Triply Existentially Complete Triangle-free Graphs

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Abstract

A triangle-free graph G is called *k-existentially complete* if for every induced k-vertex subgraph H of G, every extension of H to a $(k+1)$-vertex triangle-free graph can be realized by adding another vertex of G to H. Cherlin [8, 9] asked whether k-existentially complete triangle-free graphs exist for every k. Here we present known and new constructions of 3-existentially complete triangle-free graphs.

1 Introduction

It is well known that the Rado graph R [12, 19] is characterized by being existentially complete. Namely, for every two finite disjoint subsets of vertices $A, B \subset V(R)$ there is an additional vertex x that is adjacent to all vertices in A and to none in B. Many variations on this theme suggest themselves, specifically concerning finite and H-free graphs. First, the aforementioned extension property of R suggests a search of small finite graphs which satisfy this condition whenever $|A| + |B| \leq k$. It is known that Paley graphs and finite random graphs of order $\exp(O(k))$ have this property [2, 4, 7, 12, 5].

The Rado graph is also *homogeneous*, i.e., every isomorphism between two of its finite subsgraphs can be extended to an automorphism of R. Henson [15] has discovered the generic infinitely countable *triangle-free* graph R_3 which is homogeneous as well. It is uniquely defined by the following property. Given two finite disjoint subsets $A, B \subset V(R_3)$ with A *independent*, there is an additional vertex x that is adjacent to all vertices in A and to none in B.

The graph R_3 suggests an extremely interesting question that was raised and studied by Cherlin [8, 9]. Namely, do there exist finite graphs with a similar property? We say that a triangle-free graph is *k-existentially complete* if it satisfies this condition whenever $|A| + |B| \leq k$. Are there k-existentially complete triangle-free graphs for every k? A weaker variant of this question had been raised by Erdős and Pach [18, 11].

Cherlin's question can be viewed as an instance of a much wider subject in graph theory, namely understanding the extent to which the local behavior of infinite graphs can be emulated by finite ones. Here are some other instances of this general problem.

The infinite d-regular tree $\mathbb{T}_d$ is the ultimate d-regular expander in at least two senses. It has the largest possible number of edges emanating from every finite set of vertices. It also has the largest spectral gap that a d-regular graph can have. This observation suggests the search for finite arbitrarily large Ramanujan graphs, and for the limits on expansion in finite d-regular graphs. $\mathbb{T}_d$ is also, of course, acyclic which leads to the question how large the girth can be in a finite d-regular graph.

There are several examples of local conditions that can be satisfied in an infinite graph but not in a finite one. A nice example comes from a paper of Blass, Harari and Miller [3]. They define the link of a vertex to be the subgraph induced by its neighbors, and consider graphs in which all

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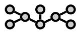
links are isomorphic to some fixed graph H. As they observe, there is an infinite graph with all links isomorphic to $H =$  , but this is impossible for finite graphs.

For all the currently known k-existentially complete triangle-free graphs, the parameter k is bounded by 3. The question is wide open for $k \geq 4$. The known finite graphs with the R-like extension property do not seem adjustable to the triangle-free case. In fact we conjecture that there is some absolute constant k_0 such that no triangle-free graph is k_0-existentially complete.

For general $p \geq 3$, Henson's universal countable K_p-free graph R_p is defined by the analogous extension property, where the subset A is only required to be $K_{(p-1)}$-free. There is a simple connection between the existential completeness properties of K_p-free graphs for different values of p. Thus, if, as we conjecture, no finite k_0-existentially complete triangle-free graphs exist, then no finite K_4-free graph can be (k_0+1)-existentially complete, etc. This follows since the link of every vertex in a $(k+1)$-existentially complete K_p-free graph is a k-existentially complete $K_{(p-1)}$-free graph. Finite random $(p-1)$-partite graphs provide an easy lower bound, since they are K_p-free and $(p-2)$-existentially complete.

Cherlin gives several examples of finite 3-existentially complete triangle-free graphs, or shortly 3ECTF, and asks which additional properties they can have. In particular, for what μ_2 it is possible that every 2 non-adjacent vertices have at least μ_2 common neighbors? Similarly, can every independent set of 3 vertices have $\mu_3 > 1$ common neighbors? Maybe the search for more robust examples for finite 3ECTF graphs, would shed light on the question of finite 4ECTF graphs and beyond.

