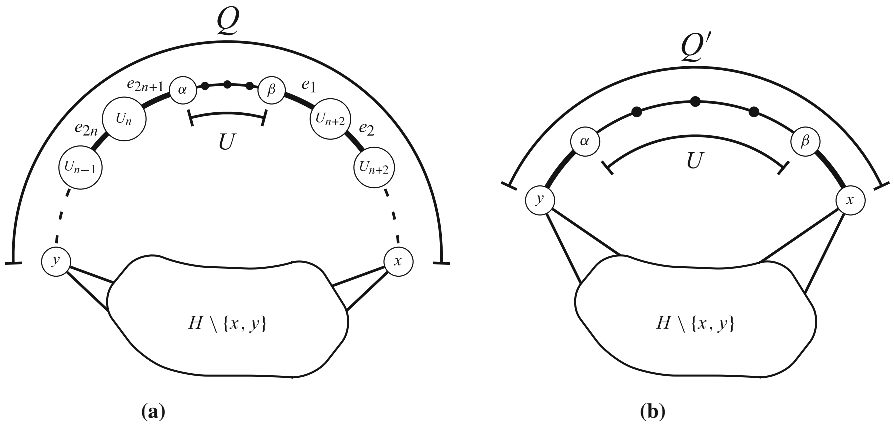


Fig. 17 The graph G vs. G'

then by ($*_2$) this weight function actually induces $\mathcal{P}$. Therefore, we can and will henceforth assume $f^{-1}(xy) \neq \emptyset$.

To construct our desired weight function we first find a function which induces those paths in $\mathcal{P}$ which have endpoints in H and U. After acquiring such a function, we see from ($*_2$) that it is then enough to focus all our attention more locally on C and to adjust this weight function so that it also induces the path system $\mathcal{P}_C$. More concretely, we construct in three steps a weight function $w\colon E(G+xy) \to (0,\infty)$ that induces $\mathcal{P}$.

1. Find a weight function $w_1\colon E(G+xy) \to [0,\infty)$ that induces $\mathcal{P}_H$, as well as every $P_{u,v} \in \mathcal{P}$ with $u \in U$ and $v \in H$.
2. Modify w_1 to a weight function $w_2\colon E(G+xy) \to [0,\infty)$ that induces $\mathcal{P}_C$ as well.
3. Alter w_2 into a strictly positive $w\colon E(G+xy) \to (0,\infty)$ while maintaining the above properties.

The resulting positive weight function w induces both $\mathcal{P}_C$ and $\mathcal{P}_H$ and in addition every path $P_{u,v} \in \mathcal{P}$ with $u \in U$ and $v \in H$ is w-geodetic. By ($*_2$) this weight function induces $\mathcal{P}$.

Before getting into the construction of w_1, we make the following simple observation concerning paths between U and H:

$$\text{If } u \in U \text{ and } v \in H \text{ then exactly one of the vertices } x \text{ and } y \text{ is in } P_{u,v}. \qquad (*_3)$$

It is clear that such a path $P_{u,v}$ contains at least one of x or y since these vertices separate U from H. We argue that if $P \in \mathcal{P}$ is a path containing some $u \in U$ then it cannot also contain both x and y. Otherwise, by the consistency of $\mathcal{P}$, the x-y subpath of P is the edge xy, while the u-x and u-y subpaths of P are $P_{u,x}$ and $P_{u,y}$, respectively. Therefore one of the paths $P_{u,x}$ or $P_{u,y}$ contains the edge xy, which contradicts that $u \in U$, with $f(u) = xy$.

From ($*_3$) we see that the paths $P_{u,v}$, with $u \in U$ and $v \in H$, actually form a partial path system in G, since none of these paths use the edge xy. This may suggest that we extend this partial system to a full system in G in a way that mirrors $\mathcal{P}$. Our approach is similar, but rather than working with a path system in G we construct a path system in G', a topological minor of G.

We wish for w_1 to induce $\mathcal{P}_H$, as well as the paths $P_{u,v} \in \mathcal{P}$ with $u \in U$ and $v \in H$. But since the other vertices in Q do not concern us at the moment, we take G' to be the graph obtained from G by suppressing the vertices in $\left(\bigcup_{i=1,i\neq n+1}^{2n+1} U_i\right) \setminus \{x, y\}$, Fig. 17. Equivalently, G' is obtained from G by replacing the path Q with the path $Q' := y\alpha U\beta x$.

To obtain w_1 we equip G' with a path system $\mathcal{P}'$. Since G' is metrizable (Proposition 5.1), there exists some $w'\colon E(G') \to (0, \infty)$ which induces $\mathcal{P}'$. We then extend w' to a function in $G + xy$ to get our desired w_1.

The path system $\mathcal{P}'$ in G', that we seek to define should be analogous to $\mathcal{P}$. The analog in G' of $P_{u,v}$ for $u \in U$ and $v \in H$ naturally suggests itself. Namely, the path obtained from $P_{u,v}$ by suppressing the vertices in $\left(\bigcup_{i=1,i\neq n+1}^{2n+1} U_i\right) \setminus \{x, y\}$. For $u, v \in U$ the path $P_{u,v}$ is just a subpath of U and is therefore a path in G'. For $u, v \in H$ then there are two cases to consider. If $xy \notin P_{u,v}$ then $P_{u,v}$ is also a path in G'. If $xy \in P_{u,v}$ the G'-analog of $P_{u,v}$ is the path obtained from $P_{u,v}$ by replacing the edge xy by Q'. In particular, whereas the x-y geodesic in $\mathcal{P}$ is the edge xy, in $\mathcal{P}'$ this geodesic is the path Q'. Thus in constructing $\mathcal{P}'$ we take every path in $\mathcal{P}$ with endpoints in G', suppress all vertices in $\left(\bigcup_{i=1,i\neq n+1}^{2n+1} U_i\right) \setminus \{x, y\}$ and replace every occurrence of the edge xy with the path Q'. The formal definition of $\mathcal{P}'$ follows:

Case 1. $u, v \in G' \setminus U$.

(i) If $xy \in P_{u,v}$, i.e., $P_{u,v} = P_{u,x}xyP_{y,v}$, we set $P'_{u,v} := P_{u,x}Q'P_{y,v}$.
(ii) If $xy \notin P_{u,v}$ then $P'_{u,v} := P_{u,v}$

Case 2. $u, v \in U$. We set $P'_{u,v} := P_{u,v}$.
Case 3. $u \in U$ and $v \in G' \setminus U$.

(i) If $x \in P_{u,v}$ then $P'_{u,v} := P_{u,\beta}\beta xP_{x,v}$.
(ii) If $y \in P_{u,v}$ then $P'_{u,v} := P_{u,\alpha}\alpha yP_{y,v}$.

We claim that this definition is valid. We need only elaborate on paths from case 3, whose definition is valid by ($*_3$). Namely, $x \in P_{u,v}$ iff $y \notin P_{u,v}$ for all $u \in U_{n+1}$ and $v \in G' \setminus U$.

We now prove that $\mathcal{P}'$ is consistent, i.e., that if $a, b \in P'_{u,v}$ for some path $P'_{u,v} \in \mathcal{P}'$, then $P'_{a,b}$ is a subpath of $P'_{u,v}$. Our analysis follows the various cases in the definition of $\mathcal{P}'$. In most cases this property is a simple consequence of the definition of $\mathcal{P}'$ and the consistency of $\mathcal{P}$. We only need to elaborate on case 1(i) where $u, v \in G' \setminus U$ and $P_{u,v}$ traverses the edge xy, namely $P_{u,v} = P_{u,x}xyP_{y,v}$, whereas by construction, $P'_{u,v} = P_{u,x}Q'P_{y,v}$. Let $a, b \in P'_{u,v}$. If either $a, b \in G' \setminus U$ or $a, b \in U$, then the definition of $\mathcal{P}'$ and the consistency of $\mathcal{P}$ readily imply that $P'_{a,b}$ is a subpath of $P'_{u,v}$. So let us assume that $a \in U$ and $b \in G' \setminus U$. W.l.o.g. $b \in P_{u,x}$ so that $P_{u,v} = P_{u,b}P_{b,x}xyP_{y,v}$ and $P'_{u,v} = P_{u,b}P_{b,x}Q'P_{y,v}$. By construction of $\mathcal{P}'$, the path $P'_{a,b}$ is either $P_{a,\beta}\beta xP_{x,b}$ or $P_{a,\alpha}\alpha yP_{y,b}$. We claim that $P'_{a,b}$ is in fact $P_{a,\beta}\beta xP_{x,b}$, which

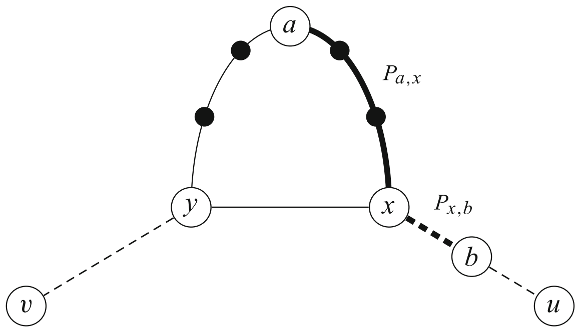


Fig. 18 The path $P_{a,b} = P_{a,x}P_{x,b}$ contains x but not y

is the a-b subpath of $P'_{u,v}$. Otherwise $P'_{a,b} = P_{a,\alpha}\alpha y P_{y,b}$ which, by the construction of $\mathcal{P}'$, implies $y \in P_{a,b}$ and $P_{a,b} = P_{a,y}P_{y,b}$. As $P_{u,v} = P_{u,b}P_{b,x}xyP_{y,v}$ and $\mathcal{P}$ is consistent, the y-b subpath $yxP_{x,b}$ of $P_{u,v}$ coincides with the path $P_{y,b}$, Fig. 18. It follows that $P_{a,b} = P_{a,y}P_{y,b} = P_{a,y}yxP_{x,b}$, and the path $P_{a,b}$ contains the edge xy. From ($*_3$) we see this contradicts the fact that $a \in U$, with $f(a) = xy$. It follows that $\mathcal{P}'$ is consistent and there exists some $w' \colon E(G') \to (0, \infty)$ which induces it.

