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Central Points for Sets in $\mathbb{R}^n$ (or: the Chocolate Ice-Cream Problem)*

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Abstract. Let A be a subset of the unit ball in $\mathbb{R}^n$, and let $0 \leq r \leq 1$ be real. Find a point x for which the intersection of the r-neighborhood of x with A has a large measure. Tight bounds on this measure are found.

1. General

After dinner a round bowl of chocolate-and-vanilla ice-cream is served. λ percent of the ice-cream is chocolate, your favorite flavor. You have a round scoop with which to probe the dessert and your goal is to maximize the percentage of chocolate ice-cream in your scoop. How well can you do?

More formally, let A be a measurable subset of the unit ball $\mathbb{B}$ in $\mathbb{R}^n$. We seek, for every $r \leq 1$, an *r-central* point x_r—i.e., a point whose r-neighborhood has a large intersection with A. Specifically we aim to maximize $\mu(A \cap B_r(x_r))/\mu(A)$.

This question can also be turned around: What is the largest measure of $A \subseteq \mathbb{B}$ in $\mathbb{R}^n$ that intersects every ball of radius r in a set of measure $\leq t$? As stated, this is a natural question in integral geometry. For analogous problems in the realm of finite graphs, see [LPRS].

We also consider the existence of a *central* point x^* for which $\mu(A \cap B_r(x^*))/\mu(A)$ is reasonably large for all $r \leq 1$.

For both questions, we seek answers that hold for every measurable set A. Our results are presented in the following two theorems:

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Theorem 1.1. *There is a constant $c > 0$ such that, for every $A \subseteq \mathbb{B}$ and every $r \leq 1$, there is a point $x_r \in \mathbb{R}^n$ for which*

$$\frac{\mu(A \cap B_r(x_r))}{\mu(A)} \geq \frac{c \cdot r^{n-1}}{1/r + \sqrt{n}\sqrt{1-r^2}}.$$

This result is tight up to the constant c.

Theorem 1.2. *There is a constant $c > 0$ such that, for every $A \subseteq \mathbb{B}$, there is a point $x^* \in \mathbb{R}^n$ such that, for every $r \leq 1$,*

$$\frac{\mu(A \cap B_r(x^*))}{\mu(A)} \geq \left(\frac{c \cdot r}{\min(\sqrt{n}, \ln(2/r))}\right)^n.$$

2. Notation

The asymptotic notations Θ, O, Ω are used throughout. Let f, g be two positive functions of n (possibly of other parameters as well). Then:

- $f = O(g)$ if there are constants $n_0, c > 0$ such that, for every $n > n_0$, $f(n) \leq c \cdot g(n)$.
- $f = \Omega(g)$ if there are constants $n_0, c > 0$ such that, for every $n > n_0$, $f(n) \geq c \cdot g(n)$.
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

We always reserve n to denote the dimension of the space $\mathbb{R}^n$ containing A, while n_0, c denote absolute constants. Also, we occasionally use a notation such as $(1 - O(1/n))^n$, where the $O(1/n)$ stands for some unspecified function that is $O(1/n)$.

Let $B_r(x) = \{y \in \mathbb{R}^n\colon \|y - x\| \leq r\}$ be the ball of radius r around x, and let

$$\Sigma_r(x) = \{y \in \mathbb{R}^n\colon \|y - x\| = r\}$$

be the sphere of radius r around x. If the center is the origin we use B_r, Σ_r instead of $B_r(0)$, $\Sigma_r(0)$.

3. For Every r There is an r-Central Point

We start with two easy attempts at proving Theorem 1.1. These approaches provide the desired proof for $r \leq 1/\sqrt{n}$. The first attempt is introduced in:

Lemma 3.1. *If $A \subseteq \mathbb{B}$, then for any $r \leq 1$ there is a point $x \in \mathbb{R}^n$ such that*

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \geq \left(\frac{r}{1+r}\right)^n.$$

Proof. The proof follows from the fact that the balls $B_r(x)$ for $x \in B_{1+r}$ cover A uniformly, so at least one of these balls must achieve the average, i.e., there is a point $x \in B_{1+r}$ such that

$$\mu(A \cap B_r(x)) \geq \frac{\mu(A)\mu(B_r)}{\mu(B_{1+r})} = \mu(A)\left(\frac{r}{1+r}\right)^n . \qquad \square$$

An asymptotic improvement is offered by:

Lemma 3.2. *If $A \subseteq \mathbb{B}$, then for every $r \leq 1$ there is a point $x \in \mathbb{R}^n$ such that*

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \geq \tfrac{1}{2}\left(\frac{r}{\sqrt{1+r^2}}\right)^n .$$