The main purpose of this article is to investigate Cherlin's examples and construct some more 3ECTF graphs. For n even we show that there are at least $2^{n^2/(16+o(1))}$ such graphs on n vertices. We show (Corollary 6) that in a 3ECTF graph the average degree can be as small as $O(\log n)$ and as high as $n/4$. For all μ_2, we find $2^{\Omega(n^2)}$ distinct 3ECTF graphs of order $\leq n$ in which every two non-adjacent vertices have at least μ_2 common neighbors (Corollary 13). However, independent triplets in these constructions do not have many common neighbors.

Although this work is meant to be self-contained, the reader is encouraged to consult part III of Cherlin's paper [9] for more background of the subject and many additional details. To simplify matters we maintain a graph-theoretic terminology, and refrain from using Cherlin's view of maximal triangle-free graphs as combinatorial geometries. We believe that this presentation makes the structure and the symmetries of the graphs more transparent.

Notation: We denote the fact that vertices x, y are adjacent in the graph under discussion by $x \sim y$. The *neighborhood* of x is $N(x) = \{y \in V | x \sim y\}$.

2 Existentially Complete Graphs

Here are some definitions and useful reductions that are due to Cherlin.

Definition (Extension Properties, Section 11.1 of [9])**.** *Let $G = (V, E)$ be a triangle-free graph, and k a positive integer. We call G*

- (E_k) *or k-existentially complete, if: For every $B \subseteq A \subseteq V$ where $|A| \leq k$ and B is independent, there exists a vertex v that is adjacent to each vertex of B and to no vertex of $A \setminus B$.*
- (E'_k)*, if: For every $B \subseteq A \subseteq V$ where A is independent of cardinality* exactly *k there exists a vertex v, adjacent to each vertex of B and to no vertex of $A \setminus B$.*
  Also, every independent set with fewer than k vertices is contained in an independent set of cardinality k.
- (Adj_k)*: Every independent set of cardinality $\leq k$ has a common neighbor.*

An (Adj_2) graph is also called *maximal triangle-free*, since the addition of any edge would create a triangle. In other words, the graph has diameter ≤ 2. In a *twin-free* graph no two vertices have the same neighborhood.

We have the following implications for triangle-free graphs (Cherlin [9], Lemmas 11.2-11.4).

Lemma 1. *For $k \geq 2$, $(E_k) \Leftrightarrow (E'_k)$.*

Lemma 2. *For $k \geq 3$, $(E_k) \Leftrightarrow (Adj_k)$ and (E_3).*

Lemma 3. $(E_2) \Leftrightarrow$ *the graph is maximal triangle-free, twin-free and contains an anti-triangle.*

In order to bridge between Lemmas 2 and 3, Cherlin investigates triangle-free graphs that are (Adj_3) and (E_2) but not (E_3). In sections 11.4-11.5 of [9] these exceptions are described in terms of linear combinatorial geometries, which can be interpreted as graphs of a certain circular structure. In general, *circular* graphs can be defined as a set of arcs in a cyclically ordered set, where adjacency means disjointness of the corresponding arcs. Such graphs, defined by disjointedness in a family of subsets, are sometimes called (general) Kneser graphs (e.g. [17]).

Here and below we only consider finite graphs. The circular graph O_{3n-1} is formed by all arcs of n consecutive elements in $\mathbb{Z}_{3n-1}$. It was independently introduced several times, e.g., Erdős and Andrásfai [1], Woodall [20], Pach [18] and van den Heuvel [16]. In terms of rational complete graphs [14], O_{3n-1} is equivalent to $K_{(3n-1)/n}$, being "almost" a triangle. As shown in [18, 6], these are the only finite triangle-free twin-free graphs where every independent set has a common neighbor. By Lemma 11.15 in [9], they are also the only finite triangle-free twin-free graphs, which are (Adj_3) but not (E_3). Note that the 5-cycle O_5 and the edge O_2 are not even (E_2).

Corollary 4 (Cherlin)**.** *A finite triangle-free graph G is k-existentially complete if and only if the following conditions hold.*

1. *Every independent set of cardinality $\leq k$ has a common neighbor.*
2. *There do not exist two vertices x, y with $N(x) \subseteq N(y)$.*
3. *G is not isomorphic to O_{3n-1} for $n \geq 1$.*

In section 12 of [9], the notion of (Adj_k) is refined to the k-th *multiplicity* of G. This is done by considering the smallest possible number of common neighbors of an independent set of k vertices:

$$\mu_k(G) = \min_{|A|=k \text{ indep.}} \#\{b \in V(G) \mid N(b) \supseteq A\}$$

Cherlin proves the following chain of implications for 3ECTF graphs:

$$\mu_4(G) \geq 1 \;\Rightarrow\; \mu_3(G) \geq 5 \;\Rightarrow\; \mu_3(G) \geq 2 \;\Rightarrow\; \mu_2(G) \geq 5 \;\Rightarrow\; \mu_2(G) \geq 2$$

