

Irreducible Non-Metrizable Path Systems in Graphs

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Abstract

A *path system* \mathcal{P} in a graph $G = (V, E)$ is a collection of paths with a unique uv path for every two vertices $u, v \in V$. We say that \mathcal{P} is *consistent* if for any path $P \in \mathcal{P}$, every subpath of P is also in \mathcal{P} . It is *metrizable* if there exists a positive weight function $w : E \rightarrow \mathbb{R}_{>0}$ such that \mathcal{P} is comprised of w -shortest paths. We call \mathcal{P} *irreducible* if there does not exist a partition $V = A \sqcup B$ such that \mathcal{P} restricts to a path system on both $G[A]$ and $G[B]$. In this paper, we construct an infinite family of non-metrizable irreducible consistent path systems on certain Paley graphs.

1 Introduction

There is a rich history of viewing graphs as geometric objects, see [13] for some of the literature. The present paper is a step in our ongoing effort to develop a geodesic geometry on graphs. The earliest work in this vein that we are aware of is due to Ore [14], who defined a graph to be *geodetic* if there is unique shortest path between any two vertices. He posed the problem of characterizing such graphs, and while some progress has been made, (e.g., [6, 7, 3, 15]) a complete characterization of such graphs remains elusive.

Every connected graph $G = (V, E)$ gives rise to a family of metric spaces by assigning positive weights to each edge, $w : E \rightarrow \mathbb{R}^+$, and defining the distance between two vertices to be the shortest length of a path which connects them. By varying the edge weights w we obtain various metrics on V . However, as we showed in [10], most graphs carry a richer family of geometric structures which extend beyond these metrics. A *path system* \mathcal{P} in a graph $G = (V, E)$ is a collection of paths in G such that for every $u, v \in V$ there is exactly one path $P_{u,v}$ in \mathcal{P} connecting u and v . We say that \mathcal{P} is *consistent* if it is closed under taking subpaths. Namely, for every path P in \mathcal{P} , any subpath of P is also a path in \mathcal{P} . Every metric on V that is induced by some $w : E \rightarrow \mathbb{R}^+$ gives rise to a consistent path system, by putting in \mathcal{P} only w -shortest paths. (If the w -shortest paths are not unique, we can always break ties between competing paths in a consistent way.) A path system that comes from such w is said to be *metrizable*. We say that $G = (V, E)$ is metrizable if every consistent path system on G is metrizable. One can ask whether all consistent path systems are metrizable. It turns out this is far from the truth. The main findings so far in this newly emerging research area can be briefly described as follows: (i) Metrizable graphs are very rare, yet (ii) all outerplanar graphs are metrizable, and

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(iii) Metrizable is polynomial time decidable. We encourage the reader to consult our paper [10] for a full account of what is currently known in this area, but note that the present paper is entirely self-contained. Some of these definitions and questions make sense also for *partial* path systems. Strict metrizable of partial path systems in both directed and undirected graphs was investigated in [4].

Here we introduce the notion of an *irreducible* path system. Such path systems are “atomic”, in that they cannot be decomposed into two smaller consistent path systems.

Definition 1.1. Let \mathcal{P} be a path system in a graph $G = (V, E)$. A partition $V = A \sqcup B$ with $A, B \neq \emptyset$ is called a reduction of \mathcal{P} if all vertices of the path $P_{u,v}$ belong to A (resp. B) whenever $u, v \in A$ (resp. $u, v \in B$). A path system with no reductions is said to be irreducible.

These notions can be illustrated on the Petersen graph Π . This graph is nonmetrizable as the following path system shows: Since $\text{diam}(\Pi) = 2$ and $\text{girth}(\Pi) = 5$, every two vertices in Π are connected by a unique path of length 1 or 2. In \mathcal{P} , the path system that we construct, most pairs are connected by these shortest paths. There are five exceptional pairs of nonadjacent pairs of vertices which are connected in \mathcal{P} by one of the five colored paths (2, 1, 6, 8), (3, 2, 7, 9), (4, 3, 8, 10), (5, 4, 9, 6), (1, 5, 10, 7) in Figure 1. It is easily verified that \mathcal{P} is consistent, and as shown in [10], it is non-metrizable. However, this path system admits a *reduction* A, B where A and B are the set of dark and light vertices, respectively, as seen in Figure 1. In fact, all of the explicit examples of non-metrizable path systems given in [10] are reducible. One can ask if this is always the case, i.e. does a non-metrizable system always admit a reduction? Said differently, is it possible to obtain any non-metrizable path system by “combining” together two simpler path systems?