Using w' we wish to construct a weight function $w_1 \colon E(G + xy) \to [0, \infty)$ that induces $\mathcal{P}_H$ and every path $P_{u,v} \in \mathcal{P}$, with $u \in U$, $v \in H$. To motivate our definition of w_1, consider what happens when $\bigl(\bigcup_{i=1, i\neq n+1}^{2n+1} U_i\bigr) \setminus \{x, y\} = \emptyset$ so that G' coincides with G and Q' coincides with Q. In this case, w' is a weight function on G and to define w_1 we need only extend this function to the edge xy. Also, in this scenario, the path system $\mathcal{P}'$ is obtained by taking the paths in $\mathcal{P}$ and replacing any occurrence of the edge xy with Q. It is not too difficult to see that if we take $w_1(xy) := w'(Q)$ and $w_1(e) := w'(e)$, $e \neq xy$, then this weight function w_1 induces $\mathcal{P}$. Indeed, since Q is a geodesic w.r.t. w' the edge xy is also a geodesic w.r.t. w_1. Thus, taking a w_1-geodesic and replacing any occurrence of the path Q with the edge xy yields another w_1-geodesic. The general definition of w_1 is similar, but slightly more care is needed.

Set $K = w'(Q)$ and fix some $0 < r < \min\,(w'(x\beta), w'(y\alpha)) \le K/2$. We define $w_1 \colon E(G + xy) \to [0, \infty)$ as follows:

- For $e \in H$, $e \neq xy$, $w_1(e) := w'(e)$.
- $w_1(xy) := K$.
- $w_1(e_1) := w'(x\beta) + r$.
- $w_1(e_{2n+1}) := w'(y\alpha) + r$.
- For $e \in U$, $w_1(e) := w'(e)$.
- For all other $e \in C$, $w_1(e) := 0$.

Consider the subpath $L := e_{2n+1}Ue_1$ of Q. Observe that

$$\begin{aligned} w_1(L) &= w_1(e_{2n+1}) + w_1(U) + w_1(e_1) = (w'(y\alpha) + r) + w'(U) + (w'(x\beta) + r) \\ &= w'(Q) + 2r = K + 2r. \end{aligned}$$

In some sense, we identify the path L in G with the path Q' in G'. Note that the edges in L and xy are the only edges of C with positive weight. Also, we could take $r = 0$

in the definition of w_1 and maintain its desired properties, but then we would have $w_1(Q) = w_1(xy)$. The choice of $r > 0$ will make it so that $w_1(xy) < w_1(Q)$ and will allow us to construct a weight function where the paths in $\mathcal{P}_C$ are *strictly* induced. This will in turn give us more flexibility in defining w.

We show that w_1 induces $\mathcal{P}_H$. Any w_1-geodesic between two vertices in H contains no vertices in Q. Otherwise, it contains all of Q and, since $w_1(xy) = K < K + 2r = w_1(L) = w_1(Q)$, replacing Q with the edge xy gives a path of strictly smaller weight. For a path P in H let P' be the path in G' obtained by replacing any occurrence of the edge xy with the path Q'. The mapping $P \mapsto P'$ is a one-to-one correspondence between the paths in H and the paths in G' with endpoints in $V(H)$. Since $w_1(xy) = w'(Q)$ these two paths have the same weight, i.e., $w_1(P) = w'(P')$. It follows that P is a w_1-geodesic iff P' is a w'-geodesic. By construction of $\mathcal{P}$, for any path P in H we have $P \in \mathcal{P}$ iff $P' \in \mathcal{P}'$. Since w' induces $\mathcal{P}'$ the claim follows.

Next we show that for $u \in U$ and $v \in H$, the path $P_{u,v} \in \mathcal{P}$ is a w_1-geodesic. Since $\mathcal{P}$ is consistent, then either $P_{u,v} = P_{u,x}P_{x,v}$ or $P_{u,v} = P_{u,y}P_{y,v}$. We first check that both $P_{u,y}$ and $P_{u,x}$ are w_1-geodesics. As u, x and y are all in the cycle C it suffices to show the inequality $|w_1(P_{u,y}) - w_1(P_{u,x})| < w_1(xy)$ holds. The only edges in C with non-zero weight are the edges in $L = e_{2n+1}P_{\alpha,u}P_{u,\beta}e_1$ and xy so that

$$\begin{aligned} w_1(P_{u,y}) &= w_1(P_{u,\alpha}e_{2n+1}) = w'(P'_{u,y}) + r, \\ w_1(P_{u,x}) &= w_1(P_{u,\beta}e_1) = w'(P'_{u,x}) + r, \end{aligned} \tag{$*_4$}$$

where $w'(P'_{u,y}) + w'(P'_{u,x}) = w(Q') = K$. Therefore,

$$\begin{aligned} |w_1(P_{u,y}) - w_1(P_{u,x})| &= |w_1(P_{u,\alpha}e_{2n+1}) - w_1(P_{u,\beta}e_1)| \\ &= |w'(P'_{u,y}) - w'(P'_{u,x})| < K = w_1(xy) \end{aligned} \tag{$*_5$}$$

and both $P_{u,y}$ and $P_{u,x}$ are w_1-geodetic. Since the paths $P_{x,v}$ and $P_{y,v}$ are in $\mathcal{P}_H$ they are also w_1-geodesics. So by ($*_1$) either $P_{u,x}P_{x,v}$ or $P_{u,y}P_{y,v}$ is a u-v geodesic w.r.t. w_1. Also, as $P_{x,v} \in \mathcal{P}_H$, from what we saw before, $w_1(P_{x,v}) = w'(P'_{x,v})$ and so

$$\begin{aligned} w_1(P_{u,x}P_{x,v}) &= w_1(P_{u,x}) + w_1(P_{x,v}) = w'(P'_{u,x}) + w'(P'_{x,v}) + r \\ &= w'(P'_{u,x}P'_{x,v}) + r. \end{aligned}$$

Similarly, $w_1(P_{y,v}) = w'(P'_{y,v})$ and

$$\begin{aligned} w_1(P_{u,y}P_{y,v}) &= w_1(P_{u,y}) + w_1(P_{y,v}) = w'(P'_{u,y}) + w'(P'_{y,v}) + r \\ &= w'(P'_{u,y}P'_{y,v}) + r. \end{aligned}$$

Suppose $P_{u,v} = P_{u,x}P_{x,v}$. By construction of $\mathcal{P}'$, $P'_{u,v} = P'_{u,x}P'_{x,v}$ and since $P'_{u,v}$ is a w'-geodesic, $w'(P'_{u,x}P'_{x,v}) = w'(P_{u,v}) \leq w'(P'_{u,y}P'_{y,v})$, which implies

$$w_1(P_{u,v}) = w_1(P_{u,x}P_{x,v}) = w'(P'_{u,x}P'_{x,v}) + r \leq w'(P'_{u,y}P'_{y,v}) + r \leq w_1(P_{u,y}P_{y,v})$$

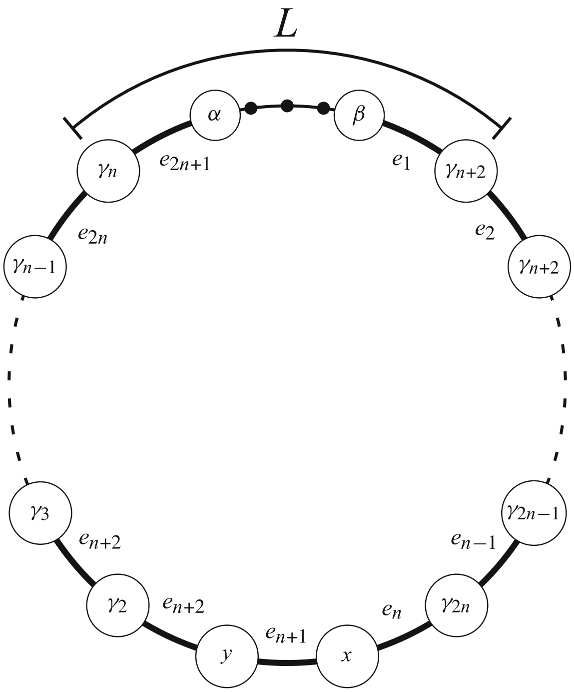


Fig. 19 The cycle C, with $U_i = \{\gamma_i\}$, $i \neq n+1$

and $P_{u,v}$ is a w_1-geodesic. The argument when $P_{u,v} = P_{u,y}P_{y,v}$ is identical.

We next establish some inequalities that we will need below. By construction $r < \min(w'(x\beta), w'(y\alpha))$, and $w_1(e_1) = w'(x\beta) + r$ and $w_1(e_{2n+1}) = w'(y\alpha) + r$ so that

$$\begin{aligned} |w_1(e_1) - 2r| &= w'(x\beta) - r < w_1(e_1), \\ |w_1(e_{2n+1}) - 2r| &= w'(y\alpha) - r < w_1(e_{2n+1}). \end{aligned} \tag{$*_6$}$$

The next step is to modify w_1 to $w_2\colon E(G+xy) \to [0,\infty)$. At this point we only need to find appropriate weights for each e_i, $i \neq 1, n+1, 2n+1$, so that w_2 strictly induces $\mathcal{P}_C$. The canonical path system of an odd cycle is induced by uniform edge weights. So, in search of a weight function which strictly induces $\mathcal{P}_C$, it is suggestive to assign the same positive weight to every edge e_i, and zero weight to all edges in U_i for $1 \leq i \leq 2n+1$. However, we need to deal with the fact that $w_1(e_1)$, $w_1(e_{n+1})$, and $w_1(e_{2n+1})$ are already defined and not necessarily equal. To solve the problem we use the fact that these three edge weights roughly equal $K/2 + r$. We define w_2 as follows:

- For $i \neq 1, n+1, 2n+1$, $w_2(e_i) := K/2 + r$.
- Otherwise, $w_2(e) := w_1(e)$.