In these terms, the strongest known example is the strongly regular Higman-Sims graph, on 100 vertices, for which $\mu_3(HS) = 2$ and $\mu_2(HS) = 6$. However, by a beautiful spectral calculation ([9], Section 12.3) no strongly regular graph has property 4ECTF. Perhaps the Higman-Sims construction should be viewed as a sporadic example. On the other hand, here we introduce a large collection of 3ECTF graphs with arbitrarily large $\mu_2(G)$ and $\mu_3(G) = 1$. Thus the two lowest levels in this hierarchy are not very restrictive.

3 Albert Graphs

We turn to describe an infinite sequence of 3ECTF graphs which Cherlin attributes to Michael Albert. The Clebsch graph is a triangle-free strongly-regular graph on 16 vertices. It can be represented as the union of four 4-cycles, where each vertex is adjacent as well to its antipodes in the other cycles. One can check directly that this graph is 3-existentially complete, e.g. by Corollary 4. Albert's construction is the extension of the Clebsch graph to any number of 4-cycles. Formally, Albert's 3ECTF graphs sequence $A(n)$ is defined by

$$\begin{aligned} &V(A(n)) = \{(i, x) \mid i \in \{1, 2, ..., n\}, x \in \mathbb{Z}_4\} \\ &(i, x) \sim (i, x + 1) \;\; \forall i, x \\ &(i, x) \sim (i', x + 2) \;\; \forall x, i \neq i' \end{aligned}$$

where addition is in $\mathbb{Z}_4$.

This construction was thoroughly generalized by Cherlin, to Albert geometries. Here we offer a different viewpoint of these graphs. Let $m, n \geq 4$ be integers. An $m \times n$ zero-one matrix M is said to be *shattered* if the submatrix corresponding to any three rows or three columns contains all four possible patterns aaa, aab, aba, baa. Namely, it must contain at least one of the strings 000 and 111, and one of 001 and 110, and so on. The *Albert graph* A_M of a shattered matrix M is obtained from an m-matching and an n-matching. The corresponding entries of M tell us how to connect these $2m+2n$ vertices.

$$\begin{aligned} &V(A_M) = \{a_1, ..., a_m\} \cup \{b_1, ..., b_m\} \cup \{c_1, ..., c_n\} \cup \{d_1, ..., d_n\} \\ &a_i \sim b_i \ \ \forall i \\ &c_j \sim d_j \ \ \forall j \\ &a_i \sim c_j, \ b_i \sim d_j \quad \text{if } M_{ij} = 1 \\ &a_i \sim d_j, \ b_i \sim c_j \quad \text{if } M_{ij} = 0 \end{aligned}$$

For example, when M is the 4×4 identity matrix, A_M is the Clebsch graph. Albert's construction corresponds to larger identity matrices.

Proposition 5. *If the matrix M is shattered, then the Albert graph A_M is 3ECTF.*

Proof. We first observe that A_M is triangle-free. Of any three vertices at least two must either come from $U = \{a_1, ..., a_m\} \cup \{b_1, ..., b_m\}$, or from $W = \{c_1, ..., c_n\} \cup \{d_1, ..., d_n\}$. But an edge in U must be of the form (a_i, b_i), and a_i and b_i have no common neighbors.

We proceed to verify the conditions in Corollary 4. It is easy to see that A_M is twin-free and not an O_{3n-1}. Property (Adj_3) follows since M is shattered. An independent triplet in U must consist of either a_i or b_i, either a_j or b_j and either a_k or b_k for some distinct i, j, k. A common neighbor exists thanks to the appropriate pattern in the restriction of M to the rows i, j and k. For an independent set with two vertices in U and one in W, the neighbor of the latter inside W is adjacent also to the first two. □

The constructions of Albert Geometries in Examples 13.1 of [9], come from explicit shattered matrices. Random matrices can be used as well. Thus, simple counting arguments yield

Corollary 6.

- *Almost every zero-one $m \times n$ matrix with $m \geq C \log n$ and $n \geq C \log m$ is shattered. Here $C > 0$ is some absolute constant.*
- *For even n, the number of n-vertex 3ECTF graphs is at least $2^{n^2/(16+o(1))}$.*

This gives some insight on the possible behavior of 3ECTF graphs. On the one hand, taking $m = n$ we get a regular graph of degree $|V|/4 + 1$. On the other hand, if $m = \Theta(\log n)$ the graph has $\Theta(\log |V|)$ vertices of degree close to $|V|/2$ while most vertices have only $\Theta(\log |V|)$ neighbors. As for multiplicities, in every Albert graph $\mu_2(G) = 2$ and $\mu_3(G) = 1$.