Proof. The proof is a slight variation on the previous one. Namely, it is shown that the family of balls $\{B_r(x) \mid x \in B_{\sqrt{1+r^2}}\}$ covers $\mathbb{B}$ almost uniformly, to within a factor of two. To see this note that the half ball $H = \{y \in B_r(x) | < x, y - x > \leq 0\}$ is contained in $B_{\sqrt{1+r^2}} \cap B_r(x)$. Of course $H \subseteq B_r(x)$ and $H \subseteq B_{\sqrt{1+r^2}}$, since, for every $y \in H$,

$$||y||^2 = ||x||^2 + ||y - x||^2 + 2\langle x, y - x\rangle \leq ||x||^2 + ||y - x||^2 \leq 1 + r^2 .$$

Now $\mu(H) = \frac{1}{2}\mu(B_r(x))$, so

$$\mu(B_r(x)) \geq \mu(B_r(x) \cap B_{\sqrt{1+r^2}}) \geq \tfrac{1}{2}\mu(B_r(x)) .$$

Rewrite this as

$$\mu(B_r(x)) \geq \mu(\{y \in B_{\sqrt{1+r^2}} | x \in B_r(y)\}) \geq \tfrac{1}{2}\mu(B_r(x)) . \tag{1}$$

Hence, the balls $\{B_r(y) | y \in B_{\sqrt{1+r^2}}\}$ cover $\mathbb{B}$ almost uniformly within a factor of two (of course the same holds for covering $A \subseteq \mathbb{B}$), i.e.,

$$\max_{x \in A} \mu(\{y \in B_{\sqrt{1+r^2}} | x \in B_r(y)\}) \leq 2 \min_{x \in A} \mu(\{y \in B_{\sqrt{1+r^2}} | x \in B_r(y)\}) .$$

This allows us to derive the desired conclusion as in Lemma 3.1 with a loss of at most a factor of two. $\square$

Remark 3.1. These simple arguments prove Theorem 1.1 for the range $r \leq 1/\sqrt{n}$. In this range, the bound in Theorem 1.1 is

$$\frac{r^{n-1}}{1/r + \sqrt{n}\sqrt{1 - r^2}} = \Theta(r^n),$$

and a point x_r as found in Lemma 3.2 yields

$$\frac{\mu(A \cap B_r(x_r))}{\mu(A)} \geq \frac{1}{2}\left(\frac{r}{\sqrt{1+r^2}}\right)^n \geq \frac{1}{2}\left(\frac{r}{\sqrt{1+\frac{1}{n}}}\right)^n = \Omega(r^n),$$

as needed. That Theorem 1.1 is tight in this range follows by considering $A = \mathbb{B}$ and observing that, for any x,

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \le \frac{\mu(B_r(x))}{\mu(A)} = r^n.$$

Therefore from now on we are only interested in the range $1/\sqrt{n} < r \le 1$.

Here spherical shells (i.e., the difference set of two concentric balls) play a central role in our considerations. We start by proving (Lemma 3.5) that if A is a spherical shell of width $\Theta(1/n)$, then the best choice of x_r achieves

$$\frac{\mu(A \cap B_r(x_r))}{\mu(A)} = \Theta\left(\frac{r^{n-1}}{1/r + \sqrt{n}\sqrt{1-r^2}}\right).$$

Since this establishes the tightness (upper bound) of the theorem, we only need to find a point x_r achieving the required bound for the general $A \subseteq \mathbb{B}$. This is done first for A being a subset of the spherical shell of width $\Theta(1/n)$ (Lemma 3.6), and then for a general A by considering its intersection with concentric spherical shells. The decomposition of $\mathbb{B}$ into concentric shells is in the same spirit as the Calderón–Zygmund decomposition [T].

First we derive certain estimations that will be needed throughout.

Fact 3.1 (see [C]). *The surface area of an n-dimensional sphere of radius* ρ *is* $S_\rho = \sigma_n \rho^{n-1}$ *where* $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$.

Lemma 3.3. *For n an integer and* $0 \le \varepsilon \le \pi/2$, *let* $I_n(\varepsilon) = \int_\varepsilon^{\pi/2} \cos^n \alpha d\alpha$. *Then*

$$I_n(\varepsilon) = \Theta\left(\frac{\cos^{n+1}\varepsilon}{\sqrt{n}\cdot(1+\sqrt{n}\sin\varepsilon)}\right).$$

Proof. Assume $n > 1$ and define $\Delta_n(\varepsilon)$ through the relation

$$\cos(\varepsilon + \Delta_n(\varepsilon)) = \left(1 - \frac{1}{n}\right)\cos\varepsilon.$$

Then

$$-\ln\left(1-\frac{1}{n}\right) = \ln\cos\varepsilon - \ln\cos(\varepsilon + \Delta_n(\varepsilon)) = \int_\varepsilon^{\varepsilon+\Delta_n(\varepsilon)} \tan x\, dx.$$

Since $\tan x$ is an increasing function, and the left-hand side does not depend on ε, $\Delta_n(\varepsilon)$ must decrease with ε.