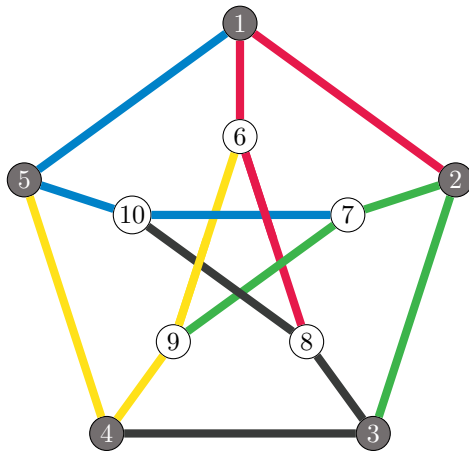


Figure 1: A reducible non-metrizable path system in the Petersen Graph

The main result of the present paper is a construction of an infinite family of non-metrizable irreducible path systems. The construction makes use of some elementary ideas from number theory which we now briefly go over. For a more in-depth overview see [12]. These path systems are defined on certain Paley graphs G_p , where $p \equiv 1 \pmod{4}$ is a prime integer. The vertex set of G_p is the finite field \mathbb{F}_p , and there is an edge connecting $a, b \in \mathbb{F}_p$ iff $a - b$ is a quadratic residue in \mathbb{F}_p . Since $p \equiv 1 \pmod{4}$, -1 is a quadratic residue of \mathbb{F}_p so that G_p is an undirected graph. For an overview of Paley graphs see [5]. We denote by R and N the sets of quadratic residues, resp. non-residues in \mathbb{F}_p . Recall that for $0 \neq a \in \mathbb{F}_p$ the Legendre symbol $\left(\frac{a}{p}\right)$ is a multiplicative

function which takes on values ± 1 with $\left(\frac{a}{p}\right) = 1$ iff a is a quadratic residue. In particular this means, $NR = N$, $NN = R$ and $RR = R$, where $AB = \{ab : a \in A, b \in B\}$. In order for our construction to work, we need to assume that $2, 3 \in N$ are both quadratic non-residues in \mathbb{F}_p . Using quadratic reciprocity it is easy to see that if $p \equiv 5 \pmod{24}$, then $-1 \in R$ and $2, 3 \in N$. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes $p \equiv 5 \pmod{24}$.

We define the following path system \mathcal{P}_p on G_p : Let $a, b \in \mathbb{F}_p$

- If $b - a \in R$ is a quadratic residue then $P_{a,b} := (a, b)$
- If $b - a = \pm 3$ then $P_{a,b} := (a, a \pm 1, a \pm 2, b)$
- If $b - a \in N$ is a quadratic non-residue and $b - a \neq \pm 3$ then $P_{a,b} := (a, \frac{b+a}{2}, b)$

We prove:

Theorem 1.2. *The path system \mathcal{P}_p is irreducible and non-metrizable for all primes $p > 5$ for which $-1 \in R$ is a quadratic residue and $2, 3 \in N$ are quadratic non-residues.*

In fact, the path system \mathcal{P}_p defines a *neighborly* path system in G_p , in that every edge of G_p is contained in \mathcal{P}_p .

2 Proof of Theorem 1.2

We first prove the path system \mathcal{P}_p is irreducible. Notice that \mathcal{P}_p is cyclically symmetric. Namely, if (a_1, \dots, a_r) is a path in \mathcal{P}_p then so is $(a_1 + x, \dots, a_r + x)$, for any $x \in \mathbb{F}_p$.