4 Hypercube Graphs

Recall that the Clebsch graph is a Cayley graph of $\mathbb{Z}_2^4$, with generators the unit vectors and the all-ones vector. Equivalently, $x \sim y$ iff their Hamming distance $d(x, y)$ is 1 or 4. Following Franek and Rödl [13], we denote this graph by $\langle \mathbb{Z}_2^4, \{1, 4\} \rangle$. Note that it also equals $\langle \mathbb{Z}_2^4, \{3, 4\} \rangle$. Here we consider the graphs

$$C_{3k+1} = \langle \mathbb{Z}_2^{3k+1}, \{2k+1, 2k+2, ..., 3k+1\} \rangle$$

which Erdős [10] used in the study of Ramsey numbers. As mentioned, C_4 is the Clebsch graph. The extension properties of these graphs were studied by Pach [18]. For future use, we record a variant of his argument in the following lemma.

Lemma 7. *If $x, y, z \in \mathbb{Z}_2^n$ satisfy*

$$d(x, y) \leq a + b, \quad d(x, z) \leq a + c, \quad d(y, z) \leq b + c.$$

for some integers $a, b, c \geq 0$, then there is some $v \in \mathbb{Z}_2^n$ for which

$$d(v, x) \leq a, \quad d(v, y) \leq b, \quad d(v, z) \leq c.$$

Proof. Define the vector m by the coordinate-wise majority vote of x, y, and z. Note that the three vectors $x' = x+m$, $y' = y+m$ and $z' = z+m$ have disjoint supports. We find a v' satisfying the claim for these three vectors and let $v = v' + m$.

If the Hamming weights satisfy $w(x') \leq a$, $w(y') \leq b$ and $w(z') \leq c$, then we are done by taking $v' = 0$. Otherwise, by assumption, at most one of these inequalities can be violated, say $w(x') > a$. We take v' to have weight $w(v') = w(x') - a$ and satisfy $x' \geq v'$ coordinate-wise. Obviously $d(v', x') \leq a$. But also,

$$d(v', y') \leq w(v') + w(y') = w(x') - a + w(y') = d(x', y') - a \leq b,$$

and similarly $d(v', z') \leq c$. □

Proposition 8. *The graphs C_{3k+1} are 3ECTF.*

Proof. To see that C_{3k+1} is triangle-free, suppose $x \sim y \sim z$, then $d(x, z) \leq d(x, \bar{y}) + d(\bar{y}, z) \leq k + k = 2k$, where $\bar{y}$ is y's antipode, namely $\bar{y} = y + (1, ..., 1)$. Therefore $x \not\sim z$.

Now we check the conditions of Corollary 4. Clearly $N(x)$ is not contained in $N(y)$ for $x \neq y$, since both are distinct Hamming balls of the same radius. Property (Adj_3) follows using Lemma 7. Indeed, if the set $\{x, y, z\}$ is independent, apply the lemma with $a = b = c = k$. For v as in the lemma, $\bar{v}$ is at least $2k + 1$ away from each of the three vectors. □

Proposition 9. $\mu_2(C_{3k+1}) = \binom{2k}{k}$, $\mu_3(C_{3k+1}) = 1$.

Proof. We check that every two non-adjacent vertices in C_{3k+1} have at least $\binom{2k}{k}$ common neighbors. Let x and y be two vectors of even Hamming distance $2t$ where $1 \leq t \leq k$. We count some of the common neighbors of x and y. Of the $3k + 1 - 2t$ coordinates in which they agree, we flip $2k + 1 - t$ coordinates of our choice, and keep the other $k - t$ unchanged. The remaining $2t$ coordinates are divided equally, t as in x and t as in y. Each resulting vector is at Hamming distance $2k + 1$ from both x and y, which hence have at least

$$\binom{3k+1-2t}{k-t}\binom{2t}{t}$$

distinct joint neighbors. This expression is decreasing in t, and hence always $\geq \binom{2k}{k}$, with equality for $t = k$.

For $d(x, y) = 2t - 1$ odd, we argue similarly. We flip $2k + 1 - (t - 1)$ of the $3k + 1 - (2t - 1)$ common coordinates, and divide the other $2t - 1$ to t and $t - 1$. This yields

$$2\binom{3k+1-(2t-1)}{k-t}\binom{2t-1}{t}$$

common neighbors. The minimum, $\binom{2k}{k}$, is again attained at $t = k$.

