Simplicial Complexes - Combinatorial and Probabilistic Challenges

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A member $A \in X$ is called a simplex or a face of dimension $\dim(A) := |A| - 1$ and $\dim(X) := \max\{\dim(A) | A \in X\}$
The elephant outside the room

The habitat of simplicial complexes in topology.

However, we will not even mention here the T word

Nor the H and C words - Homology and Cohomology.
We see that a graph is synonymous with a one-dimensional simplicial complex. It has zero-dimensional faces (aka vertices)
Familiar objects with new names

We see that a **graph** is synonymous with a **one-dimensional simplicial complex**. It has zero-dimensional faces (aka vertices) and one dimensional faces = edges.
What is expressible in this language?

How do you say a tree in the language of simplicial complexes?
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Of course, a tree is a connected and acyclic graph.
What is expressible in this language?

How do you say a tree in the language of simplicial complexes?

Of course, a tree is a connected and acyclic graph.

Does the language of simplicial complexes provide analogous terms?
ISRAEL JOURNAL OF MATHEMATICS, Vol 45, No. 4, 1983

ENUMERATION OF Q-ACYCLIC SIMPLICIAL COMPLEXES

BY
GIL KALAI

ABSTRACT
Let $\mathcal{C} = \mathcal{C}(n, k)$ be the class of all simplicial complexes $C$ over a fixed set of $n$ vertices ($2 \leq k \leq n$) such that: (1) $C$ has a complete $(k - 1)$-skeleton, (2) $C$ has precisely $\binom{n-1}{k-1}$ $k$-faces, (3) $H_k(C) = 0$. We prove that for $C \in \mathcal{C}$, $H_{k-1}(C)$ is a finite group, and our main result is:

$$\sum_{C} |H_{k-1}(C)|^2 = n^{\binom{n-2}{k-1}}.$$
Recall the incidence matrix of a graph

\[ V \times E \quad \text{Vertices vs. edges.} \]

\[ A_G = \begin{pmatrix} 
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i & \ldots & +1 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
j & \ldots & -1 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix} \]
G is connected iff $A_G$ has a trivial left kernel.
Incidence matrices tell many things

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  - Because $A_G$’s left kernel is the linear span of the indicator vectors of $G$’s connected components.
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- The cycle space of $G$ is the right kernel of $A_G$. 
Incidence matrices tell many things

- $G$ is **connected** iff $A_G$ has a **trivial left kernel**.
  - Because $A_G$’s left kernel is the linear span of the indicator vectors of $G$’s connected components.

- The **cycle space** of $G$ is the **right kernel** of $A_G$.
  - Because $A_G$’s right kernel is the linear span of the indicator vectors of $G$’s cycle.
Recall: Equivalent descriptions of trees

**Theorem**

If $G = (V, E)$ is a graph with $n$ vertices and $n - 1$ edges, then TFAE

1. $G$ is connected.
2. $G$ is acyclic.
Why $G$ is connected iff it is acyclic

For every $G$ \( \text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1 \)
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1. $G$ is connected $\iff$ $A_G$ has a **trivial** left kernel.
2. $G$ is acyclic $\iff$ $A_G$ has a **zero** right kernel.

$A_G$ is an $n \times (n - 1)$ matrix, hence this basic theorem is just an exercise in linear algebra...
We need a high-dimensional analog of the incidence matrix.
Boundary operator of a simplicial complex

$(d - 1)$-dimensional faces vs. $d$-dimensional faces.

$$\partial_2 = \begin{pmatrix}
... & ... & ijk & ... & ... & ... & ... & ... \\
... & ... & ... & ... & ... & ... & ... & ... \\
ij & ... & ... & ... & ... & ... & ... & ... \\
... & ... & ... & ... & ... & ... & ... & ... \\
i & ... & ... & ... & ... & ... & ... & ... \\
... & ... & ... & ... & ... & ... & ... & ... \\
jk & ... & ... & ... & ... & ... & ... & ... \\
... & ... & ... & ... & ... & ... & ... & ... \\
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\end{pmatrix}$$
Does this suggest what a hypertree is?

We know where to start:
Does this suggest what a **hypertree** is?

We know where to start:

Q: What is the rank of $\partial^K_n$?
Does this suggest what a hypertree is?

