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Note

A New Derivation of the Counting Formula for Young Tableaux*

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A new very short proof of the counting formula for Young tableaux is given. Its equivalence with the hook formula is easy to establish.

Given integers $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ the corresponding Young tableau is that part of an $a_1 \times n$ matrix consisting of all (i,j) entries with $n \geqslant j \geqslant 1$, $a_j \geqslant i \geqslant 1$. The problem which we discuss here is: In how many ways can the integers $1,\dots, \sum_1^n a_i$ be placed into the entries of the $a_1,\dots, a_n$ Young tableau so that any row and any column forms an increasing sequence? We call this number $F(a_1,\dots, a_n)$ and we give a new formula for $F(a_1,\dots, a_n)$. The known solution for this problem (see [1, 2] for more details) is by the so-called *hook formula* which can be easily derived from our formula.

Notice that $F(a_1,\dots, a_n)$ is the unique solution for difference equation (1) under initial conditions (2)–(4).

$$\begin{aligned}\varphi(a_1,\dots, a_n) = \varphi(a_1 - 1, a_2,\dots, a_n) + \varphi(a_1, a_2 - 1,\dots, a_n)&\\ + \cdots + \varphi(a_1,\dots, a_{n-1}, a_n - 1),&\qquad (1)\end{aligned}$$

$$\varphi(a_1,\dots, a_{n-1}, 0) = \varphi(a_1,\dots, a_{n-1}), \qquad (2)$$

$$\varphi(0,\dots, 0) = 1, \qquad (3)$$

If for some

$$1 \leqslant i \leqslant n-1, \qquad a_{i+1} = a_i + 1, \qquad \text{then} \quad \varphi = 0. \qquad (4)$$

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Equation (1) states that the integer $\sum_1^n a_i$ has to be placed at the lower end of the columns in the tableau. Conditions (2)–(4) are obvious. Note also that (1)–(4) define a unique solution. Now we state our

THEOREM.

$$F(a_1,\ldots, a_n)= \sum_{\pi\in S_n} \binom{S}{a_1+\pi(1)-1,\ldots, a_n+\pi(n)-n} \sigma(\pi), \tag{5}$$

where S_n *is the symmetric group acting on* $\{1,\ldots, n\}$, $S=\sum_1^n a_i$, $\binom{N}{n_1,\ldots,n_t}=N!/(n_1!\cdots n_t!)$ *and* $\sigma(\pi)$ *is the sign of the permutation* $\pi\in S_n$.

Proof. We only have to show that (5) satisfies (1)–(4). That is satisfies (1) is clear because it is a linear combination of multinomial coefficients and multinomial coefficients satisfy (1). To see (2), notice that if $a_n=0$, then the only nonzero terms in (5) are those where $\pi\in S_n$ satisfies $\pi(n)=n$, and so it reduces to

$$\sum_{\pi\in S_{n-1}} \binom{S}{a_1+\pi(1)-1,\ldots, a_{n-1}+\pi(n-1)-(n-1)} \sigma(\pi)=F(a_1,\ldots, a_{n-1}).$$

If $a_1=\cdots=a_n=0$ in (3) the only nonzero term is that for the identity permutation so we get 1. To see (4), let τ be the transposition of $i, i+1$. The term in (5) for $\pi\in S_n$ and that for $\pi\tau$ differ only in sign. Since $\tau^2=id$ this pairs all terms in (5) and so the sum reduces to zero and (4) follows.

Next we want to exhibit the equivalence of (5) and the hook formula. We refer the reader to [1] for the explanation and conventional proof of the hook formula. After grouping terms in the hook formula, it can be written as

$$F(a_1,\ldots, a_n)=\frac{S!}{\prod (a_i+n-i)!}\prod_{i<j} (a_i-i-a_j+j) \tag{6}$$

and we want to show the equivalence of (5) and (6). Letting $b_i=a_i+n-i$, we can rewrite (6) as

$$F(a_1,\ldots, a_n)=\frac{S!}{\prod b_i!}\prod_{i<j} (b_i-b_j). \tag{7}$$

Now $\prod_{i<j} (b_i-b_j)$ is the Van der Monde determinant

$$\begin{vmatrix} 1 & & 1 \\ b_n & & b_1 \\ \vdots & & \vdots \\ b_n^{n-1} & \cdots & b_1^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & & 1 \\ b_n & & b_1 \\ \vdots & & \vdots \\ b_b(b_n-1)\cdots(b_n-n+1) & \cdots & b_1(b_1-1)\cdots(b_1-n+1) \end{vmatrix}$$

So (7) can be rewritten as

$$F(a_1,\ldots,a_n) = S!\begin{vmatrix} \frac{1}{b_n!} & & \frac{1}{b_1!} \\ \frac{1}{(b_n-1)!} & & \frac{1}{(b_1-1)!} \\ \vdots & \cdots & \vdots \\ \frac{1}{(b_n-n+1)!} & & \frac{1}{(b_1-n+1)!} \end{vmatrix}$$

$$= S!\begin{vmatrix} \frac{1}{a_n!} & & \frac{1}{(a_1+n-1)!} \\ \vdots & \cdots & \vdots \\ \frac{1}{(a_n-n+1)!} & & \frac{1}{a_1!} \end{vmatrix}.$$

Expand this determinant and (5) follows.

A nice feature of formula (5) is that it has an inclusion–exclusion form. This may indicate that similar formulas can be found for the following more general question: Given a partially ordered set $(P, \geqslant)$, how many linear extensions does it have? The problem of counting Young tableaux is a special case, where P is an upper ideal in the product of two chains. We hope to apply the methods of this note to other instances of this general problem.

References

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2. R. P. Stanley, Theory and applications of plane partitions, *Stud. Appl. Math.* **50** (1971), 167–88; 259–79.