Finally, it's easy to demonstrate three vectors in C_{3k+1} with a single joint neighbor. Take three vectors, x, y and z, of Hamming weight k and disjoint supports. For a common neighbor v we have

$$3(2k + 1) \leq d(x, v) + d(y, v) + d(z, v) \leq 2 \cdot 3k + 3 = 3(2k + 1)$$

since in $3k$ coordinates at most two of the distances can contribute. But equality is reached only by the all-ones vector $v = (1, ..., 1)$. □

By Propositions 8-9, the sequence C_{3k+1} constitutes an example for 3ECTF graphs with $\mu_2(C_{3k+1}) \to \infty$. Here $\mu_2 = n^{2/3-o(1)}$, where $n = 2^{3k+1}$ is the number of vertices. Also, these graphs are $n^{\lambda-o(1)}$-regular, where $\lambda = \log_2 3 - \frac{2}{3} \approx 0.918$. A neighborhood of a vertex in these graphs is also a largest possible independent set. See [10].

Here are two variations where the 2-multiplicity and the degree are traded off. We first apply Albert's idea to C_{3k+1}. For $x \in \mathbb{Z}_2^k$ denote $parity(x) = \sum_{i=1}^{k} x_i \bmod 2$. We first partition C_{3k+1} as follows.

$$V_1 = \{x \in \mathbb{Z}_2^{3k+1} \mid parity(x) = x_1 = x_2\}$$
$$V_2 = \{x \in \mathbb{Z}_2^{3k+1} \mid parity(x) \neq x_1 = x_2\}$$
$$V_3 = \{x \in \mathbb{Z}_2^{3k+1} \mid parity(x) = x_1 \neq x_2\}$$
$$V_4 = \{x \in \mathbb{Z}_2^{3k+1} \mid parity(x) = x_2 \neq x_1\}$$

Of course, we may forget the first two coordinates, and regard the elements of each V_i as $\mathbb{Z}_2^{3k-1}$. The adjacencies within each V_i correspond to Hamming distances $2k-1$ and $2k+1, ..., 3k-1$. Also, two vertices form distinct V_i are adjacent iff their Hamming distance is between $2k$ and $3k-1$. This leads to the following definition of $C_{3k-1}(m)$.

$$\begin{aligned}
&V(C_{3k-1}(m)) = \{(v,i) \mid v \in \mathbb{Z}_2^{3k-1}, 1 \le i \le m\} \\
&(v,i) \sim (u,i) \quad \text{if } d(v,u) \in \{2k-1, 2k+1, 2k+2, ..., 3k-1\} \\
&(v,i) \sim (u,j) \quad \text{if } i \ne j \text{ and } d(v,u) \in \{2k, 2k+1, 2k+2, ..., 3k-1\}
\end{aligned}$$

By the above discussion $C_{3k-1}(4) = C_{3k+1}$, and $C_{3k-1}(m)$ is 3ECTF for $m \ge 4$, since any three vertices belong to an isomorphic copy of C_{3k+1}. For the same reason $\mu_2(C_{3k-1}(m)) \ge \binom{2k}{k}$, and for constant k and large m the graph is $\Theta(n)$-regular.

To introduce the second variation, consider the following graph.

$$C'_{4k} = \left\langle \mathbb{Z}_2^{4k}, \{1, 3, 5, ..., 2k-1, 4k\} \right\rangle$$

By considering the two matchings on odd-parity and on even-parity vectors we see that this is an Albert graph. In fact, each odd distance d can be separately replaced by $4k-d$ to yield another Albert graph.

The following family of 3ECTF graphs which we describe without proof interpolates between C_{3k+1} and C'_{4k}. We do not give the general construction, only demonstrate the case $k = 5$.

$$\begin{gathered}
\left\langle \mathbb{Z}_2^{16}, \{11, 12, 13, 14, 15, 16\} \right\rangle \\
\left\langle \mathbb{Z}_2^{17}, \{11, 13, 14, 15, 16, 17\} \right\rangle \\
\left\langle \mathbb{Z}_2^{18}, \{11, 13, 15, 16, 17, 18\} \right\rangle \\
\left\langle \mathbb{Z}_2^{19}, \{11, 13, 15, 17, 18, 19\} \right\rangle \\
\left\langle \mathbb{Z}_2^{20}, \{11, 13, 15, 17, 19, 20\} \right\rangle
\end{gathered}$$

5 Twisted Graphs

Having seen 3ECTF graphs with unbounded μ_2, we move to the next construction in search of many such graphs.