Proposition 2.1. *For $p \geq 30$, the path system \mathcal{P}_p is irreducible.*

Proof. Suppose toward contradiction that $V = A \sqcup B$ is a reduction of \mathcal{P}_p . Consider the Hamiltonian cycle $C = (0, 1, 2, 3, \dots, p-1, 0)$ in G_p . This reduction splits this cycle into $2k$ segments $C = \bigcup_1^{2k} R_i$, where $V(R_{2i-1}) \subset A$ and $V(R_{2i}) \subset B$ for all $1 \leq i \leq k$. Let l be the smallest length of these segments, where the length of a segment is the number of its edges. Due to \mathcal{P}_p 's rotational symmetry, we can and will assume that the shortest segment is R_1 , and $R_1 = (-\frac{l}{2}, \dots, -1, 0, 1, \dots, \frac{l}{2})$ for even l and $R_1 = (-\lfloor \frac{l}{2} \rfloor, -\lfloor \frac{l}{2} \rfloor + 1, \dots, -1, 0, 1, \dots, \lfloor \frac{l}{2} \rfloor, \lfloor \frac{l}{2} \rfloor + 1)$ for odd l .

If $l = 0$, then $R_1 = (0)$ and $-1 \in R_{2k}$, $1 \in R_2$. By assumption $V(P_{-1,1}) \subset B$, since both $V(R_{2k}), V(R_2) \subset B$. However, $P_{-1,1} = (-1, 0, 1)$, whereas $V(R_1) = \{0\} \subset A$. A similar argument works for $1 \leq l \leq 3$.

We next consider the case where l is even and $4 \leq l \leq \frac{p}{4}$. Again, by symmetry we may assume that $R_1 = (-\frac{l}{2}, \dots, -1, 0, 1, \dots, \frac{l}{2})$. Since l is minimal, $|R_2| \geq l$ and $|R_{2k}| \geq l$, and hence $(\frac{l}{2} + 1, \frac{l}{2} + 2, \dots, \frac{3l}{2} + 1) \subseteq R_2$ and $(-\frac{l}{2} - 1, -\frac{l}{2} - 2, \dots, -\frac{3l}{2} - 1) \subseteq R_{2k}$. We claim next that the range $[\frac{l}{2} + 1, \frac{3l}{2} + 1]$ is comprised only of non-residues. For if $b \in [\frac{l}{2} + 1, \frac{3l}{2} + 1]$, is a quadratic residue, then by construction $P_{-b,b} = (-b, 0, b)$. This is a contradiction, since $-b \in V(R_{2k}) \subset B, b \in V(R_1) \subset B, 0 \in V(R_0) \subset A$. On the other hand, the interval $[\frac{l}{2} + 1, \frac{3l}{2} + 1]$ contains both $\frac{l}{2} + 1$ and $l + 2$, where one is a quadratic residue and the other is not, since their ratio is $2 : 1$, and by assumption $2 \in N$. When l is odd an identical argument works for the same range.

There remains the range $l > \frac{p}{4}$. Again, we assume that l is even since the proof when l is odd is essentially identical. As l is the length of the smallest segment and C contains an even number of segments, it necessarily follows that C splits into exactly two segments $C = R_1 \sqcup R_2$, where $R_1 = (-\frac{l}{2}, \dots, -1, 0, 1, \dots, \frac{l}{2})$, $l \leq \frac{p}{2}$.

Set $K = \frac{p-1}{2}$ and notice that $K \in V(R_2) = B$. If $P_{K-a, K+a+1} = (K-a, 0, K+a+1)$ for some small $a > 0$, we again encounter a contradiction, since this path starts and ends in $V(R_2) = B$ and its middle vertex is in $V(R_1) = A$. The choice $a = 1$ won't do, since $P_{K-1, K+2} = (K-1, K, K+1, K+2)$. Also, for $P_{K-a, K+a+1} = (K-a, 0, K+a+1)$ to hold, $2a+1$ must be a non-residue, for otherwise $P_{K-a, K+a+1} = (K-a, K+a+1)$. Thus, if $5 \in N$, we can use $a = 2$. If not, and $5 \in R$, then $15 \in N$ because we are assuming $3 \in N$. We can, therefore, take $a = 7$, which is "small enough" under the assumption $p \geq 30$. \square

Let L_p be the maximum length of a consecutive segment of non-residues in \mathbb{F}_p . We remark that upper bounds on L_p can be used to derive shorter proofs of Proposition 2.1. E.g., Hummel [11] showed that $L_p \leq \sqrt{p}$ for every prime $p \neq 13$, and Burgess [8] proved that $L_p \leq O(p^{1/4} \log p)$. We remark that the smallest prime satisfying the conditions of Theorem 1.2 is 29. The path system \mathcal{P}_{29} can be shown to be irreducible with an argument similar to the one used in the proof of Proposition 2.1.