Define $\varepsilon_0 = \varepsilon, \varepsilon_{i+1} = \varepsilon_i + \delta_i$ where $\delta_i = \Delta_n(\varepsilon_i)$. Observe that $\cos^n \varepsilon_{i+1} = (1-1/n)^n \cos^n \varepsilon_i = \Theta(\cos^n \varepsilon_i)$. This allows us to estimate $I_n(\varepsilon) = \sum_{i=0}^\infty \int_{\varepsilon_i}^{\varepsilon_{i+1}} \cos^n \alpha\, d\alpha = \Theta(\sum_{i=0}^\infty \delta_i \cos^n \varepsilon_i)$. The last sum is bounded below by its first term $\delta_0 \cos^n \varepsilon$. Since δ_i are decreasing, an upper bound for this sum is $\delta_0(\cos^n\varepsilon)\cdot\sum_{i=0}^{\infty}(1-1/n)^{ni} = O(\delta_0\cos^n\varepsilon)$, i.e.,

$$I_n(\varepsilon) = \Theta(\delta_0\cos^n\varepsilon) = \Theta(\Delta_n(\varepsilon)\cos^n\varepsilon). \tag{2}$$

To estimate $\Delta_n(\varepsilon)$ consider

$$\frac{\cos\varepsilon}{n} = \cos\varepsilon - \cos(\varepsilon+\Delta_n(\varepsilon)) = 2\sin\left(\varepsilon+\frac{\Delta_n(\varepsilon)}{2}\right)\sin\left(\frac{\Delta_n(\varepsilon)}{2}\right).$$

The fact that $\sin z = \Theta(z)$ for $0 \le z \le \pi/2$ implies

$$\Delta_n(\varepsilon)(\varepsilon+\Delta_n(\varepsilon)) = \Theta\left(\frac{\cos\varepsilon}{n}\right).$$

So

$$\begin{aligned}\Delta_n(\varepsilon) &= \sqrt{\frac{\varepsilon^2}{4}+\Theta\left(\frac{\cos\varepsilon}{n}\right)}-\frac{\varepsilon}{2}\\ &= \frac{\Theta((\cos\varepsilon)/n)}{\sqrt{\varepsilon^2/4+\Theta((\cos\varepsilon)/n)}+\varepsilon/2}\\ &= \Theta\left(\frac{\cos\varepsilon}{n\sin\varepsilon+\sqrt{n\cos\varepsilon}}\right).\end{aligned}$$

Therefore

$$I_n(\varepsilon) = \Theta\left(\frac{\cos^{n+1}\varepsilon}{\sqrt{n}\cdot(\sqrt{n}\sin\varepsilon+\sqrt{\cos\varepsilon})}\right) = \Theta\left(\frac{\cos^{n+1}\varepsilon}{\sqrt{n}\cdot(\sqrt{n}\sin\varepsilon+1)}\right)$$

as needed. The last step is justified by separately considering the cases $\varepsilon < 1/\sqrt{n}$ and $\varepsilon \ge 1/\sqrt{n}$. □

Define for $0 \le t \le 1$ the function

$$\Psi(t) = \frac{t^{n-1}}{1+\sqrt{n}\sqrt{1-t^2}}.$$

Lemma 3.4. *Consider the n-dimensional spherical cap of radius ρ having a head angle $2\alpha_0 \le \pi$. Its surface area S_{ρ,α_0} is*

$$S_{\rho,\alpha_0} = \Theta(S_\rho\Psi(\sin\alpha_0)).$$

Proof. $S_{\rho,\alpha_0} = \int_0^{\alpha_0}\sigma_{n-1}(\rho\sin\alpha)^{n-2}\rho\,d\alpha$. Therefore,

$$\frac{S_{\rho,\alpha_0}}{S_\rho} = \frac{\sigma_{n-1}}{\sigma_n}\int_0^{\alpha_0}\sin^{n-2}\alpha\,d\alpha = \frac{\sigma_{n-1}}{\sigma_n}\int_{\pi/2-\alpha_0}^{\pi/2}\cos^{n-2}\alpha d\alpha.$$

By Lemma 3.3, and the fact that $\sigma_{n-1}/\sigma_n = \Theta(\sqrt{n})$ we get the required result. □

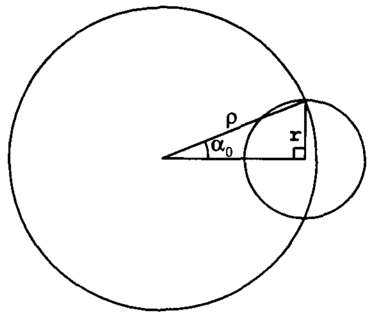

Fig. 1. Maximizing α_0.