We start with variation of a 3ECTF construction from Section 13.2 of [9]. Given positive integers m_0, m_1, m_2 and m_3, we define the twisted graph $G(m_0, m_1, m_2, m_3)$ as follows.

$$\begin{aligned}
&V(G(m_0, m_1, m_2, m_3)) = \{(i,j,x) \mid i \in \{0,1,2,3\}, 1 \le j \le m_i, x \in \mathbb{Z}_4\} \\
&(i,j,x) \sim (i,j,x+1) \ \ \forall i,j,x \\
&(i,j,x) \sim (i,j',x+2) \ \ \forall i,x,j \ne j' \\
&(i,j,x) \sim (i',j',x+3) \ \ \forall x,j,j',(i,i') \in \{(0,1),(0,2),(0,3),(1,2),(2,3),(3,1)\}
\end{aligned}$$

Remark: the graph $G(1, m_1, m_2, m_3)$ differs from $G(m_1, m_2, m_3)$ in Example 13.3 of [9], unless we switch two edges inside V_0. We do not know how to place both graphs on a common ground.

The following proposition reveals some of the structure of these graphs. Although it is a special case of Proposition 12, we believe that it is easier to follow and would make the more complicated Proposition 12 more transparent.

Proposition 10. *For $m_0, m_1, m_2, m_3 \ge 2$, the graph $G(m_0, m_1, m_2, m_3)$ is 3ECTF.*

Proof. For $i \in \{0,1,2,3\}$ let

$$V_i = \{(i,j,x) \in V(G) \mid x \in \mathbb{Z}_4, 1 \le j \le m_i\}.$$

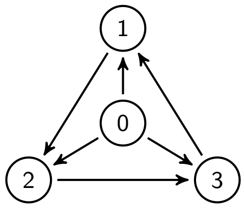
Note that each V_i spans a copy of $A(m_i)$ (see section 3). Moreover, the subgraph induced on $V_i \cup V_{i'}$ is $A(m_i + m_{i'})$. To see this, "twist" the $m_{i'}$ last 4-cycles in $A(m_i + m_{i'})$, by sending $x \mapsto x + 1 \bmod 4$. This affects only cross edges with $x + 2 \to x + 3$. Therefore, we only need to consider triplets from distinct V_i's.

We first verify triangle-freeness. Along an edge of such a triangle, the $\mathbb{Z}_4$-coordinate changes from x to $x \pm 3$, but three such numbers do not sum to zero mod 4.

We next find a common neighbor for three independent vertices (i,j,x), (i',j',x') and (i'',j'',x'') for distinct i, i' and i''. Suppose first that the parity of x differs from those of x' and x''. Which is the neighbor of (i',j',x') in the 4-cycle $(i,j,\cdot)$? From parity considerations, and since $(i,j,x) \not\sim (i',j',x')$, it must be $(i,j,x+2)$. For similar reasons $(i,j,x+2)$ is the neighbor of (i'',j'',x'') in that 4-cycle. This implies that $(i,j''',x+2)$ is a common neighbor of (i,j,x), (i',j',x') and (i'',j'',x'') for any $j''' \neq j$.

Now if x, x' and x'' have equal parities, then each of (i',j',x') and (i'',j'',x'') is adjacent to one of $\{(i,j,x-1),(i,j,x+1)\}$. If both are adjacent to the same one, it is the common neighbor and we are done. A similar solution may exist in different roles of the three vertices. If all fails, then either $(i,i',i'') = (1,2,3)$ and $x = x' = x''$ or $(i,i',i'') \in \{(0,1,2),(0,2,3),(0,3,1)\}$ and $x = x'' = x'+2$. In this case, the common neighbor is in the fourth part $V_{i'''}$, being $(i''',j''',x+1)$ and $(i''',j''',x+3)$ respectively, for any j'''. □

A *tournament* is an orientation of a complete graph. Edges $(i,i') \in E(T)$ are denoted $i \to i'$. Note that the set of ordered pairs $\{(0,1),(0,2),(0,3),(1,2),(2,3),(3,1)\}$, that appears in the above definition of $G(m_0,m_1,m_2,m_3)$, can be encoded by the following tournament that we call T_4.



The tournament T_4' is obtained from T_4 by reversing all edges. These two four-vertex tournaments are characterized by the property that every two vertices are connected by exactly one path of length two. In other words, for each pair of vertices one remaining vertex is attached to them in the same way, while the other one is attached in opposite ways. It is exactly this property that allowed us to find the common neighbor in the previous argument. A tournament is *shattered* if every three of its vertices extend to a copy of either T_4 or T_4'.

