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A LOWER BOUND FOR THE CIRCUMFERENCE OF A GRAPH

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Let $G = (V, E)$ be a block of order n, different from K_n. Let $m = \min\{d(x) + d(y): [x, y] \notin E\}$. We show that if $m \leqslant n$ then G contains a cycle of length at least m.

1. Introduction and notations

We discuss only finite undirected graphs without loops and multiple edges. We prove the main theorem and show how Ore's theorem [3] on Hamiltonian graphs is easily deducible.

Let $G = (V, E)$ be a graph where V, E are the vertex and edge sets respectively. The cardinality of a set S is denoted by $|S|$. n stands throughout for $|V|$, the order of G. $\Gamma(x)$ is the set of vertices adjacent to x. $d(x) = |\Gamma(x)|$, the degree of x. A *block*, or equivalently a 2-connected graph is a connected graph which remains connected after the deletion of any of its vertices. The *circumference* of G, $c(G)$ is the length of the longest cycle contained in G. [The length of a path or a cycle is the number of edges it contains].

We also define $m(G)$ by $m(G) = \min\{d(x) + d(y): [x, y] \notin E\}$. $m(G)$ is undefined for complete graphs.

Let $P = (x_1, ..., x_l)$ be a path in G. $P(x_i, x_j)$ $(j \geqslant i)$ is the subpath of P connecting x_i to x_j. We use P^* for the reverse path $P^* = (x_l, ..., x_1)$.

2. The theorem

Theorem. *Let G be a block, then* $c(G) \geqslant \min\{n, m(G)\}$.

Proof. We prove the theorem when $m = m(G) \leqslant n$. From this the case $m > n$ easily follows. Let G be a counter-example, and let $P = (x_1, ..., x_l)$ be the longest path in G. Evidently $\Gamma(x_1) \cup \Gamma(x_l) \subsetneq \{x_1, ..., x_l\}$. G does not contain a cycle of length l, unless $l = n$ in which case there is nothing to prove. Let c be such a cycle. As c does not contain all vertices of G and as G is connected, we can find $x \in c$, $y \notin c$ that are adjacent. The path starting with $[y, x]$, then following along c has length l. This contradicts the maximality of l. So $[x_1, x_l] \notin E$ and $d(x_1) + d(x_l) \geqslant m$. We may now deduce that $l \geqslant m+1$, for if $l \leqslant m$, there is an index i such that $[x_1, x_{i+1}], [x_i, x_l] \in E$.

The following cycle has length l:

$$P(x_1, x_i)\,[x_i, x_l]\,P^*(x_l, x_{i+1})\,[x_{i+1}, x_1].$$

Further we show that if $[x_1, x_j], [x_i, x_l] \in E$, $(j > i)$, then $j - i > l - m + 1$. Else, the following cycle,

$$P(x_1, x_i)\,[x_i, x_l]\,P^*(x_l, x_{j+1})[x_{j+1}, x_1]$$

has length $l - (j - i - 1) \geqslant m$.

We now show that if $[x_1, x_j] \in E$, then for $i < j$, $[x_i, x_l] \notin E$. Let $[x_1, x_j]$, $[x_i, x_l] \in E$ with $i < j$, and let $j - i$ be minimal. We already know that $j - i \geqslant l - m + 2$. Also $[x_1, x_t] \in E$ implies $[x_{t-1}, x_l] \notin E$. By the last two arguments at least $d(x_1) + l - m$ vertices are not adjacent to x_l. As $\Gamma(x_l) \subsetneq \{x_1, ..., x_{l-1}\}$

$$d(x_l) \leqslant l - 1 - (d(x_1) + l - m) = m - 1 - d(x_1),$$

$$d(x_1) + d(x_l) \leqslant m - 1, \quad \text{a contradiction.}$$

We now denote

$$u = \max\{t\colon [x_1, x_t] \in E\}, \qquad v = \min\{t\colon [x_l, x_t] \in E\}.$$

Since G is a block, there exist integers s_1, t_1 such that $s_1 < u < t_1$, for which there is a path $P_1(x_{s_1}, x_{t_1})$ having no other vertices in common with P. We assume that t_1 is maximal with respect to this property. Suppose s_i, t_i, P_i are already defined. By the same reasoning there is a maximal integer t_{i+1} for which there is an integer s_{i+1} such that $s_{i+1} < t_i < t_{i+1}$ and there is a path $P_{i+1}(x_{s_{i+1}}, x_{t_{i+1}})$ having with P only end vertices in common. Moreover, the P_i thus defined are mutually disjoint, except possibly for end vertices. If $(P_i \cap P_j) \setminus P \neq \emptyset$ $(j > i)$, there would be a path connecting x_{t_j} to x_{s_i}. This path has no vertices in common with P except for x_{t_j}, x_{s_i}. But