Corollary 3.1. *The largest area of a spherical cap* $B_r(x) \cap \Sigma_\rho$ *where* $0 \le r \le \rho$ *is*

$$\Theta\left(S_\rho \Psi\left(\frac{r}{\rho}\right)\right).$$

Proof. The estimate of S_{ρ,α_0} in Lemma 3.4 is an increasing function of α_0. The largest possible angle α_0 of the cap is obtained when $\sin\alpha_0 = r/\rho$, as can be seen in Fig. 1. □

Lemma 3.5. *Let* A *be the shell* $\mathbb{B}\backslash B_{1-\varepsilon}$ *in* $\mathbb{R}^n$ *where* $\varepsilon = \Theta(1/n)$ *and* $1/\sqrt{n} \le r \le 1$. *Then*

$$\max_x \frac{\mu(A \cap B_r(x))}{\mu(A)} = \Theta(\Psi(r)).$$

Proof. Define $\lambda = \max_x(\mu(A \cap B_r(x))/\mu(A))$. Then our goal is to show that $\lambda = \Theta(\Psi(r))$. Also let $\alpha(\rho, \|x\|)$ be half the head angle of the spherical cap $B_r(x) \cap \Sigma_\rho$ (obviously only the norm of x matters). Then

$$\lambda = \max_x \frac{\int_{1-\varepsilon}^1 S_{\rho,\alpha(\rho,\|x\|)}\, d\rho}{\int_{1-\varepsilon}^1 S_\rho\, d\rho}. \tag{3}$$

Since $\varepsilon = O(1/n)$ the variable surface areas S_ρ differ from the constant S_1 only by a constant factor. This allows us to write

$$\lambda = \Theta\left(\max_x \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \frac{S_{\rho,\alpha(\rho,\|x\|)}}{S_\rho}\, d\rho\right) = \Theta\left(\max_d \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \frac{S_{\rho,\alpha(\rho,d)}}{S_\rho}\, d\rho\right). \tag{4}$$

Consider first $r \le 1 - 2\varepsilon$. Then the angle $\alpha(\rho, d)$ is determined by the triangle shown in Fig. 2. Now $r \le 1 - 2\varepsilon < 1 - \varepsilon \le \rho \le 1$ implies that the angle $\alpha(\rho, d) \le \pi/2$, so Lemma 3.4 can be applied, and thus

$$\lambda = \Theta\left(\max_d \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \Psi(\sin\alpha(\rho, d))\, d\rho\right). \tag{5}$$

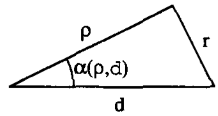

Fig. 2. The angle $\alpha(\rho, d)$.

Our intention is to show that the integrand $\Psi(\sin\alpha(\rho, d))$ varies at most by a constant factor over the integration range $1 - \varepsilon \le \rho \le 1$. More specifically, we claim that $\sin\alpha(\rho, d)$ varies by no more than a $1 \pm O(1/n)$ factor, while $\cos\alpha(\rho, d)$ varies by no more than a constant factor over that range.

To get our analysis started we consider d_m, the d that maximizes λ. For fixed ρ and r, $\alpha(\rho, d)$ is a unimodal function of d that is maximized when $d^2 + r^2 = \rho^2$. Then d_m must satisfy

$$(1 - \varepsilon)^2 \le d_m^2 + r^2 \le 1. \tag{6}$$

Otherwise, say $d_m^2 + r^2 > 1 \ge \rho^2$. Then by decreasing d slightly, the unimodal $\alpha(\rho, d)$ increases for all ρ's, and since Ψ and $\sin(\cdot)$ are increasing the integral must grow. The other case is handled similarly. From now on $d := d_m$ is fixed.

We turn to estimating the change in $\sin\alpha(\rho, d)$ and $\cos\alpha(\rho, d)$ over the integration range $1 - \varepsilon \le \rho \le 1$. Now

$$\begin{aligned} 1 - r^2 &\ge (1 - \varepsilon)^2 - r^2 = (1 - \varepsilon - r)(1 - \varepsilon + r) \\ &= \tfrac{1}{4}(1 - r + (1 - 2\varepsilon - r))(1 + r + (1 + r - 2\varepsilon)) \\ &\ge \tfrac{1}{4}(1 - r)(1 + r) = \tfrac{1}{4}(1 - r^2). \end{aligned}$$