We recall that for $u \in U_i$, $f(u) = e_i$. To prove that w_2 strictly induces $\mathcal{P}_C$, by Lemma 7.4, it suffices to show that $|w_2(P_{u,\alpha_i}) - w_2(P_{u,\beta_i})| < w_2(e_i)$ for every

$u \in U_i$, where $e_i = \alpha_i \beta_i$. Moreover, for $i \neq n+1$ the edges in U_i have zero w_2-weight. So if the inequality holds for some vertex in U_i, it holds for all of them. It therefore suffices to show that $\mathcal{P}_C$ is strictly induced by w_2 in the case where $U_i = \{\gamma_i\}$ is a singleton, $i \neq n+1$ with $\gamma_1 = y$ and $\gamma_{2n+1} = x$, Fig. 19. In this scenario, $L = e_{2n+1} U_{n+1} e_1 = \gamma_n \alpha U_{n+1} \beta \gamma_{n+2} = \gamma_n P_{\alpha,\beta} \gamma_{n+2}$.

We start with the range $1 < i \leq n$, so $e_i = \gamma_{n+i}\gamma_{n+i+1}$. To calculate $w_2(P_{\gamma_i,\gamma_{n+i}})$ and $w_2(P_{\gamma_i,\gamma_{n+i+1}})$ we write

$$P_{\gamma_i,\gamma_{n+i}} = \gamma_i \gamma_{i+1} \cdots \gamma_n L \gamma_{n+2} \cdots \gamma_{n+i} \quad \text{and}$$
$$P_{\gamma_i,\gamma_{n+i+1}} = \gamma_i \gamma_{i-1} \cdots \gamma_1 \gamma_{2n+1} \cdots \gamma_{n+i+2} \gamma_{n+i+1}.$$

In $E(P_{\gamma_i,\gamma_{n+i}}) \setminus E(L)$ there are $n-2$ edges, and each such edge e satisfies $w_2(e) = K/2 + r$. Therefore, $w_2(P_{\gamma_i,\gamma_{n+i}}) = (n-2)(K/2+r) + w_2(L)$. Similarly, $w_2(e) = K/2 + r$ for every edge $e \in P_{\gamma_i,\gamma_{n+i+1}}$, where $e \neq \gamma_1\gamma_{2n+1} = xy$. The number of such edges is $n-1$, and therefore $w_2(P_{\gamma_i,\gamma_{n+i+1}}) = (n-1)(K/2+r) + w_2(xy)$. Also, by construction $w_2(L) = w_1(L) = K + 2r$ and $w_2(xy) = w_1(xy) = K$. Since $0 < r < K/2$, there holds

$$|w_2(P_{\gamma_i,\gamma_{n+i}}) - w_2(P_{\gamma_i,\gamma_{n+i+1}})| = \frac{K}{2} - r < \frac{K}{2} + r = w_2(e_i).$$

Now we consider the case $i = 1$, $e_1 = \beta\gamma_{n+2}$. Since $\gamma_1 = y$ we can write

$$P_{y,\beta} = y\gamma_2 \cdots \gamma_n P_{\alpha,\beta}, \qquad P_{y,\gamma_{n+2}} = yx\gamma_{2n} \cdots \gamma_{n+3}\gamma_{n+2}.$$

Clearly,

$$w_2(P_{y,\gamma_{n+2}}) = (n+1)\frac{K}{2} + (n-1)r,$$

since every edge in $P_{y,\gamma_{n+2}}$ other than xy weighs $K/2 + r$.

We turn to calculate $w_2(P_{y,\beta})$. We first observe:

$$\begin{aligned} w_2(\gamma_n P_{\alpha,\beta}) &= w_2(\gamma_n P_{\alpha,\beta}\gamma_{n+2}) - w_2(\beta\gamma_{n+2}) = w_2(L) - w_2(\beta\gamma_{n+2}) \\ &= K + 2r - w_2(e_1). \end{aligned}$$

But aside of the edges in $\gamma_n P_{\alpha,\beta}$ there are $n-1$ edges in $P_{y,\beta}$ with weight $K/2 + r$. Therefore:

$$w_2(P_{y,\beta}) = (n+1)\frac{K}{2} + (n+1)r - w_2(e_1).$$

Since $w_2(e_1) = w_1(e_1)$, from ($*_6$) we get

$$|w_2(P_{y,\beta}) - w_2(P_{y,\gamma_{n+2}})| = |w_2(e_1) - 2r| = |w_1(e_1) - 2r| < w_1(e_1) = w_2(e_1).$$

The argument when $n+2 \le i \le 2n+1$ is similar. There remains the case $i = n+1$, $e_{n+1} = xy$. For $u \in U_{n+1}$ we can write

$$P_{u,x} = P_{u,\beta}\gamma_{n+2}\gamma_{n+3}\cdots\gamma_{2n}x \quad \text{and} \quad P_{u,y} = P_{u,\alpha}\gamma_n\gamma_{n-1}\cdots\gamma_2 y.$$

Note that

$$w_2(P_{u,x}) = (n-1)\left(\frac{K}{2}+r\right) + w_2(P_{u,\beta}\gamma_{n+2}),$$

since $P_{u,x}$ has $n-1$ edges of weight $K/2+r$ in addition to the edges contains in $P_{u,\beta}\gamma_{n+2}$. Similarly, $w_2(P_{u,y}) = (n-1)(K/2+r) + w_2(P_{u,\alpha}\gamma_n)$. Since $w_2(P_{u,\beta}\gamma_{n+2}) = w_1(P_{u,\beta}e_1)$ and $w_2(P_{u,\alpha}\gamma_n) = w_1(P_{u,\alpha}e_{2n+1})$, by ($*_5$) it follows

$$|w_2(P_{u,x}) - w_2(P_{u,y})| = |w_1(P_{u,\alpha}e_{2n+1}) - w_1(P_{u,\beta}e_1)| < w_1(xy) = w_2(xy).$$

Finally, we construct a strictly positive $w\colon E(G+xy) \to (0,\infty)$ which induces $\mathcal{P}$. This is accomplished by perturbing w_2. Using ($*_2$), we see that w_2 is a non-negative weight function which induces $\mathcal{P}$. The strictly metrizable version of Theorem 7.2 already follows now, using Lemma 7.3. However, the non-strictly metrizable statement requires a bit more work.

Set $N_1 := \sum_{i=1}^{n} |E(U_i)|$ and $N_2 := \sum_{i=n+2}^{2n+1} |E(U_i)|$ and fix some small $\delta > 0$. We construct w as follows:

- For $e \in U_i$, $i \neq n+1$, $w(e) := \delta$.
- $w(e_1) := w_2(e_1) + N_1\delta$.
- $w(e_{2n+1}) := w_2(e_{2n+1}) + N_2\delta$.
- Otherwise, $w(e) := w_2(e)$

Since w_2 strictly induces $\mathcal{P}_C$, by Lemma 7.3, w also strictly induces $\mathcal{P}_C$. Moreover, as w_1 induces $\mathcal{P}_H$ and w and w_1 agree over H, this implies that w also induces $\mathcal{P}_H$. If every path in $\mathcal{P}$ between U_i and H is w-geodetic then by ($*_2$), w induces $\mathcal{P}$. Let $u \in U_i$ and $v \in H$. By ($*_1$), either $P_{u,x}P_{x,v}$ or $P_{u,y}P_{y,v}$ is a w-geodesic. From ($*_4$), $w_2(P_{u,\beta}e_1) = w_1(P_{u,\beta}e_1) = w_1(P_{u,x})$. Also, we saw $w_2(P_{u,x}) = (n-1)(K/2+r) + w_2(P_{u,\beta}e_1)$. The definition of w yields

$$w(P_{u,x}) = w_2(P_{u,x}) + N_1\delta + N_2\delta = w_1(P_{u,x}) + \tilde{K},$$

where $\tilde{K} = (n-1)(K/2+r) + N_1\delta + N_2\delta$. Since w and w_1 agree on H

$$\begin{aligned} w(P_{u,x}P_{x,v}) &= w(P_{u,x}) + w(P_{x,v}) = w_1(P_{u,x}) + \tilde{K} + w_1(P_{x,v}) \\ &= w_1(P_{u,x}P_{x,v}) + \tilde{K}. \end{aligned}$$

In the same way,

$$w(P_{u,y}P_{y,v}) = w_1(P_{u,y}P_{y,v}) + \tilde{K}.$$

Since either $P_{u,v} = P_{u,x}P_{x,v}$ or $P_{u,v} = P_{u,y}P_{y,v}$, this implies $P_{u,v}$ is w-geodetic if it is w_1-geodetic. But it was already shown that $P_{u,v}$ is w_1-geodetic. □

Based on Theorem 7.2 we conclude:

Corollary 7.5 *Every outerplanar graph G is strictly metrizable.*

Proof As usual we can assume w.l.o.g. that G is 2-connected. We argue by induction on $|E(G)| - |V(G)|$. In the base case G is a cycle. For the induction step we apply Theorem 7.2 to $G \setminus e$, where $e = uv$ is an internal edge such that u and v are connected by a suspended path. That such an edge exists can be easily shown by considering ear decompositions of G. □

Next we show that all small graphs are metrizable.