It remains to show that:

Proposition 2.2. *The path system \mathcal{P}_p is not metrizable.*

Had \mathcal{P}_p been metrizable, there would exist a weight function $w : E \rightarrow \mathbb{R}_{>0}$ such that

$$w(P_{u,v}) \leq w(Q) \text{ for every } u, v \in V \text{ and every } uv \text{ path } Q. \quad (1)$$

In particular, this weight function must satisfy the following inequalities:

$$w(u, \frac{u+v}{2}) + w(\frac{u+v}{2}, v) \leq w(u, z) + w(z, v)$$

where $u, v, z \in \mathbb{F}_p$, $u-v \in N$, $u-z, v-z \in R$, $u-v \neq \pm 3$ and

$$w(u, u+1) + w(u+1, u+2) + w(u+2, v) \leq w(u, z) + w(z, v)$$

where $u, v, z \in \mathbb{F}_p$, $u-v \in N$, $u-z, v-z \in R$, $v-u = 3$.

Let φ_x be the rotation-by- x map

$$\varphi_x(a_1, \dots, a_r) = (a_1 + x, \dots, a_r + x).$$

Again we use the cyclic symmetry, i.e., invariance under φ_x of G_p and \mathcal{P}_p .

Due to cyclic symmetry, if w satisfies (1), so does $w \circ \varphi_x$. Moreover, the set of all w that satisfy (1) is a convex cone. Therefore, if w satisfies (1), then so does the weight function

$$\tilde{w} = \sum_{x \in \mathbb{F}_p} w \circ \varphi_x.$$

Note also that $\tilde{w}(x, y)$ depends only on $|x - y|$.

We associate a formal variable x_a with every quadratic residue $a \in R$. By the above discussion, if \mathcal{P}_p is metrizable then the following system of linear equations and inequalities is feasible:

$$\begin{aligned}
2x_a &\leq x_b + x_c & a, b, c \in R, \quad 2a = b + c \neq \pm 3 \\
3x_1 &\leq x_b + x_c & a, b, c \in R, \quad b + c = \pm 3 \\
x_a &= x_{-a} & a \in R \\
x_a &> 0 & a \in R
\end{aligned} \tag{*}$$

Therefore, to show that \mathcal{P}_p is non-metrizable it suffices to show that (*) is infeasible.

We consider instead a more general setup and ask if the following system of linear inequalities is feasible. It has $M + N$ inequalities in x_1, \dots, x_N , and it is given that $a_1, \dots, a_M \geq 2$.

$$\begin{aligned}
a_m x_{i_m} &\leq x_{j_m} + x_{k_m} & m = 1, \dots, M \\
x_i &> 0 & i = 1, \dots, N.
\end{aligned} \tag{**}$$

We associate a digraph \vec{D} with this system of linear inequalities. Its vertex set is $V(\vec{D}) = 1, 2, \dots, N$. The edge set is defined as follows: Any inequality $a_m x_\alpha \leq x_\beta + x_\gamma$ that appears in the system gives rise to an edge from α to β and one from α to γ . We say the system of inequalities is *strongly connected* if \vec{D} is strongly connected. We observe:

Lemma 2.3. *If (**) is a strongly connected system, then it is feasible if and only if $a_1 = a_2 = \dots = a_M = 2$. In this case $x_1 = x_2 = \dots = x_N$ is the only feasible solution.*

Proof. Clearly, $x_1 = x_2 = \dots = x_N$ is a feasible solution when $a_1 = a_2 = \dots = a_M = 2$. Consider a feasible solution $(x_1, \dots, x_N) \in \mathbb{R}^N$ and say that $x_\alpha = \max_{1 \leq j \leq N} x_j$. If there is a directed edge $(\alpha, \beta) \in E(\vec{D})$, then some inequality in the system must have the form $a x_\alpha \leq x_\beta + x_\gamma$. Here $a \geq 2$, and since x_α is maximal, necessarily $a = 2$ and $x_\alpha = x_\beta = x_\gamma$. By the same reasoning $x_\alpha = x_\delta$ whenever there is a directed path from α to δ in H . The claim follows since \vec{D} is strongly connected. \square