Whence $1 - r^2 = \Theta((1 - \varepsilon)^2 - r^2)$. Rewrite (6) as

$$(1 - \varepsilon)^2 - r^2 \le d^2 \le 1 - r^2,$$

which by the last remark says that

$$d^2 = \Theta((1 - \varepsilon)^2 - r^2) = \Theta(1 - r^2),$$

so, for every $1 - \varepsilon \le \rho \le 1$, this implies

$$\rho^2 - r^2 = \Theta(d^2). \tag{7}$$

By the cosine theorem,

$$\cos\alpha(\rho, d) = \frac{\rho^2 + d^2 - r^2}{2\rho d}. \tag{8}$$

Using (7) and $\rho = \Theta(1)$ we can rewrite $(\rho^2 + d^2 - r^2)/2\rho d = (d^2 + \Theta(d^2))/2\rho d = \Theta(d)$, so

$$\cos\alpha(\rho, d) = \Theta\left(\sqrt{1 - r^2}\right)$$

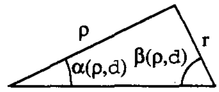

Fig. 3. The angle $\beta(\rho, d)$.

and

$$\frac{1}{1+\sqrt{n}\cos\alpha(\rho,d)} = \Theta\left(\frac{1}{1+\sqrt{n}\sqrt{1-r^2}}\right). \tag{9}$$

To estimate the change in $\sin\alpha(\rho, d)$ over $1-\varepsilon \leq \rho \leq 1$, define also the angle $\beta(\rho, d)$ by the triangle shown in Fig. 3. By high-school trigonometry

$$\sin\alpha(\rho,d) = \frac{r}{\rho}\sin\beta(\rho,d) = \frac{r}{\rho}\sqrt{1-\cos^2\beta(\rho,d)} = \frac{r}{\rho}\sqrt{1-\frac{(d^2+r^2-\rho^2)^2}{4r^2d^2}}.$$

From $(1-\varepsilon)^2 \leq \rho^2, d^2+r^2 \leq 1$ we conclude

$$\frac{(d^2+r^2-\rho^2)^2}{4r^2d^2} \leq \frac{(1-(1-\varepsilon)^2)^2}{4r^2((1-\varepsilon)^2-r^2)} \leq \frac{4\varepsilon^2}{4r^2((1-\varepsilon)^2-r^2)}.$$

The denominator is unimodal as a function of r over $1/\sqrt{n} \leq r \leq 1-2\varepsilon$, so the minimum is obtained at either end. Since $\varepsilon = \Theta(1/n)$, it can be checked that at both ends the denominator is $\Theta(1/n)$, so $(d^2+r^2-\rho^2)^2/4r^2d^2 = O(1/n)$, and we conclude that

$$\sin\alpha(\rho,d) = r\cdot\left(1 \pm O\left(\frac{1}{n}\right)\right). \tag{10}$$

Therefore $\sin^{n-1}\alpha(\rho,d) = \Theta(r^{n-1})$. Together with (9) this yields the desired $\Psi(\sin\alpha(\rho,d)) = \Theta(\Psi(r))$, and by (5) we get $\lambda = \Theta(\Psi(r))$ as needed.

Now turn to the case $1-2\varepsilon \leq r \leq 1$ and show that both $\Psi(r)$ and λ are $\Theta(1)$ in this range, so $\lambda = \Theta(\Psi(r))$ as needed.

Since $\sqrt{1-r^2} \leq \sqrt{1-(1-2\varepsilon)^2} = \sqrt{4\varepsilon(1-\varepsilon)}$and $\varepsilon = \Theta(1/n)$, it follows that $\sqrt{1-r^2} = O(1/\sqrt{n})$, i.e., $1+\sqrt{n}\sqrt{1-r^2} = \Theta(1)$. Therefore

$$\Psi(r) = r^{n-1}\frac{1}{1+\sqrt{n}\sqrt{1-r^2}} = \Theta(r^{n-1}) = \Theta(1).$$

Obviously $\lambda \leq 1$, so it is enough to show that $\lambda = \Omega(1)$. To see that, choose x such that $d = ||x|| = \sqrt{1-r^2}$, and consider the possibilities for the angle $\alpha(\rho, d)$:

- If $\alpha(\rho,d)$ is obtuse, then at least a hemisphere of Σ_ρ is contained in $B_r(x)$ (indeed $\alpha(\rho,d)$ may get as large as π if $\Sigma_\rho \subset B_r(x)$). Then $S_{\rho,\alpha(\rho,d)}/S_\rho \geq \frac{1}{2} = \Omega(1)$.