Proposition 7.6 *Every graph of order at most* 4 *is strictly metrizable.*

Proof Every graph other than K_4 with at most four vertices is outerplanar and therefore strictly metrizable by Corollary 7.5. There remains the neighborly path systems of K_4 which is strictly induced by unit edge weights. □

Corollary 7.7 *Every* (*strictly*) *non-metrizable graph contains a subdivision of $K_{2,3}$.*

Proof Every 2-connected non-outerplanar graph other than K_4 contains a subdivision of $K_{2,3}$. □

We next introduce infinitely many graphs which are metrizable but not strictly metrizable.

Proposition 7.8 *The graph $K_{2,n}$, $n \geq 4$, is metrizable but not strictly metrizable.*

Proof Let $a_1, a_2; b_1, \ldots, b_n$ be the vertex bipartition of $K_{2,n}$. We show that $K_{2,n}$ is metrizable by induction on n. The base case $K_{2,2}$ coincides with the 4-cycle which we already know to be metrizable. Let $\mathcal{P}$ be a path system in $K_{2,n+1}$, $n \geq 2$. If $\mathcal{P}$ is not neighborly, then some edge, say a_1b_{n+1} is not in $\mathcal{P}$, and $\mathcal{P}$ can be considered a path system in $K_{2,n+1} \setminus a_1b_{n+1}$. But $K_{2,n+1} \setminus a_1b_{n+1}$ has two biconnected components, namely, $K_{2,n}$ and the edge a_2b_{n+1}. By induction $K_{2,n}$ is metrizable, and since both its biconnected components are metrizable so is $K_{2,n+1} \setminus a_1b_{n+1}$. If $\mathcal{P}$ is neighborly it is easily verified that in this case every path in $\mathcal{P}$ has length 1 or 2 and is induced by constant edge weights.

Next we show that $K_{2,4}$ is not strictly metrizable. Again we write $V(K_{2,4}) = (a_1, a_2; b_1, b_2, b_3, b_4)$ and consider the path system that includes all the edges of $K_{2,4}$ and the paths

$$P_{b_1,b_2} = b_1a_2b_2, \qquad P_{b_3,b_4} = b_3a_2b_4, \qquad P_{b_1,b_3} = b_1a_1b_3,$$
$$P_{b_2,b_3} = b_2a_2b_3, \qquad P_{b_1,b_4} = b_1a_1b_4, \qquad P_{a_1,a_2} = a_1b_1a_2,$$
$$P_{b_2,b_4} = b_2a_1b_4.$$

Assume towards a contradiction that this system is strictly metrizable under some weight function w. The first row of paths yields the following inequalities:

$$\begin{aligned} w(a_2b_1) + w(a_2b_2) &< w(a_1b_1) + w(a_1b_2),\\ w(a_2b_3) + w(a_2b_4) &< w(a_1b_3) + w(a_1b_4),\\ w(a_1b_1) + w(a_1b_3) &< w(a_2b_1) + w(a_2b_3),\\ w(a_1b_2) + w(a_1b_4) &< w(a_2b_2) + w(a_2b_4). \end{aligned}$$

Adding these inequalities and canceling terms we find $0 < 0$, a contradiction.

8 Structural Description of Metrizability

As shown above Proposition 5.1, the sets of metrizable and strictly metrizable graphs are closed under taking a topological minor. It follows that there is a minimal set of graphs $\mathcal{F}_M$ such that a graph G is metrizable if and only if no graph in $\mathcal{F}_M$ is a topological minor of G. This set is minimal in the sense that if $H_1 \neq H_2$ belong to $\mathcal{F}_M$, then H_1 is not a topological minor of H_2. Likewise, there is a minimal set $\mathcal{F}_{SM}$ such that G is strictly metrizable if and only if no graph from $\mathcal{F}_{SM}$ is a topological minor of G.

Theorem 8.1 *Both $\mathcal{F}_M$ and $\mathcal{F}_{SM}$ are finite.*

We start with some preliminary comments. In *each* of our proofs in Sect. 6 where we show that some graph G is non-metrizable we find some subgraph of G that is a subdivision of a graph in Fig. 22. If G is not one of the graphs in Fig. 22, and G is shown to be non-metrizable by invoking one these theorems, it necessarily follows that G is not minimal, i.e., $G \notin \mathcal{F}_M$. In particular, if a graph G with at least ten vertices is shown to be non-metrizable using one of these theorems, then $G \notin \mathcal{F}_M$, since all the graphs in Fig. 22 have at most nine vertices. Similarly, it follows that $G \notin \mathcal{F}_{SM}$ since a graph which is not metrizable is clearly also not strictly metrizable. We make the following crucial observation which is an easy consequence of Theorem 7.2:

Observation 8.2 *If $G \in \mathcal{F}_M$ or $G \in \mathcal{F}_{SM}$, and if $xy \in E(G)$, then G cannot contain a suspended path of length greater than 1 with endpoints x and y.*

Proof If $G \in \mathcal{F}_M$, then G is non-metrizable. By Theorem 7.2 the graph $G \setminus xy$ is non-metrizable as well. An identical argument works when $G \in \mathcal{F}_{SM}$.

Recall that we call a vertex essential if it has degree at least 3. We prove next an upper bound on the number of essential vertices in a minimal graph.

Proposition 8.3 *Every graph G in $\mathcal{F}_M$ or in $\mathcal{F}_{SM}$ has at most twelve essential vertices.*

Proof Let G be a 2-connected graph with at least thirteen essential vertices and let G' be the multigraph obtained by suppressing all vertices of degree 2. Clearly, G' has at least thirteen vertices and it contains no loops, since G is 2-connected. If G' is actually

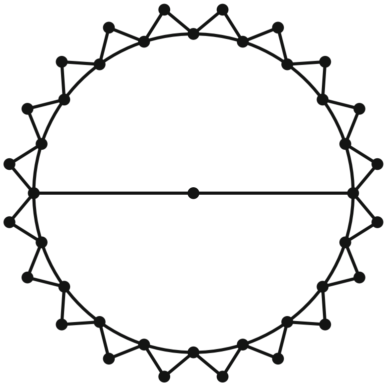


Fig. 20 This graph is not in $\mathcal{F}_M$ by Observation 8.2

a simple graph then it is not metrizable by Theorem 6.14 and therefore not minimal, implying that neither is G.

Suppose that G' has a set of four distinct edges $\{e_1, e_1', e_2, e_2'\}$, where e_1, e_1' connect the same two vertices u_1, v_1 and likewise for e_2, e_2' and u_2, v_2. These edges correspond to suspended u_1-v_1 paths P_1, P_1' and suspended u_2-v_2 paths P_2, P_2', respectively. By Observation 8.2 all these paths have length bigger than 1. These paths are suspended in G which contains at least thirteen essential vertices, so there is an essential vertex outside of P_1, P_1', Q_1, or Q_1'. By Lemma 6.11, G is not metrizable and therefore not minimal.

In the only remaining case there is exactly one pair of essential vertices u and v in G' with two or three parallel edges between them. These edges correspond to suspended paths in G between u and v, of length bigger than 1 (again, by Observation 8.2). By eliminating one of these paths, if necessary, we can assume that there are precisely two suspended paths between u and v. Replacing these parallel edges in G' by the corresponding suspended paths yields a graph G'' which is a subdivision of G. It is easily verified that G'' has at least eleven essential vertices outside these two suspended paths. By Lemma 6.13, G'' is not metrizable. Consequently G'' and hence G is not minimal. □

We remark that in the proof of Proposition 8.3 the bound 12 comes from use of Theorem 6.14. As was stated before, the constant 12 in Theorem 6.14 can be reduced to 8 with a more involved argument. With this optimized version of Theorem 6.14 the bound in Proposition 8.3 can also be reduced to 8.

We note that Observation 8.2, and therefore Theorem 7.2, is crucial in the proof of Proposition 8.3. It allows us discount graphs we would otherwise be unsure how to deal with, see for example Fig. 20.

A *quasi-order* is a binary relation which is reflexive and transitive. We say that X is a *well-quasi-ordered set* (wqo for short) if there a quasi-order $\leq$ on X such that for any sequence $x_1, x_2, x_3, \ldots$ in X there exists $i < j$ satisfying $x_i \leq x_j$. The set of all finite graphs is not well-quasi-ordered with respect to the topological minor relation. We note that Theorem 8.1 can be derived from deep results of Liu and Thomas [13], who characterized collections of graphs which are well-quasi-ordered with respect to the topological minor relation. However, there is a much more elementary route to the same goal.

Proposition 8.4 *Fix $n \geq 0$. If $G_1, G_2, G_2, \ldots$ is an infinite sequence of graphs, each with at most n essential vertices, then there exists $1 \leq i' < i$ such that G_i contains a subdivision of $G_{i'}$.*

Proof We start with some general observations: Let $M = (a_k \geq a_{k-1} \geq \ldots \geq a_1 \geq 1)$ and $M' = (b_\ell \geq b_{\ell-1} \geq \ldots \geq b_1 \geq 1)$ be multisets of positive integers. We say that $M \succ M'$ if there is an injection $\sigma : [\ell] \to [k]$ such that $b_i \leq a_{\sigma(i)}$ for all i. We recall that multisets of positive integers are well-quasi-ordered by the relation $\succ$, and that wqo sets are closed under Cartesian product [11].

We turn to prove the proposition. By passing to a subsequence, if necessary we can assume that each G_i, $i \geq 1$, has precisely $n' \leq n$ essential vertices. Assume w.l.o.g. that $n' = n$. For each i we label G_i's essential vertices, i.e., we fix a bijection from these n vertices to $\{1, 2, \ldots, n\}$. For $1 \leq j < k \leq n$ let $N_{i,j,k}$ be the multiset consisting of the lengths of all the suspended j-k paths in G_i. Let $N_i = \prod_{1 \leq j < k \leq n} N_{i,j,k}$. Clearly G_i contains a subdivision of $G_{i'}$ iff $N_i \succ N_{i'}$. The conclusion follows from the general comments on wqo sets. □

With these results the proof of Theorem 8.1 is clear.