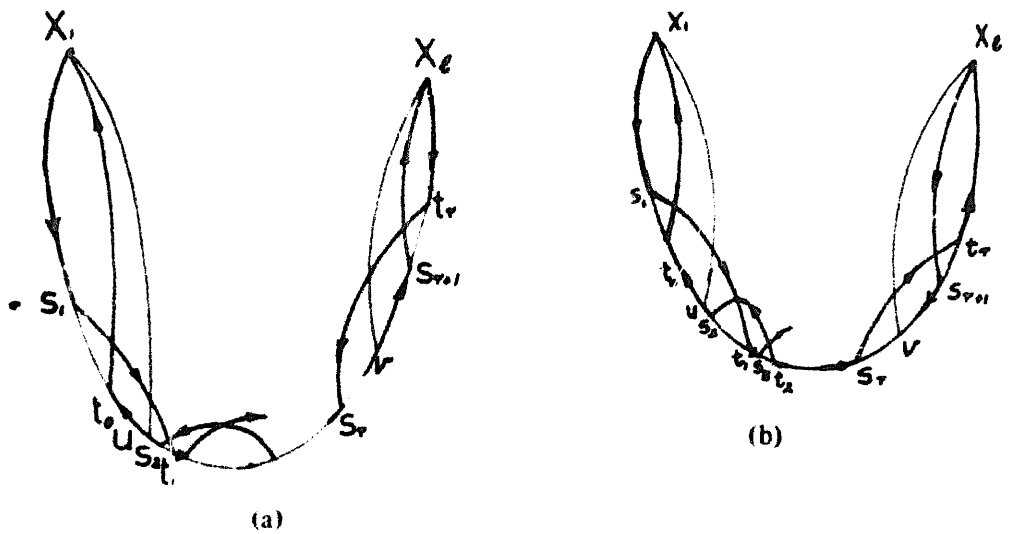

Fig. 1. (a) r even, (b) r odd.

this contradicts the maximality of t_i. Furthermore $s_{i+2} \geqslant t_i$ by the maximality of t_{i+1}. The situation is described in Fig. 1.

The sequence t_i is increasing, so for some f, $t_f > v$. Choose $r = \min\{f: t_f > v\}$. We need the following two definitions:

$$t_0 = \min\{t: [x_1, x_t] \in E, t > s_1\},$$

$$s_{r+1} = \max\{s: [x_s, x_l] \in E, s < t_r\}.$$

We now have the following cycle whose length we show to be $\geqslant m$.

$$([x_1, t_0]\ P(t_0, s_2)\ P_2(s_2, t_2)\ P(t_2, s_4)\ P_4(s_4, t_4) \ldots)^*$$

$$P(x_1, s_1)\ P_1(s_1, t_1)\ P(t_1, s_3)\ P_3(s_3, t_3) \ldots$$

There is a slight difference between the two cases: r is even and r is odd as is described above.

Let this cycle be denoted by C. By definition of t_0, s_{r+1} we have $V(C) \supseteq \{x_1, x_l\} \cup \Gamma(x_1) \cup \Gamma(x_l)$. Hence $|V(C)| \geqslant 2 + d(x_1) + d(x_l) - |\Gamma(x_1) \cap \Gamma(x_l)|$. But $|\Gamma(x_1) \cap \Gamma(x_l)| \leqslant 1$, so the length of C exceeds $d(x_1) + d(x_l) \geqslant m$. Note that we proved: $d(x) + d(y) \geqslant m$ for nonadjacent x, y implies $c(G) \geqslant m$ if G is a block. This formulation settles the case $m > n$.

3. Ore's theorem and concluding remarks

Ore's theorem is one of the earlier results in Hamiltonian graph theory. In our notation it states: *For any G satisfying $m(G) \geqslant n$, $c(G) = n$ (G is Hamiltonian).*

In order to get Ore's theorem we only have to show that $m(G) \geqslant n$ implies that G is a block. Suppose on the contrary that $V(G) = \{x\} \cup A \cup B$ with $A \cap B = \emptyset$, and no edge joining a vertex in A to vertex in B. If $u \in A$, $v \in B$, we have $d(u) \leqslant |A|$, $d(v) \leqslant |B|$ so $d(u) + d(v) \leqslant |A| + |B| = n - 1$.

The case $m(G) \geqslant n$ has been discussed by Kronk [2](see also Berge [1, p.204]).

J.A. Bondy (private communication) noted that by properly altering the proof of Theorem 1 in his paper "Large cycles in graphs" (Discrete Math. 1 (1971) 121–132) it is possible to obtain a proof of the main theorem in this paper.

References

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[3] O. Ore, Arc covering of graphs, Ann. Math. Pura Appl. 55 (1961) 315–321.