We show that the following subsystem of (*) is strongly connected, and hence by Lemma 2.3, it is infeasible. Let $R' := R \setminus \left\{ \frac{p+3}{2}, \frac{p-3}{2} \right\}$

$$\begin{aligned}
I_{a;b,c} : & \quad 2x_a \leq x_b + x_c & a, b, c \in R', \quad 2a = b + c \neq 3 \\
I_1 : & \quad 3x_1 \leq x_b + x_c & a, b, c \in R', \quad 3 = b + c \\
I_0 : & \quad x_a > 0 & a \in R'
\end{aligned} \tag{***}$$

The following theorem of Burgess [9, 8] shows that Paley graphs resemble random $G(n, 1/2)$ graphs (see also, e.g., [2, 1]).

Theorem 2.4. *Let $a_1, \dots, a_k \in \mathbb{F}_p$ be distinct. Then*

$$\left| \sum_{x=0}^{p-1} \prod_{i=1}^k \left(\frac{x - a_i}{p} \right) \right| \leq (k-1)\sqrt{p}$$

We also make use of the following well known lemma:

Lemma 2.5. For $a, b \in \mathbb{F}_p$, $a \neq b$,

$$\sum_{x=0}^{p-1} \binom{x-a}{p} \binom{x-b}{p} = -1$$

Proof. Since $\binom{x-a}{p} = \binom{(x-a)^{-1}}{p}$,

$$\sum_{x=0}^{p-1} \binom{x-a}{p} \binom{x-b}{p} = \sum_{\substack{x=0 \\ x \neq a}}^{p-1} \binom{(x-a)^{-1}}{p} \binom{x-b}{p} = \sum_{\substack{x=0 \\ x \neq a}}^{p-1} \binom{(x-b)(x-a)^{-1}}{p}.$$

Notice that $(x-b)(x-a)^{-1} = 1 + \frac{a-b}{x-a}$ takes on all values in \mathbb{F}_p except 1. Therefore

$$\sum_{\substack{x=0 \\ x \neq a}}^{p-1} \binom{x-a}{p} \binom{x-b}{p} = \sum_{\substack{x=0 \\ x \neq a}}^{p-1} \binom{(x-b)(x-a)^{-1}}{p} = -\binom{1}{p} + \sum_{x=0}^{p-1} \binom{x}{p} = -1.$$

□

We denote neighbor sets in G_p by Γ .

Lemma 2.6. Let $x, y, z \in G_p$ distinct vertices. Then

$$|\Gamma(x) \cap \Gamma(y) \setminus \Gamma(z)| = \frac{p}{8} + O(\sqrt{p})$$

in fact

$$\left| |\Gamma(x) \cap \Gamma(y) \setminus \Gamma(z)| - \frac{p}{8} \right| \leq 2\sqrt{p} + 4$$

Proof. Recall that $z \in \Gamma(a)$ iff $a - z \in R$. Therefore

$$\begin{aligned} |\Gamma(x) \cap \Gamma(y) \setminus \Gamma(z)| &= \frac{1}{8} \sum_{\substack{i=0 \\ i \neq x, y, z}}^{p-1} \left(1 + \binom{x-i}{p} \right) \left(1 + \binom{y-i}{p} \right) \left(1 - \binom{z-i}{p} \right) \\ &\quad + \frac{1}{4} \left(1 + \binom{x-z}{p} \right) \left(1 + \binom{y-z}{p} \right) \\ &= \frac{1}{8} \sum_{i=0}^{p-1} \left(1 + \binom{x-i}{p} \right) \left(1 + \binom{y-i}{p} \right) \left(1 - \binom{z-i}{p} \right) + K, \end{aligned}$$

where $|K| \leq 3$. Expanding the sum we get

$$\frac{p}{8} + \sum_{i=0}^{p-1} \left[\binom{x-i}{p} \binom{y-i}{p} - \binom{x-i}{p} \binom{z-i}{p} - \binom{y-i}{p} \binom{z-i}{p} - \binom{x-i}{p} \binom{y-i}{p} \binom{z-i}{p} \right].$$

The conclusion follows from Theorem 2.4 and Lemma 2.5. □

Lemma 2.7. The system $(***)$ is strongly connected for $p > 400$.