Proof of Theorem 8.1 By Proposition 8.3 the set $\mathcal{F}_M$ consists of graphs with at most twelve essential vertices. Since each graph in $\mathcal{F}_M$ is a minimal element with respect to the topological minor relation, from Proposition 8.4 this implies that there can only be finitely many such graphs. The proof dealing with $\mathcal{F}_{SM}$ is identical. □

9 A Continuous Perspective

The metrizability problem for the cycle can be stated in an appealing continuous form. We remark that it is very possible that the results in this section have been previously studied under a different setting, but nonetheless, we think it is worthwhile to examine them through the lens of metrizability. We say that a map $T : S^1 \to S^1$ is *crossing* if for any $x, y \in S^1$ the segments $[x, T(x)]$, $[y, T(y)]$ intersect. We say a crossing map T is *metrizable* if T has a *compatible* non-atomic probability measure μ over S^1. Namely for every $x \in S^1$, the points x and $T(x)$ split S^1 into two arcs of μ-measure of $1/2$ each. We prove the following:

Proposition 9.1 *A crossing map $T : S^1 \to S^1$ is metrizable if and only if there exists a T-invariant non-atomic probability measure.*

We start with two simple observations.

Observation 9.2 *If $T : S^1 \to S^1$ is crossing, then for any $w \in S^1$ the set $T^{-1}(w)$ is either empty, a single point or a connected arc of S^1.*

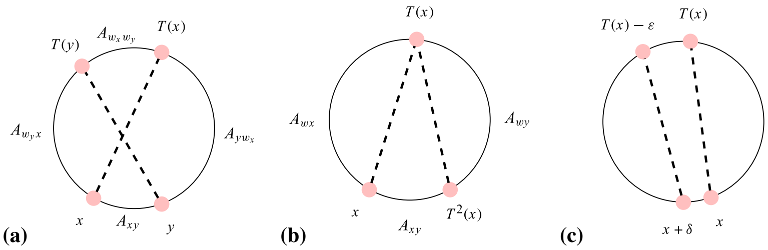


Fig. 21 **a** The points $x, y, T(x), T(y)$ partition S^1 into fours arcs. **b** The crossing function T has a point of discontinuity at $T(x)$. **c** A transformation which is not locally monotonic at points of continuity is not crossing

Proof Say that $T(x) = T(y) = w$ for some $x \neq y$. The points x, y, w split S^1 into three arcs A_{xy}, A_{yw}, A_{wx}. We claim that $f(z) = w$ for every $z \in A_{xy}$. Indeed, if $f(z) \neq w$ then $f(z)$ is in either A_{xy}, A_{yw}, or A_{wx}. But then y, z (resp. x, z, resp. both) fails the crossing condition. □

Observation 9.3 *If μ is compatible with T then $\mu(T^{-1}(x)) = 0$ for all $x \in S^1$.*

Proof In view of Observation 9.2 and the fact that μ is atom-free it suffices to consider the case where $T^{-1}(x) = A$ is an arc. But if $\mu(A) > 0$, we can find some $x' \neq x''$ in A so that the (x', x'') subarc of A has a positive μ measure. The partitions of S^1 that x, x' induce and the one induced by x, x'' cannot both satisfy the compatibility requirement. □

Proof of Proposition 9.1 We first show that every T-invariant measure μ is compatible. We need to show that $\mu(A_1) = \mu(A_2)$, where A_1 and A_2 be the two arcs defined by x and $T(x)$ for an arbitrary $x \in S^1$. Since T is crossing, for every $y \in A_2$ there holds $T(y) \in A_1 \cup \{T(x), x\}$, whence $T(A_2) \subseteq A_1 \cup \{T(x), x\}$. By Observation 9.3 both $T^{-1}(T(x))$ and $T^{-1}(x)$ are μ zero-sets, implying

$$\mu(A_2) \leq \mu(T^{-1}A_1) + \mu(T^{-1}T(x)) + \mu(T^{-1}(x)) = \mu(T^{-1}A_1) = \mu(A_1).$$

By symmetry we also have $\mu(A_1) \leq \mu(A_2)$.

The reverse statement says that if μ is compatible with T, then for every arc A there holds

$$\mu(T^{-1}A) = \mu(A).$$

This statement for A and for its complement A^c are equivalent, since $\mu(T^{-1}(A^c)) = \mu((T^{-1}A)^c) = 1 - \mu(T^{-1}A)$ and $\mu(A^c) = 1 - \mu(A)$. Also by Observation 9.3 this holds for A a singleton. So consider an arc with endpoints $x \neq y$, where $T(x) = w_x$, $T(y) = w_y$, and $w_x \neq w_y$. The points x, y, w_x, w_y split S^1 into four arcs A_{xy}, A_{yw_x}, $A_{w_x w_y}$, and $A_{w_y x}$, Fig. 21a.

By the crossing condition we must have

$$A_{w_x w_y} \subseteq T^{-1}(A_{xy} \setminus \{x, y\}) \subseteq A_{w_x w_y} \cup \{T(x), T(y)\},$$

which implies

$$\mu(A_{w_x w_y}) = \mu(T^{-1}A_{xy}).$$

Moreover, since μ is T-compatible we have

$$\mu(A_{xy}) + \mu(A_{yw_x}) = \frac{1}{2} = \mu(A_{w_y x} + \mu(A_{w_x w_y}) \qquad \text{and}$$
$$\mu(A_{w_y x}) + \mu(A_{xy}) = \frac{1}{2} = \mu(A_{yw_x}) + \mu(A_{w_x w_y}),$$

which implies $\mu(A_{xy}) = \mu(A_{w_x w_y}) = \mu(T^{-1}A_{xy})$. □

Observation 9.4 *Every continuous crossing mapping $T\colon S^1 \to S^1$ is an involution.*

Proof Suppose towards a contradiction that $y = T \circ T(x) \neq x$ for some $x \in S^{-1}$. We show that T is not continuous at $w = T(x)$. The points x, y, and w split S^1 into three arcs A_{xy}, A_{yw}, and A_{wx}, Fig. 21b. By the crossing condition, $T(z) \in A_{wx}$ for every $z \in A_{yw}$. Consider a sequence of points in A_{yw} which converges to w to conclude that T is discontinuous at w. □

Identifying S^1 with $[0, 1]/(0 \sim 1)$, we say a function $f\colon S^1 \to S^1$ is locally monotonic increasing at $x \in S^1$, if for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. $f([x, x+\delta)) \subseteq [f(x), f(x)+\varepsilon)$.

Observation 9.5 *Let $T\colon S^1 \to S^1$ be a crossing function and suppose that T is continuous at $x \in S^1$. Then T is locally monotonic increasing at x.*

Proof Suppose that T is not locally monotonic increasing at x. Since T is continuous at x this means we can find small $\varepsilon > 0$, $\delta > 0$ such that $f(x+\delta) = f(x) - \varepsilon$. But this implies that T is not crossing, see Fig. 21c. □

Proposition 9.6 *Let $T\colon S^1 \to S^1$ be a C^1 crossing map and $f\colon S^1 \to [0, \infty)$ a continuous map. The measure defined by $d\mu(x) = f(x)\,dx$, where dx denotes the Lebesgue measure, is T-compatible if and only if for all $y \in S^1$*

$$(f \circ T(y)) \cdot T'(y) = f(y).$$

Proof First we note, since T is continuous and crossing for all $x, y \in S^1$, $T([x, y]) = [T(x), T(y)]$. Now assume that μ is T-compatible. Then by Proposition 9.1 μ is T invariant. Therefore,

$$\int_y^{y+\delta} f(x)\,dx = \mu([y, y+\delta]) = \mu(T^{-1}([y, y+\delta]))$$
$$= \mu(T([y, y+\delta])) = \mu([T(y), T(y+\delta)])) = \int_{T(y)}^{T(y+\delta)} f(x)\,dx.$$

Since T is C^1, $T(y+\delta) = T(y) + \delta T'(y) + o(\delta)$. Moreover, we are dealing with continuous functions and therefore

$$\int_y^{y+\delta} f(x)\,dx = f(y)\delta + o(\delta)$$

and

$$\int_{T(y)}^{T(y+\delta)} f(x)\,dx = \int_{T(y)}^{T(y)+\delta T'(y)+o(\delta)} f(x)\,dx = f \circ T(y) \cdot T'(y)\delta + o(\delta).$$

Taking $\delta \to 0$ we find

$$f(y) = f \circ T(y) \cdot T'(y).$$

Now assume $f(y) = f \circ T(y) \cdot T'(y)$ for all $y \in S^1$, and let $a, b \in S^1$. Then

$$\mu(T^{-1}([a,b])) = \mu(T([a,b])) = \mu([T(a), T(b)]) = \int_{T(a)}^{T(b)} f(x)\,dx$$
$$\overset{x\mapsto T(x)}{=} \int_{T^2(a)}^{T^2(b)} f \circ T(x) \cdot T'(x)\,dx = \int_a^b f(x)\,dx = \mu([a,b]).$$

The second to last line follows from the fact that T is an involution and $f(y) = f \circ T(y) \cdot T'(y)$. That μ is T-compatible follows from Proposition 9.1. □

Corollary 9.7 *Let $T\colon S^1 \to S^1$ be a C^1 crossing map. Then the measure μ defined by $d\mu = (T'(x))^{1/2}dx$ is T-compatible.*

Proof Observe that

$$1 = [x]' = [T \circ T(x)]' = T'(x) \cdot T' \circ T(x) \quad \Longrightarrow \quad T'(x) = \frac{1}{T' \circ T(x)}.$$

so that T' is non-zero. Since by Observation 9.5, $T' \geq 0$, T' is strictly positive and μ is a well-defined positive measure. Moreover,

$$(T'(x))^{1/2} = (T'(x))^{-1/2} \cdot T'(x) = (T' \circ T(x))^{1/2} \cdot T'(x).$$

The rest follows from Proposition 9.6, taking $f(x) = (T'(x))^{1/2}$. □

10 The Computational Perspective

In this section we discuss the computational complexity of the following decision problems:

- (Path System Metrizability) Decide if a given path system $\mathcal{P}$ in a graph G is metrizable.
- (Graph Metrizability) Decide if a given graph G is metrizable.