- If $\alpha(\rho, d)$ is acute, then, by Lemma 3.4, $S_{\rho,\alpha(\rho,d)}/S_\rho = \Theta(\Psi(\sin\alpha(\rho, d)))$. By (8),

$$\cos\alpha(\rho, d) = \frac{\rho^2 + d^2 - r^2}{2\rho d} = \frac{\rho^2 + 1 - 2r^2}{2\rho\sqrt{1 - r^2}} = \frac{\rho - (2r^2 - 1)/\rho}{2\sqrt{1 - r^2}}.$$

This expression increases with ρ and $\rho \leq 1$, and since $r \geq 1 - 2\varepsilon$ we get $\cos\alpha(\rho, d) \leq \sqrt{1 - r^2} \leq \sqrt{1 - (1 - 2\varepsilon)^2} \leq \sqrt{4\varepsilon} = O(1/\sqrt{n})$. So,

$$\Psi(\sin\alpha(\rho, d)) = \frac{\sin^{n-1}\alpha(\rho, d)}{1 + \sqrt{n}\cos\alpha(\rho, d)}$$

is $\Theta(1)$. The denominator is $\Theta(1)$ and for the numerator

$$\sin^{n-1}\alpha(\rho, d) = (1 - \cos^2(\alpha(\rho, d)))^{(n-1)/2} = (1 - O(1/n))^{(n-1)/2} = \Theta(1).$$

Then, for all $1 - \varepsilon \leq \rho \leq 1$, $S_{\rho,\alpha(\rho,d)}/S_\rho = \Omega(1)$, and (4) yields $\lambda = \Omega(1)$ as needed. □

Remark 3.2. This lemma proves the upper bound (tightness) of Theorem 1.1 for $1/\sqrt{n} \leq r \leq 1$ since in this range $\Psi(r) = \Theta(r^{n-1}/(1/r + \sqrt{n}\sqrt{1 - r^2}))$. This is verified by separately considering $r < \frac{1}{2}$ and $r \geq \frac{1}{2}$.

Remark 3.3. In the above proof, it turned out that for all $1/\sqrt{n} \leq r \leq 1$ the choice of x such that $||x|| = \sqrt{1 - r^2}$ is optimal up to a constant factor, and that for this choice $S_{\rho,\alpha(\rho,||x||)}/S_\rho = \Theta(\Psi(r))$ for all $1 - \varepsilon \leq \rho \leq 1$.

We now move on to sets A which are *contained* in some spherical shell.

Lemma 3.6. *Let $A \subseteq \mathbb{B}\backslash B_{1-\varepsilon}$ in $\mathbb{R}^n$ where $\varepsilon = \Theta(1/n)$. Then, for every $1/\sqrt{n} \leq r \leq 1$,*

$$\max_x \frac{\mu(A \cap B_r(x))}{\mu(A)} = \Omega(\Psi(r)).$$

Proof. The proof is by averaging over all $x \in \Sigma_d$, where $d = \sqrt{1 - r^2}$. Let μ_t be the normalized Haar measure on Σ_t. We saw in the proof of Lemma 3.5 that

$$\mu_\rho(B_r(x) \cap \Sigma_\rho) = \frac{S_{\rho,\alpha(\rho,d)}}{S_\rho} = \Theta(\Psi(r)),$$

where $||x|| = d$ $(= \sqrt{1 - r^2})$. Integrating over $x \in \Sigma_d$ with respect to μ_d yields

$$\mu(\{(x, y) \in \Sigma_d \times \Sigma_\rho\colon ||x - y|| \leq r\}) = \Theta(\Psi(r)), \tag{11}$$

where $\mu = \mu_d \times \mu_\rho$.

We need to evaluate $\mu_d(B_r(y)\cap\Sigma_d)$ for $y\in\mathbb{B}\backslash B_{1-\varepsilon}$. Keeping r, d fixed this expression depends only on $||y||=\rho$. Integrating over all $y\in\Sigma_\rho$ and using (11) we conclude that

$$\mu_d(B_r(y)\cap\Sigma_d)=\Theta(\Psi(r)).$$

Since the result is independent of ρ, the numbers $\mu_d(B_r(y)\cap\Sigma_d)$ as y varies over the shell $\mathbb{B}\backslash B_{1-\varepsilon}$ change only by a constant factor. In particular,

$$\max_{y\in A}\mu_d(\{x\in\Sigma_d|y\in B_r(x)\})=\Theta\left(\min_{y\in A}\mu_d(\{x\in\Sigma_d|y\in B_r(x)\})\right).$$

An averaging argument as in the proof of Lemma 3.2 implies that there is a point $x\in\Sigma_d$ such that $\mu(A\cap B_r(x))/\mu(A)\geq c\Psi(r)$ for some constant c, as needed. □

Proof of Theorem 1.1. By Remarks 3.1 and 3.2 we only need to find the desired $x=x_r$, and we may assume $1/\sqrt{n}<r\leq 1$. Let k, $\{R_i\}_{i=0}^k$ be such that

$$R_0=1,\quad R_i\left(1-\frac{1}{n}\right)\leq R_{i+1}\leq R_i\left(1-\frac{1}{10n}\right)\quad\text{for } i=0,\ldots,k-1 \text{ and } R_k=r.$$