We next associate a graph $G_T(m)$ to a shattered tournament T and a positive integer m.

$$
\begin{aligned}
&V(G_T(m)) = \{(i,j,x) \mid i \in V(T), 1 \le j \le m, x \in \mathbb{Z}_4\} \\
&(i,j,x) \sim (i,j,x+1) \ \ \forall i,j,x \\
&(i,j,x) \sim (i,j',x+2) \ \ \forall i,x,j \neq j' \\
&(i,j,x) \sim (i',j',x+3) \ \ \forall x,j,j',i \to i'
\end{aligned}
$$

It is easy to apply the proof of Proposition 10 to general shattered tournaments and conclude that

Corollary 11. *The graph $G_T(m)$ is 3ECTF for every shattered tournament T and every $m \ge 2$.*

The graphs that were constructed so far in this section have $\mu_2(G) = 2$ and $\mu_3(G) = 1$. The next example integrates them with $C_{3k-1}(m)$, thus increasing the 2-multiplicity. To this end, we first define a similar *twist* function in the hypercube. The vector $\tau(x)$ is obtained by applying the following transformation on the first two coordinates of $x \in \mathbb{Z}_2^n$, leaving the remaining $n-2$ coordinates unchanged:

$$(0,0) \mapsto (0,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,0)$$

We turn to extend this construction and associate a graph $G_T(m,k)$ with every shattered tournament T and positive numbers m,k.

$$
\begin{aligned}
&V(G_T(m,k)) = \{(i,j,x) \mid i \in V(T), 1 \le j \le m, x \in \mathbb{Z}_2^{3k-1}\} \\
&(i,j,x) \sim (i,j,x') \quad \text{if } d(x,x') \in \{2k-1, 2k+1, 2k+2, ..., 3k-1\} \\
&(i,j,x) \sim (i,j',x') \quad \text{if } j \neq j' \text{ and } d(x,x') \in \{2k, 2k+1, ..., 3k-1\} \\
&(i,j,x) \sim (i',j',x') \quad \text{if } i \to i' \text{ and } d(x,\tau(x')) \in \{2k, 2k+1, ..., 3k-1\}
\end{aligned}
$$

Note that $G_T(m)$ is isomorphic to $G_T(m,1)$.

Proposition 12. *For $m \geq 2$, $k \geq 1$, and a shattered tournament T, the graph $G_T(m,k)$ is 3ECTF.*

Proof. We adapt the proof of Proposition 10 to $k > 1$. For $i \in V(T)$, we partition the vertices as follows:

$$V_i = \{(i,j,x) \mid x \in \mathbb{Z}_2^{3k-1}, 1 \leq j \leq m\}.$$

The induced graph on each V_i is isomorphic to $C_{3k-1}(m)$. Since τ is an isometry, also $V_i \cup V_{i'}$ induces a copy of $C_{3k-1}(2m)$. By our previous comments these subgraphs are 3ECTF's.

We next show that three vertices from distinct parts never form a triangle. For each edge between parts $(i,j,x) \sim (i',j',x')$ we have $d(x,\tau(x')) \geq 2k$. Therefore at least $2k-2$ of the last $3k-3$ coordinates of x and x' differ. Suppose, toward a contradiction, that a triangle of such three edges exists, and consider two cases.

- Suppose that along some of the edges the Hamming distance of the last $3k-3$ coordinates is at least $2k-1$. Then the three differences cannot add up to 0, because at least one of the last $3k-3$ coordinates flips three times.
- If, on the other hand, they all equal $2k-2$, then in each such edge $d(x,\tau(x')) = 2k$, and hence $\text{parity}(x) = \text{parity}(\tau(x'))$. But τ switches parity, so the parity changes three times around the triangle – a contradiction.

Next, we seek a common neighbor for the independent set $\{(i_1,j_1,x_1),(i_2,j_2,x_2),(i_3,j_3,x_3)\}$, with i_1,i_2,i_3 distinct. Define $\tau_{12} = \tau$ if $i_1 \to i_2$, and $\tau_{12} = \tau^{-1}$ otherwise. Correspondingly define τ_{23} and so on. Also, define the following two sets of three distances.

$$d_{12} = d(x_1,\tau_{12}(x_2)) \qquad d^*_{12} = d(\tau_{31}(x_1),\tau_{32}(x_2))$$
$$d_{23} = d(x_2,\tau_{23}(x_3)) \qquad d^*_{23} = d(\tau_{12}(x_2),\tau_{13}(x_3))$$
$$d_{31} = d(x_3,\tau_{31}(x_1)) \qquad d^*_{31} = d(\tau_{23}(x_3),\tau_{21}(x_1))$$

By the independence assumption, $d_{12}, d_{23}, d_{31} \leq 2k-1$. Note that d_{12} and d^*_{12} are defined by the same two vectors up to three applications of τ or τ^{-1}, so that $d^*_{12} = d_{12} \pm 1$, $d^*_{23} = d_{23} \pm 1$ and $d^*_{31} = d_{31} \pm 1$.