It is also of interest to determine if a path system/graph is *strictly* metrizable.

Theorem 10.1 *Path System Metrizability and Strict Path System Metrizability can be decided in polynomial time.*

The strict case of Theorem 10.1 is already proven in [2]. In that paper, Bodwin characterizes strict metrizability in terms of flow and uses that characterization to obtain a procedure which decides whether or not a path system is strictly metrizable by solving a linear program with only polynomially many constraints. Moreover, it seems likely that a similar approach can be used to deal with the non-strict version of the problem. However, our approach is different and builds on the classical theory of Grötschel, Lovász, and Schrijver [10]. Let us recall some basic definitions from that theory. A *strong separation oracle* for a polyhedron $K \subseteq \mathbb{R}^n$ receives as input a point $x \in \mathbb{Q}^n$ and either asserts that $x \in K$ or returns a vector $c \in \mathbb{Q}^n$ s.t. $c^T x < c^T y$ for all $y \in K$. The *encoding length* of an integer s or a simplified fraction $q = s/t$ is the least number of bits needed to express s resp. q. The encoding length of a vector or a matrix the is the sum over their entries. Here is our main tool:

Theorem 10.2 [10] *Suppose that the polyhedron $K = \{x \in \mathbb{R}^n : Ax \leq b\}$ has a strong separation oracle, where $A \in M_{m\times n}(\mathbb{Q})$, $b \in \mathbb{Q}^m$. If each of the inequalities $\langle a_i, x\rangle \leq b_i$ has encoding length $\leq \varphi$, then it is possible to determine whether or not K is empty in time* $\mathrm{poly}(n, \varphi)$.

Proof of Theorem 10.1 We first deal with non-strict metrizability. Let $\mathcal{P}$ be a path system in G, and let $\mathcal{Q}_{u,v}$ denote the collection of all the simple u-v paths in G not equal to $P_{u,v} \in \mathcal{P}$. Then

$$A_{u,v} := \left\{ x \in \mathbb{R}^E : x > 0,\ \forall\, Q \in \mathcal{Q}_{u,v} \sum_{e\in P_{u,v}} x_e - \sum_{e\in Q} x_e \leq 0 \right\}$$

is the collection of all positive edge weights which induce $P_{u,v}$ as a u-v geodesic. But clearly $x \in \mathbb{R}^E$ and αx induce the same path system for any $\alpha > 0$. So $\mathcal{P}$ is metrizable iff it is induced by some $x \geq 1$. Let $B := \{x \in \mathbb{R}^E : x_e \geq 1 \text{ for every } e \in E\}$. Therefore $\mathcal{P}$ is metrizable if and only if the polyhedron

$$K = \bigcap_{u,v\in V} A_{u,v} \cap B$$

is not empty. Also, $\varphi \leq O(n)$ since all the coefficients in K's defining inequalities are $1, 0, -1$ and each such inequality is supported on at most $2(n-1)$ coordinates. By Theorem 10.2 all that remains is to find a poly-time strong separation oracle for K. On input $w \in \mathbb{Q}^E$ we need to decide whether $w \in K$, and if not, provide a violated inequality. If $w \notin B$ then one of the inequalities $x_e \geq 1$ is violated, so let us assume

$w \in B$. We calculate the distance $d_w(u, v)$ for each pair of vertices $u, v \in V$. If $w(P_{u,v}) = d_w(u, v)$ for each $u, v \in V$ then $w \in K$. Otherwise there exist $u, v \in V$ and a u-v path Q in G s.t. $w(Q) = d_w(u, v) < w(P_{u,v})$, which means that the inequality $\sum_{e \in P_{u,v}} x_e - \sum_{e \in Q} x_e \le 0$ is violated.

For the strict case define K in the same way except for $u, v \in V$ we set

$$A_{u,v} := \left\{ x \in \mathbb{R}^E : \forall\, Q \in \mathcal{Q}_{u,v} \sum_{e \in P_{u,v}} x_e - \sum_{e \in Q} x_e \le -1 \right\}.$$

But now we need to verify, given $w \in \mathbb{Q}^E$ not only that $w(P_{u,v}) = d_w(u, v)$ but that $P_{u,v}$ is the unique shortest path between u and v. If this is not the case, and $w(P_{u,v}) = d_w(u, v) = w(Q)$ for some u-v path $Q \ne P_{u,v}$, then the inequality $\sum_{e \in P_{u,v}} x_e - \sum_{e \in Q} x_e \le -1$ is violated. To find such Q if one exists, we calculate $\min d_w(u, z) + d_w(z, v)$ over all vertices $z \notin P_{u,v}$. □

It follows from Robertson and Seymour's forbidden minor theory that the metrizability of graphs can be efficiently decided.

Theorem 10.3 *Graph Metrizability and Strict Graph Metrizability can be decided in polynomial time.*

Here is what we need from the graph minors theory.

Theorem 10.4 [18] *Fix a graph H. It can be decided in polynomial time whether a given graph G contains a subdivision of H.*

Proof of Theorem 10.3 This follows immediately from Theorem 10.4: The graph G is not metrizable if and only if it contains a subdivision of some $H \in \mathcal{F}_M$, and the set $\mathcal{F}_M$ is finite, by Theorem 8.1. The proof for the strict case is identical. □

10.1 Poly-Time vs. Practical Algorithms

While Theorem 10.3 proves the existence of a polynomial time algorithm to decide graph metrizability, we still lack a practical algorithm that achieves this. There are several reasons for this lacuna. While we know that $\mathcal{F}_M$ and $\mathcal{F}_{SM}$ are both finite, we are far from having the complete catalog. Even if we get to know the entire list of these minimal graphs, it is not inconceivable that the sheer size of these sets makes the algorithm in the proof of Theorem 10.3 impractical. Thus, the search of a viable algorithm to decide metrizability is still on. There are really two problems at hand. It is reasonable to expect (but we do not know whether or not this is the case) that current LP solvers can practically find a certificate for the non-metrizability of a given non-metrizable path system. In some cases reported throughout the paper such a certificate was found by hand. In contrast, we have only a brute force algorithm[2] to prove that a given graph G is metrizable. Namely it generates all possible consistent path systems

[2] https://github.com/dcizma1/testing-graph-metrizability.

in G and checks each for metrizability using a linear program as in the proof of Theorem 10.1. Needless to say, this is practical only with small graphs. Indeed, this is how we found the non-metrizable graphs in Fig. 22 as well as path systems realizing their non-metrizability. Note, however, that we do have humanly verifiable proofs that all the graph in Fig. 22 are non-metrizable. These proofs can be found in Appendix A.

11 Open Problems

This paper suggests numerous open problems and new avenues of research. Below we list some of those.

Here is the issue that we consider most pressing. We have seen throughout the paper several certificates that certain graphs are non-metrizable. These proofs proceed by comparing the weights of chosen paths to alternative ones. These inequalities are then combined to conclude that certain edge weights are non-positive.

Open Problem 11.1 *Do there exist humanly verifiable certificates that certain graphs are metrizable?*

Open Problem 11.2 *Can it be decided in polynomial time whether a given consistent partial path system can be extended to a full consistent path system?*

Open Problem 11.3 *The graph $\Theta_{a,b,c}$ has two vertices of degree* 3 *that are connected by three openly disjoint paths of a, b, c edges respectively. By Corollary 7.5, $\Theta_{a,b,c}$ is metrizable when* $\min\{a, b, c\} = 1$. *Also, $\Theta_{3,3,4} \in \mathcal{F}_M$ (see Fig. 22k). However, we do not know whether $\Theta_{a,b,c}$ is metrizable or not when* $\min\{a, b, c\} = 2$ *or* $a = b = c = 3$.

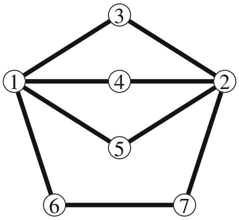
Can we quantify the level of a graph's non-metrizability? Let Π_G be the collection of all consistent path systems in G, and let $\mathcal{M}_G \subseteq \Pi_G$ be the collection of all those which are metrizable. Whether or not G is metrizable is expressed by this inclusion being proper or not.