Proof. To prove this lemma we show that the following digraph \vec{D} on the vertex set R' is strongly connected. Every triple $x, y, z \in R'$ with $2x = y + z \neq 3$ gives rise to the edges $(x, y), (x, z) \in E(\vec{D})$. Also, whenever $y + z = 3$ for $y \neq z$ in R' we get the edges $(1, y), (1, z) \in E(\vec{D})$.

For any two vertices $a, b \in R'$, we find a path in \vec{D} from a to b . We first assume $4a \neq b$ and we find three distinct elements $\alpha, \beta, \gamma \in R'$ such that $2a = \alpha + \beta$ and $2\beta = \gamma + b$. By construction, $(a, \beta), (\beta, b) \in E(\vec{D})$ is a 2-step path from a to b . Let S be the set of $\beta \in R$ for which there exist $\alpha, \gamma \in R$ with $2a = \alpha + \beta$ and $2\beta = \gamma + b$.

$$S = (\Gamma(2a) \cap R) \setminus \Gamma(b/2)$$

Indeed, fix $\beta \in R$. Notice that $2a = \alpha + \beta$ for some $\alpha \in R$ iff $\beta \in \Gamma(2a)$. Similarly, $2\beta = \gamma + b$ for some $\gamma \in R$ iff $2\beta \in \Gamma(b)$, and since 2 is a non-residue $2\beta \in \Gamma(b)$ iff $\beta \notin \Gamma(b/2)$. As $R = \Gamma(0)$ and $0, 2a, \frac{b}{2}$ are all distinct we can use Lemma 2.6 and to get

$$|S| = |(\Gamma(2a) \cap R) \setminus \Gamma(b/2)| \geq \frac{p}{8} - 2\sqrt{p} - 4.$$

But aside of the requirement that $\beta \in S$ we also insist that $\beta, 2a - \beta, 2\beta - b \notin \left\{ \frac{p+3}{2}, \frac{p-3}{2} \right\}$. This rules out at most 6 eligible values for β . Such a β exists provided that $\frac{p}{8} - 2\sqrt{p} - 4 > 6$. This inequality holds when $p > 400$.

When $4a = b$ there is a 4-step path from a to b via β for any $\beta \in R' \setminus \{a, b, 4b, a/4\}$. \square

It remains to show that \mathcal{P}_p is non-metrizable for primes $5 < p < 400$ satisfying the conditions of Theorem 1.2, of which there are 10. This can be proven directly by showing the corresponding system of linear inequalities (*) is infeasible, see appendix A.

3 Open Problems

The notion of irreducible path systems suggests many open questions and conjectures. Here are some of them:

Conjecture 3.1. *Asymptotically almost every graph has an irreducible non-metrizable path system. Specifically, this holds with probability $1 - o_n(1)$ for $G(n, 1/2)$ graphs.*

Perhaps even more is true. Let Π_G be the collection of all consistent path systems in G , and let $\mathcal{R}_G \subseteq \Pi_G$ be the collection of all those which are metrizable or reducible.

Open Problem 3.2. *Is it true that $|\mathcal{R}_G| = o_n(|\Pi_G|)$ for asymptotically almost every n -vertex graph?*

If we wish to know whether a given path system is irreducible, we currently must resort to brute force searching. So we ask:

Open Problem 3.3. *Is there an efficient algorithm to determine if a given path system is irreducible?*

A reduction $V = A \sqcup B$ as in Definition 1.1 still does not tell the whole story.

Open Problem 3.4. *Let $G = (V, E)$ be a graph, a bipartition $V = V_1 \sqcup V_2$ of its vertex set, and path systems $\mathcal{Q}_1, \mathcal{Q}_2$ on $G(V_1), G(V_2)$ respectively. Under what conditions is there a consistent non-metrizable path system on G whose restriction to G_1, G_2 coincides with $\mathcal{Q}_1, \mathcal{Q}_2$ respectively?*

Other, more general notions of irreducibility suggest themselves. Let $k \geq 2$ be an integer and \mathcal{P} a path system on a graph G . We say \mathcal{P} is k -reducible if V can be partitioned into k nonempty parts $V = A_1 \sqcup \cdots \sqcup A_k$ s.t. $P_{u,v} \subseteq G[A_i]$ whenever $u, v \in A_i$. Otherwise we say \mathcal{P} is k -irreducible. Notice that the case for $k = 2$ coincides with the original definition of irreducibility.