Let us assume first $d^*_{12} \leq 2k-2$. Apply Lemma 7 to the vectors x_3, $\tau_{31}(x_1)$ and $\tau_{32}(x_2)$, with $a = k$ and $b = c = k-1$ and obtain some $v \in \mathbb{Z}_2^{3k-1}$ with

$$d(v,x_3) \leq k, \qquad d(v,\tau_{31}(x_1)) \leq k-1, \qquad d(v,\tau_{32}(x_2)) \leq k-1.$$

If $d(v,x_3) = k$ then $(i_3,j_3,\bar{v})$ is a common neighbor, while for $d(v,x_3) = k-1$ it is $(i_3,j,\bar{v})$ for any $j \neq j_3$. For $d(v,x_3) \leq k-2$ any j would do.

By applying the same argument to d^*_{23} and d^*_{31}, we may assume that $d^*_{12}, d^*_{23}, d^*_{31} \geq 2k-1$. In particular, they exceed d_{12}, d_{23} and d_{31} by one, respectively.

The *restricted parity* of $x \in \mathbb{Z}_2^n$, is defined as the parity of its first two coordinates. We claim that, under the above assumption, x_1, x_2 and x_3 all have the same restricted parity. Otherwise, say x_3 is the exception. This implies several further properties.

1. The vectors $\tau_{21}(x_1)$ and $\tau_{23}(x_3)$ have different restricted parity.
2. Since $d^*_{12} = d_{12} + 1$, at least one of the first two coordinates of $\tau_{31}(x_1)$ and $\tau_{32}(x_2)$ which appear in the definition of d^*_{12} must differ. But then, from restricted parity considerations, they both differ.
3. Also by restricted parity and by the previous property, x_3 agrees either with $\tau_{31}(x_1)$ or with $\tau_{32}(x_2)$ on the first two coordinates, and disagrees on them with the other one. Say it disagrees with $\tau_{31}(x_1)$.

Properties 1 and 3 imply that $d^*_{31} < d_{31}$, an already settled case. Therefore, the three vectors must have the same restricted parity. Now, by the reasoning of property 2, each of the three following vector pairs differs in both of the first two coordinates.

$$\tau_{31}(x_1),\ \tau_{32}(x_2) \qquad \tau_{12}(x_2),\ \tau_{13}(x_3) \qquad \tau_{23}(x_3),\ \tau_{21}(x_1)$$

Here the shattered tournament comes in. There is an i_4 for which $\{i_1, i_2, i_3, i_4\}$ induces either a T_4 or a T_4' tournament. Therefore exactly one of (τ_{41}, τ_{42}) differs from its counterpart in (τ_{31}, τ_{32}). Consequently, $\tau_{41}(x_1)$ and $\tau_{42}(x_2)$ agree on their first two coordinates. Denoting their distance by d^+_{12}, we have

$$d^+_{12} = d^*_{12} - 2 = d_{12} - 1 \leq 2k - 2,$$

and likewise $d^+_{23}, d^+_{31} \leq 2k - 2$. Apply Lemma 7 to $\tau_{41}(x_1)$, $\tau_{42}(x_2)$ and $\tau_{43}(x_3)$ with $a = b = c = k - 1$, to obtain $v \in \mathbb{Z}_2^{3k-1}$ at distance at most $k - 1$ from each of the three. This yields the desired common neighbor $(i_4, j, \bar{v})$ for any possible j. □

As every two vertices of $G_T(2, k)$ are covered by some embedded copy of C_{3k+1}, we have $\mu_2(G_T(2, k)) \geq \binom{2k}{k}$ by Proposition 9. One can verify that this is in fact an equality. Since there are $2^{\Omega(n^2/2^{6k})}$ non-isomorphic tournaments on $n/2^{3k}$ vertices, and for large n almost all of them are shattered, Proposition 12 yields

Corollary 13. *There are*

$$2^{\Omega\left(\frac{n^2}{(\mu \log \mu)^3}\right)}$$

3ECTF graphs with up to n vertices, in which every pair of independent vertices has at least μ common neighbors.

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