Open Problem 11.4 *We suspect that there exist* 2*-connected n-vertex graphs G for which $|\mathcal{M}_G| = o_n(|\Pi_G|)$. Actually we even believe that this is the case for* most *graphs.*

Open Problem 11.5 *Associated with every connected graph G is $\mathcal{A}_{\mathcal{G}}$, a hyperplane arrangement in $\mathbb{R}^{E(G)}$ which encodes a lot of information on metrizable path systems in G. If P and Q are openly disjoint paths between the same two vertices in G, then the hyperplane $\left\{\sum_{e\in P} x_e = \sum_{f\in Q} x_f\right\}$ is in $\mathcal{A}_{\mathcal{G}}$. It would be interesting to investigate the basic features of such arrangements.*

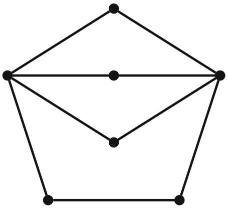
Open Problem 11.6 *Find a complete list of the graphs in $\mathcal{F}_M$ and $\mathcal{F}_{SM}$.*

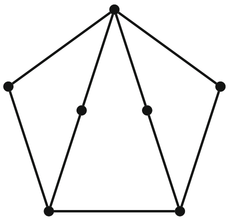
Open Problem 11.7 *It makes sense to speak of consistent path systems in* 1*-dimensional CW complexes. The case of S^1 viewed as a* 1*-dimensional CW complex with a single vertex and a single edge is considered in Sect. 9. Is there an interesting theory of metrizability in this broader context?*

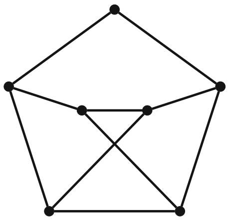


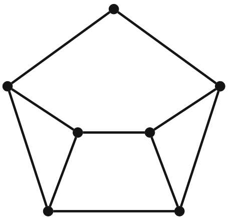
(132), (13), (14), (15), (16), (167), (23),
(24), (25), (276), (27), (314) (325), (316),
(3167), (415), (4276), (427), (516), (527), (67)

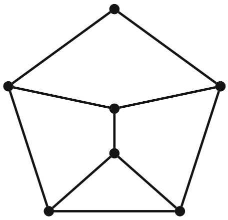
$$\begin{aligned} w_{2,3} + w_{2,5} &\le w_{1,3} + w_{1,5} \\ w_{1,4} + w_{1,5} &\le w_{2,4} + w_{2,5} \\ w_{2,4} + w_{2,7} + w_{6,7} &\le w_{1,4} + w_{1,6} \\ w_{1,3} + w_{1,6} + w_{6,7} &\le w_{2,3} + w_{2,7} \end{aligned} \quad \Longrightarrow \quad w_{6,7} \le 0$$

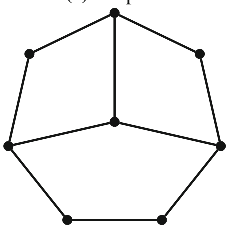

(a) Graph 22a

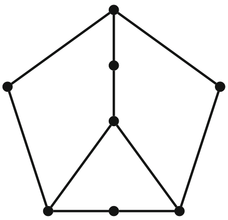

(b) Graph 22b

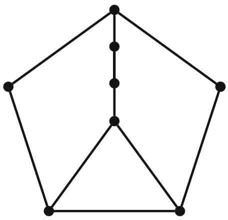

(c) Graph 22c

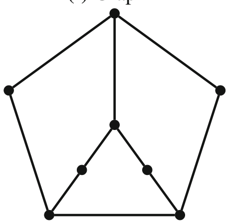

(d) Graph 22d

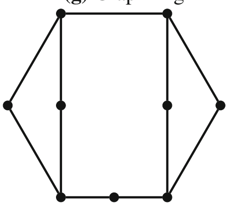

(e) Graph 22e


(f) Graph 22f


(g) Graph 22g


(h) Graph 22h


(i) Graph 22i


(j) Graph 22j

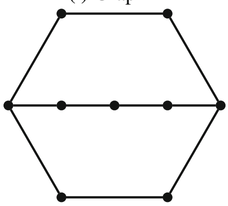
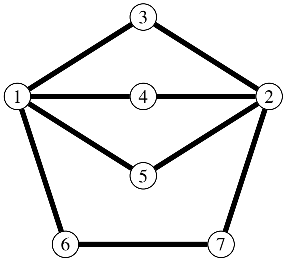

(k) Graph 22k

Fig. 22 Currently known topologically minimal non-metrizable graphs

Open Problem 11.8 *Which graphs $G = (V, E)$ have the property that every* ***neighborly*** *consistent path system is metrizable? I.e., we assume that for every $xy \in E$, this edge is the chosen x-y geodesic.*

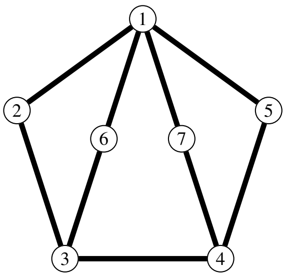
Appendix A: Certificates of Non-Metrizability

For each graph G in Fig. 22 we give a path system in G along with a system of inequalities a weight function inducing this path system must satisfy. In each case, these inequalities imply at least one edge in the graph must have a non-positive weight, showing the graph in not metrizable.



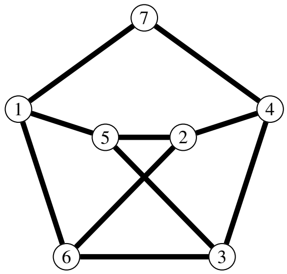
(132), (13), (14), (15), (16), (167), (23),
(24), (25), (276), (27), (314) (325), (316),
(3167), (415), (4276), (427), (516), (527), (67)

$$\begin{aligned} w_{2,3} + w_{2,5} &\leq w_{1,3} + w_{1,5} \\ w_{1,4} + w_{1,5} &\leq w_{2,4} + w_{2,5} \\ w_{2,4} + w_{2,7} + w_{6,7} &\leq w_{1,4} + w_{1,6} \\ w_{1,3} + w_{1,6} + w_{6,7} &\leq w_{2,3} + w_{2,7} \end{aligned} \implies w_{6,7} \leq 0$$



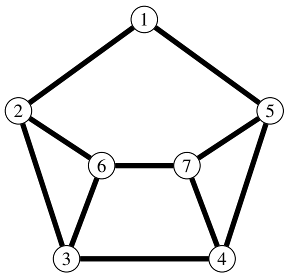
(12), (163), (174), (15), (16), (17), (23)
(234), (2345), (216), (217), (34), (345), (36)
(347), (45), (436), (47), (5436), (517), (6347)

$$\begin{aligned} w_{2,3} + w_{3,4} + w_{4,5} &\leq w_{1,2} + w_{1,5} \\ w_{1,2} + w_{1,6} &\leq w_{2,3} + w_{3,6} \\ w_{1,5} + w_{1,7} &\leq w_{4,5} + w_{4,7} \\ w_{3,6} + w_{3,4} + w_{4,7} &\leq w_{1,6} + w_{1,7} \end{aligned} \implies w_{3,4} \leq 0$$



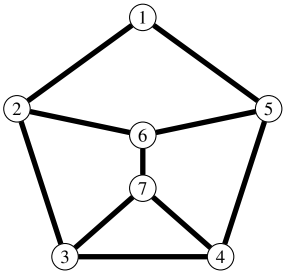
(1742), (163), (174), (15), (16), (17),
(263), (24), (25), (26), (247), (34), (35), (36),
(347), (435), (436), (47), (516), (5347), (617)

$$\begin{aligned} w_{2,6} + w_{3,6} &\leq w_{2,4} + w_{3,4} \\ w_{1,5} + w_{1,6} &\leq w_{3,5} + w_{3,6} \\ w_{1,7} + w_{4,7} + w_{2,4} &\leq w_{1,6} + w_{2,6} \\ w_{3,5} + w_{3,4} + w_{4,7} &\leq w_{1,5} + w_{1,7} \end{aligned} \quad \Longrightarrow \quad w_{4,7} \leq 0$$



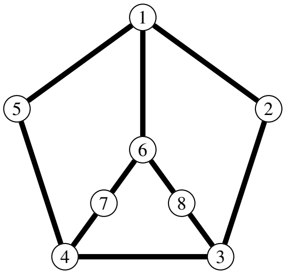
(12), (123), (1234), (15), (126), (157), (23),
(234), (215), (26), (2157), (34), (345), (36),
(367), (45), (476), (47), (5126), (57), (67)

$$\begin{aligned} w_{1,2} + w_{2,3} + w_{3,4} &\leq w_{1,5} + w_{4,5} \\ w_{1,2} + w_{1,5} + w_{5,7} &\leq w_{2,6} + w_{6,7} \\ w_{1,5} + w_{1,2} + w_{2,6} &\leq w_{5,7} + w_{6,7} \\ w_{3,6} + w_{6,7} &\leq w_{3,4} + w_{4,7} \\ w_{4,7} + w_{6,7} &\leq w_{3,4} + w_{3,6} \\ w_{3,4} + w_{4,5} &\leq w_{1,5} + w_{1,2} + w_{2,3} \end{aligned} \quad \Longrightarrow \quad w_{1,2} \leq 0$$



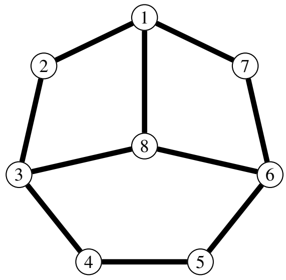
(12), (1543), (154), (15), (156), (1567),
(23), (2154), (215), (26), (237), (34), (345)
(326), (37), (45), (47), (476), (56), (567), (67)

$$\begin{aligned} w_{1,5} + w_{4,5} + w_{3,4} &\le w_{1,2} + w_{2,3} \\ w_{1,2} + w_{1,5} + w_{4,5} &\le w_{2,3} + w_{3,4} \\ w_{2,3} + w_{3,7} &\le w_{2,6} + w_{6,7} \\ w_{2,3} + w_{2,6} &\le w_{3,7} + w_{6,7} \\ w_{5,6} + w_{6,7} &\le w_{4,5} + w_{4,7} \\ w_{4,7} + w_{6,7} &\le w_{4,5} + w_{5,6} \end{aligned} \quad \Longrightarrow \quad w_{1,5} \le 0$$



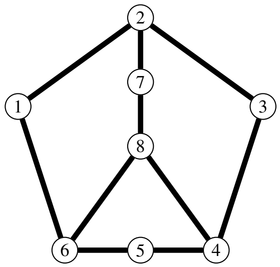
(12), (1543), (154), (15), (16), (167), (168), (23),
(234), (2345), (216), (2167), (2168), (34), (345),
(386), (347), (38), (45), (4516), (47), (438),
(516), (547), (5168), (67), (68), (7438)

$$\begin{aligned} w_{4,7} + w_{3,4} + w_{3,8} &\le w_{6,7} + w_{6,8} \\ w_{2,3} + w_{3,4} + w_{4,5} &\le w_{1,2} + w_{1,5} \\ w_{1,5} + w_{1,6} + w_{6,8} &\le w_{4,5} + w_{3,4} + w_{3,8} \\ w_{1,2} + w_{1,6} + w_{6,7} &\le w_{2,3} + w_{3,4} + w_{4,7} \end{aligned} \quad \Longrightarrow \quad w_{1,6} \le 0$$