Open Problem 3.5. *Do there exist k -irreducible path systems for all $k \geq 2$? Is it even the case that most graphs on $n > n_0(k)$ vertices have this property?*

A Certificates of Infeasibility

By the discussion in section Section 2, to prove \mathcal{P}_p is non-metrizable it suffices to show the systems (*) is infeasible. Below are certificates of infeasibility for (*) for all $5 < p < 400$ satisfying the conditions of Theorem 1.2. In each case, adding up all inequalities yields that some weight, x_a , must be non-positive, contradicting $x_a > 0$. Note we identify $x_{-a} = x_a$.

$p = 29$

$$\begin{aligned} 3x_1 &\leq x_4 + x_1 \\ 2x_4 &\leq x_1 + x_7 \\ 2x_5 &\leq x_1 + x_9 \\ 2x_9 &\leq x_4 + x_7 \\ 2x_7 &\leq x_5 + x_9 \end{aligned}$$

$p = 53$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_{24} &\leq x_1 + x_4 \\ 2x_4 &\leq x_7 + x_{15} \\ 2x_7 &\leq x_{15} + x_{24} \\ 2x_{15} &\leq x_1 + x_{24} \end{aligned}$$

$p = 101$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_4 &\leq x_{17} + x_{25} \\ 2x_{36} &\leq x_4 + x_{25} \\ 2x_{17} &\leq x_1 + x_{33} \\ 2x_{33} &\leq x_1 + x_{36} \\ 2x_{25} &\leq x_{17} + x_{33} \end{aligned}$$

$p = 149$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_4 &\leq x_1 + x_9 \\ 2x_{67} &\leq x_4 + x_{19} \\ 2x_9 &\leq x_1 + x_{19} \\ 2x_{19} &\leq x_{29} + x_{67} \\ 2x_{29} &\leq x_9 + x_{67} \end{aligned}$$

$p = 173$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_{84} &\leq x_1 + x_4 \\ 2x_4 &\leq x_1 + x_9 \\ 2x_9 &\leq x_{22} + x_{40} \\ 2x_{22} &\leq x_{40} + x_{84} \\ 2x_{40} &\leq x_9 + x_{84} \end{aligned}$$

$p = 197$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_4 &\leq x_{16} + x_{24} \\ 2x_{90} &\leq x_{16} + x_{33} \\ 2x_{16} &\leq x_1 + x_{33} \\ 2x_{24} &\leq x_1 + x_{47} \\ 2x_{33} &\leq x_{24} + x_{90} \\ 2x_{47} &\leq x_4 + x_{90} \end{aligned}$$

$p = 269$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_4 &\leq x_{37} + x_{45} \\ 2x_{127} &\leq x_1 + x_{14} \\ 2x_{14} &\leq x_{45} + x_{73} \\ 2x_{37} &\leq x_1 + x_{73} \\ 2x_{45} &\leq x_{37} + x_{127} \\ 2x_{73} &\leq x_4 + x_{127} \end{aligned}$$

$p = 293$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_{144} &\leq x_1 + x_4 \\ 2x_4 &\leq x_{141} + x_{144} \\ 2x_{141} &\leq x_{60} + x_{71} \\ 2x_{71} &\leq x_1 + x_{141} \\ 2x_{102} &\leq x_{60} + x_{144} \\ 2x_{60} &\leq x_{71} + x_{102} \end{aligned}$$

$p = 317$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_{156} &\leq x_1 + x_4 \\ 2x_4 &\leq x_1 + x_7 \\ 2x_7 &\leq x_{63} + x_{77} \\ 2x_{77} &\leq x_7 + x_{156} \\ 2x_{49} &\leq x_{63} + x_{156} \\ 2x_{63} &\leq x_{49} + x_{77} \end{aligned}$$

$p = 389$

$$\begin{aligned} 3x_1 &\leq x_1 + x_4 \\ 2x_4 &\leq x_{79} + x_{87} \\ 2x_{17} &\leq x_1 + x_{35} \\ 2x_{35} &\leq x_{17} + x_{87} \\ 2x_{157} &\leq x_4 + x_{79} \\ 2x_{79} &\leq x_1 + x_{157} \\ 2x_{87} &\leq x_{17} + x_{157} \end{aligned}$$

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