It is easily verified that such R_i can be found. Define the shells $C_i=B_{R_i}\backslash B_{R_{i+1}}$ for $0\leq i<k$, and the core $C_k=B_{R_k}=B_r$. Consider the following situations:

- There is shell C_j such that

$$\frac{\mu(A\cap C_j)}{\mu(A)}\geq 0.01\frac{\Psi(r)}{\Psi(r/R_j)}. \tag{12}$$

  In such a situation, since $r/R_j\geq r\geq 1/\sqrt{n}$ apply Lemma 3.6 to $A\cap C_j$ to find a point x for which

$$\frac{\mu(A\cap C_j\cap B_r(x))}{\mu(A\cap C_j)}\geq c\Psi\left(\frac{r}{R_j}\right).$$

  So

$$\frac{\mu(A\cap B_r(x))}{\mu(A)}\geq\frac{\mu(A\cap C_j\cap B_r(x))}{\mu(A\cap C_j)}\cdot 0.01\frac{\Psi(r)}{\Psi(r/R_j)}\geq c\cdot 0.01\Psi(r).$$

- The core $C_k=B_r$ satisfies

$$\frac{\mu(A\cap C_k)}{\mu(A)}\geq 0.01\Psi(r). \tag{13}$$

  Here the obvious choice $x=0$ yields the required result.

Otherwise, if (12) holds for no $0\leq j<k$ and (13) does not hold, then summing up the (reverse) inequalities, we conclude

$$\sum_{j=0}^{k}\frac{\mu(A\cap C_j)}{\mu(A)}<0.01\cdot\sum_{j=0}^{k}\frac{\Psi(r)}{\Psi(r/R_j)}.$$

The left-hand side is 1 since $A \cap C_j$ for $j = 0, \ldots, k$ constitute a partition of A. The right-hand side is

$$\begin{aligned} 0.01 \cdot \sum_{j=0}^{k} \frac{\Psi(r)}{\Psi(r/R_j)} &= 0.01 \cdot \sum_{j=0}^{k} \frac{r^{n-1}}{1+\sqrt{n}\sqrt{1-r^2}} \cdot \frac{1+\sqrt{n}\sqrt{1-(r/R_j)^2}}{(r/R_j)^{n-1}} \\ &= 0.01 \cdot \sum_{j=0}^{k} R_j^{n-1} \cdot \frac{1+\sqrt{n}\sqrt{1-(r/R_j)^2}}{1+\sqrt{n}\sqrt{1-r^2}} \leq 0.01 \cdot \sum_{j=0}^{k} R_j^{n-1} \\ &\leq 0.01 \cdot \sum_{j=0}^{\infty} \left(1 - \frac{1}{10n}\right)^{(n-1)\cdot j} \leq 0.01 \cdot \sum_{j=0}^{\infty} e^{-j/10} \\ &= \frac{0.01}{1-e^{-1/10}} < 1, \end{aligned}$$

a contradiction. □

4. There Is a Center Good for Every r

The proof of Theorem 1.2 is split into Theorems 4.1 and 4.2.

Theorem 4.1. *For every $A \subseteq \mathbb{B}$, there is a point $x \in \mathbb{R}^n$ such that, for all $0 \leq r \leq 1$,*

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \geq \left(\frac{r}{4\sqrt{n}}\right)^n.$$

Proof. We construct a sequence of cubes $\{I_i\}_i$ where I_i has side $2^{-(i-1)}$ and satisfies $\mu(A \cap I_i)/\mu(A) \geq 2^{-ni}$. To begin, $I_0 = [-1, 1]^n$ will do. Now, given I_i, decompose it to 2^n disjoint cubes $I_i^{(1)} \cdots I_i^{(2^n)}$ by halving it along each coordinate. By an averaging argument we can choose $I_{i+1} = I_i^{(j)}$ such that $\mu(A \cap I_i^{(j)}) \geq 2^{-n}\mu(A \cap I_i) \geq 2^{-n(i+1)}$. Since I_i is a decreasing sequence of closed sets, there is a point $x \in \bigcap_{i\geq 0} I_i$. Given r, choose the least i such that $I_i \subseteq B_r(x)$. Whence $r \leq 4\sqrt{n}/2^i$ (two times the diameter of I_i), and we have

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \geq \frac{\mu(A \cap I_i)}{\mu(A)} \geq 2^{-ni} = \left(\frac{r}{4\sqrt{n}}\right)^n. \qquad \square$$