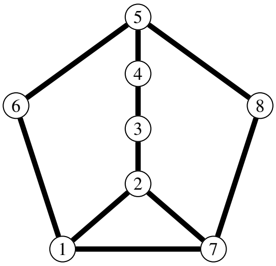
(12), (123), (17654), (1765), (176), (17), (18),
(23), (234), (2345), (23456), (217), (218), (34),
(345), (3456), (3217), (38), (45), (456), (4567),
(438), (56), (567), (5438), (67), (68), (768)

$$\left.\begin{aligned} w_{4,5}+w_{3,4}+w_{3,8} &\le w_{5,6}+w_{6,8}\\ w_{1,2}+w_{1,8} &\le w_{2,3}+w_{3,8}\\ w_{6,7}+w_{6,8} &\le w_{1,7}+w_{1,8}\\ w_{1,7}+w_{6,7}+w_{5,6}+w_{4,5} &\le w_{1,2}+w_{2,3}+w_{3,4}\\ w_{2,3}+w_{3,4}+w_{4,5}+w_{5,6} &\le w_{1,2}+w_{1,7}+w_{6,7}\\ w_{2,3}+w_{1,2}+w_{1,7} &\le w_{3,4}+w_{4,5}+w_{5,6}+w_{6,7} \end{aligned}\right. \implies \quad w_{4,5}\le 0$$



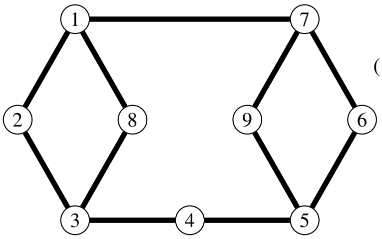
(12), (123), (1234), (165), (16), (1687), (168),
(23), (234), (2165), (216), (27), (278), (34),
(345), (3456), (327), (3278), (45), (456), (487),
(48), (56), (5487), (548), (67), (68), (78)

$$\left.\begin{aligned} w_{1,6}+w_{6,8}+w_{7,8} &\le w_{1,2}+w_{2,7}\\ w_{2,3}+w_{2,7}+w_{7,8} &\le w_{3,4}+w_{4,8}\\ w_{4,5}+w_{4,8} &\le w_{5,6}+w_{6,8}\\ w_{1,2}+w_{2,3}+w_{3,4} &\le w_{1,6}+w_{5,6}+w_{4,5}\\ w_{1,2}+w_{1,6}+w_{5,6} &\le w_{2,3}+w_{3,4}+w_{4,5}\\ w_{3,4}+w_{4,5}+w_{5,6} &\le w_{2,3}+w_{1,2}+w_{1,6} \end{aligned}\right. \implies \quad w_{7,8}\le 0$$



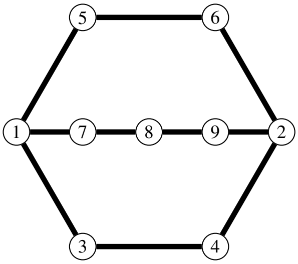
(12), (16543), (1654), (165), (16), (17), (178),
(23), (234), (2345), (216), (27), (23458), (34),
(345), (3456), (327), (3458), (45), (456), (4327),
(458), (56), (587), (58), (6587), (658), (78)

$$\begin{aligned} w_{1,6}+w_{5,6}+w_{4,5}+w_{3,4} &\le w_{1,2}+w_{2,3} \\ w_{2,3}+w_{3,4}+w_{4,5}+w_{5,8} &\le w_{2,7}+w_{7,8} \\ w_{5,6}+w_{5,8}+w_{7,8} &\le w_{1,6}+w_{1,7} \\ w_{3,4}+w_{2,3}+w_{2,7} &\le w_{4,5}+w_{5,8}+w_{7,8} \\ w_{1,7}+w_{7,8} &\le w_{1,6}+w_{5,6}+w_{5,8} \\ w_{1,2}+w_{1,6} &\le w_{2,3}+w_{3,4}+w_{4,5}+w_{5,6} \end{aligned} \quad \Longrightarrow \quad w_{3,4}\le 0$$



(12), (183), (1834), (1765), (176), (17), (18), (179),
(23), (234), (2345), (23456), (217), (218), (2179),
(34), (345), (3456), (3817), (38), (3459), (45),
(456), (43817), (438), (459), (56), (567), (5438),
(59), (67), (6718), (679), (718), (79), (83459)

$$\begin{aligned} w_{2,3}+w_{3,4}+w_{4,5}+w_{5,6} &\le w_{1,2}+w_{1,7}+w_{6,7} \\ w_{3,4}+w_{3,8}+w_{1,8}+w_{1,7} &\le w_{4,5}+w_{5,6}+w_{6,7} \\ w_{3,8}+w_{3,4}+w_{4,5}+w_{5,9} &\le w_{1,8}+w_{1,7}+w_{7,9} \\ w_{1,7}+w_{6,7}+w_{5,6} &\le w_{1,8}+w_{3,8}+w_{3,4}+w_{4,5} \\ w_{1,2}+w_{1,8} &\le w_{2,3}+w_{3,8} \\ w_{6,7}+w_{7,9} &\le w_{5,6}+w_{5,9} \end{aligned} \quad \Longrightarrow \quad w_{3,4}\le 0$$



(17892), (13), (134), (15), (156), (17), (178), (1789),
(243), (24), (265), (26), (2987), (298), (29), (34),
(315), (3156), (317), (3178), (3429), (4265), (426),
(4317), (43178), (429), (56), (517), (5178),
(51789), (62987), (6298), (629), (78), (789), (89)

$$\begin{aligned} w_{2,6}+w_{2,9}+w_{8,9}+w_{7,8} &\le w_{5,6}+w_{1,5}+w_{1,7}\\ w_{3,4}+w_{1,3}+w_{1,7}+w_{7,8} &\le w_{2,4}+w_{2,9}+w_{8,9}\\ w_{1,5}+w_{1,7}+w_{7,8}+w_{8,9} &\le w_{5,6}+w_{2,6}+w_{2,9}\\ w_{2,4}+w_{2,6}+w_{5,6} &\le w_{3,4}+w_{1,3}+w_{1,5}\\ w_{1,3}+w_{1,5}+w_{5,6} &\le w_{3,4}+w_{2,4}+w_{2,6}\\ w_{3,4}+w_{2,4}+w_{2,9} &\le w_{1,3}+w_{1,7}+w_{7,8}+w_{8,9} \end{aligned} \quad \Longrightarrow \quad w_{7,8} \le 0$$

References

1. Blokhuis, A., Brouwer, A.E.: Geodetic graphs of diameter two. Geom. Dedic. **25**(1–3), 527–533 (1988)
2. Bodwin, G.: On the structure of unique shortest paths in graphs. In: 13th Annual ACM-SIAM Symposium on Discrete Algorithms (San Diego 2019), pp. 2071–2089. SIAM, Philadelphia (2019)
3. Bosák, J., Kotzig, A., Znám, Š.: Strongly geodetic graphs. J. Combin. Theory **5**(2), 170–176 (1968)
4. Bridgland, M.F.: Geodetic Graphs and Convexity. PhD thesis, Louisiana State University and Agricultural & Mechanical College (1983). https://digitalcommons.lsu.edu/gradschool_disstheses/3917/
5. Chartrand, G., Harary, F.: Planar permutation graphs. Ann. Inst. H. Poincaré Sect. B **3**(4), 433–438 (1967)
6. Dawes, R.W.: Minimally 3-connected graphs. J. Combin. Theory Ser. B **40**(2), 159–168 (1986)
7. Dekel, Y., Lee, J.R., Linial, N.: Eigenvectors of random graphs: nodal domains. Random Struct. Algorithms **39**(1), 39–58 (2011)
8. Diestel, R.: Graph Theory. Graduate Texts in Mathematics, vol. 173. Springer, Berlin (2018)
9. Dirac, G.A.: A property of 4-chromatic graphs and some remarks on critical graphs. J. Lond. Math. Soc. **27**(1), 85–92 (1952)
10. Grötschel, M., Lovász, L., Schrijver, A.: Geometric Algorithms and Combinatorial Optimization. Algorithms and Combinatorics: Study and Research Texts, vol. 2. Springer, Berlin (1988)
11. Higman, G.: Ordering by divisibility in abstract algebras. Proc. Lond. Math. Soc. **2**, 326–336 (1952)
12. Hoory, Sh., Linial, N., Wigderson, A.: Expander graphs and their applications. Bull. Am. Math. Soc. **43**(4), 439–561 (2006)
13. Liu, Ch.-H., Thomas, R.: Robertson's conjecture I. Well-quasi-ordering bounded tree-width graphs by the topological minor relation (2020). arXiv:2006.00192
14. Lovász, L.: Graphs and Geometry. American Mathematical Society Colloquium Publications, vol. 65. American Mathematical Society, Providence (2019)
15. Ollivier, Y.: Ricci curvature of Markov chains on metric spaces. J. Funct. Anal. **256**(3), 810–864 (2009)
16. Ore, O.: Theory of Graphs. American Mathematical Society Colloquium Publications, vol. 38. American Mathematical Society, Providence (1962)
17. Parthasarathy, K.R., Srinivasan, N.: Geodetic blocks of diameter three. Combinatorica **4**(2–3), 197–206 (1984)
18. Robertson, N., Seymour, P.D.: Graph minors. XIII. The disjoint paths problem. J. Combin. Theory Ser. B **63**(1), 65–110 (1995)