Theorem 4.2. *For every $A \subseteq \mathbb{B}$, there is a point $x \in \mathbb{R}^n$ such that, for all $0 \leq r \leq 1$,*

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \geq \left(\frac{r}{c \cdot \ln(2/r)}\right)^n.$$

Proof. Fix $A \subseteq \mathbb{B}$, $0 < r \leq 1$ and introduce a parameter $0 < \alpha < 1$ to be determined later. Construct sequences $\{x_i\}_{i\geq 0}$, $\{X_i\}_{i\geq 0}$ so that

$$X_i \subseteq B_{\alpha^i}(x_i) \quad \text{and} \quad \frac{\mu(X_i \cap A)}{\mu(A)} \geq \left(\frac{\alpha}{1+\alpha}\right)^{ni}. \tag{14}$$

For $i = 0$, the choice $X_0 = \mathbb{B}$ and $x_0 = 0$ clearly satisfies (14). To proceed by induction on i, let (14) hold with x_i, X_i. By Lemma 3.1 (scaled by α^i and shifted by x_i), there is a point $x \in \mathbb{R}^n$ such that

$$\frac{\mu(B_{\alpha^{i+1}}(x) \cap X_i \cap A)}{\mu(X_i \cap A)} \geq \left(\frac{\alpha^{i+1}}{\alpha^i + \alpha^{i+1}}\right)^n = \left(\frac{\alpha}{1+\alpha}\right)^n. \tag{15}$$

Setting $x_{i+1} = x$, $X_{i+1} = B_{\alpha^{i+1}}(x) \cap X_i$ yields

$$\frac{\mu(X_{i+1} \cap A)}{\mu(A)} = \frac{\mu(X_{i+1} \cap A)}{\mu(X_i \cap A)} \frac{\mu(X_i \cap A)}{\mu(A)} \geq \left(\frac{\alpha}{1+\alpha}\right)^{n(i+1)},$$

completing the inductive step. Since $\{X_i\}$ is a decreasing sequence of closed sets, we can choose $x \in \bigcap_{i=0}^{\infty} X_i$. Let $i = \lceil \log_\alpha r/2 \rceil$. Then $X_i \subseteq B_{r/2}(x_i) \subseteq B_r(x)$. Therefore

$$\frac{\mu(A \cap B_r(x))}{\mu(A)} \geq \frac{\mu(A \cap X_i)}{\mu(A)} \geq \left(\frac{\alpha}{1+\alpha}\right)^{n\lceil \log_\alpha r/2 \rceil} \geq \left(\frac{\alpha}{1+\alpha}\right)^{n \log_\alpha (\alpha r/2)} = \left(\frac{r}{2\gamma}\right)^n,$$

where $\gamma = (1/\alpha)(1+\alpha)^{\log_\alpha(\alpha r 2)}$. Then

$$\ln \gamma = \ln\left(\frac{1}{\alpha}\right) + \ln(1+\alpha) + \frac{\ln(1+\alpha)}{\ln(1/\alpha)} \ln\left(\frac{2}{r}\right) \leq \ln\left(\frac{1}{\alpha}\right) + \ln 2 + \frac{\ln(2/r)}{(1/\alpha)\ln(1\alpha)}.$$

Choose $0 < \alpha_0 < 1$ to be the (unique) solution of the equation

$$\frac{1}{\alpha_0} \ln \frac{1}{\alpha_0} = \ln\left(\frac{2}{r}\right).$$

Then

$$\ln \gamma \leq \ln \frac{1}{\alpha_0} + \ln 2 + 1.$$

Now, $\ln(1/\alpha_0) \geq \frac{1}{10}$ since otherwise

$$-1 < \ln\ln 2 \leq \ln\ln\left(\frac{2}{r}\right) = \ln\frac{1}{\alpha_0} + \ln\ln\frac{1}{\alpha_0} < \frac{1}{10} + \ln\frac{1}{10} < -1.$$

So

$$\ln \gamma \leq \ln\ln\left(\frac{2}{r}\right) - \ln\ln\frac{1}{\alpha_0} + \ln 2 + 1 \leq \ln\ln\left(\frac{2}{r}\right) - \ln\tfrac{1}{10} + \ln 2 + 1 < \ln\ln\left(\frac{2}{r}\right) + 6.$$

Therefore

$$\gamma \leq c \cdot \ln\left(\frac{2}{r}\right)$$

for some $c > 0$, as claimed. □

Remark 4.1. We still do not know the best possible bound for Theorem 1.2. However, we do know it to be weaker than that of Theorem 1.1. Consider the situation where the set A is a very thin shell. Since the center x will have to be good for all $0 \leq r \leq 1$, we have to choose $x \in A$ although for any r we can do asymptotically better by letting $||x|| = \sqrt{1-r^2}$.

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