We know where to start:

Q: What is the rank of $\partial_d^{K_n}$?

A: $\binom{n-1}{d}$,
Does this suggest what a hypertree is?

We know where to start:

Q: What is the rank of $\partial^K_n$?

A: $\binom{n-1}{d}$, let’s prove it

That $\text{rank}(\partial_d) \leq \binom{n-1}{d}$ follows from $\partial_{d-1}\partial_d = 0$.

To show that $\text{rank}(\partial_d) \geq \binom{n-1}{d}$ we exhibit an explicit acyclic set of $\binom{n-1}{d}$ columns, i.e., with zero right kernel. This is the set of $d$-faces of a $d$-dimensional hypertree.
So, what is a $d$-dimensional hypertree?

It is a $d$-dimensional simplicial complex with

- A full $(d-1)$-dimensional skeleton.
- $n-1$ $d$-dimensional faces.
- Whose boundary operator $\partial_d$ has
  - a trivial left kernel.
  - zero right kernel.
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The simplest example of a hypertree

Arguably the simplest one-dimensional (=graphic) tree is a star, i.e., all 1-dimensional faces (=edges) that contain, say, vertex $n$. 
The simplest example of a hypertree

Arguably the simplest one-dimensional (＝graphic) tree is a star, i.e., all 1-dimensional faces (＝edges) that contain, say, vertex \( n \).
The same works in every dimension: Take all \( d \)-faces (＝sets of size \( d + 1 \)) which contain the vertex \( n \).
Let’s see how this works.
The $d$-dimensional hyperstar

- The **boundary operator** $\partial^X_d$ of a $d$-dimensional simplicial complex $X$ is the signed inclusion matrix of its $(d - 1)$ vs. $d$-dimensional faces.
- Rows are indexed by $X$'s faces of dimension $d - 1$ and columns by its $d$-faces.
- The signing reflects the chosen **orientation** of the faces, but here we skip this issue.
- In the case of the hyperstar the rows naturally fall in two categories:
Boundary operator of the hyperstar

- The top rows are the $d$-sets (equal to $(d - 1)$-faces) that do not contain the element $n$.
- The bottom rows are the $d$-sets that contain the element $n$.
- Columns: All $(d + 1)$-sets that contain the element $n$.

Therefore:

\[ \partial_d S^n = \left( \frac{I}{\partial_{d-1} S^{n-1}} \right) \]
This complex, the $d$-dimensional hyperstar is **acyclic**, i.e., has zero right kernel due to the identity matrix part of its boundary operator.

and since the number of its $d$-faces is $\binom{n-1}{d}$ it is a hypertree.
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In some cases you can draw a similar conclusion even when no such identity matrix sits there and without any linear-algebraic calculations.
Collapsibility

Let $X$ be a $d$-dimensional complex.
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Collapsibility

Let $X$ be a $d$-dimensional complex. If some $(d - 1)$-dimensional face $\tau$ is contained in a unique $d$-dimensional face $\sigma$, then in the corresponding elementary collapse we remove both $\tau$ and $\sigma$ from $X$.

$X$ is $d$-collapsible if it is possible to eliminate all its $d$-faces by a series of elementary collapses. A simple linear algebra argument shows that collapsibility implies acyclicity.
Let $\partial^X$ be the (matrix representing the) boundary operator of the complex $X$. If $\sigma$ is the one and only $d$-face that contains the $(d-1)$-face $\tau$, then $(\tau, \sigma)$ is the only nonzero entry in row $\tau$. This yields the elementary collapse which erases row $\tau$ and column $\sigma$ of $\partial^X$. But column $\sigma$ cannot participate in any linear dependency of the columns, since no other column can help us zero out the $\tau$-th coordinate.
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Therefore, whether $X$ is acyclic or not, does not change due to an elementary collapse.

$X$ is *d-collapsible* if it is possible to eliminate all columns of $A_X$ by a series of elementary collapses.
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\( X \) is \( d \)-collapsible if it is possible to eliminate all columns of \( A_X \) by a series of elementary collapses.

Consequently, if \( X \) is collapsible then it is acyclic. Thus, collapsing is a way to prove that the right kernel is empty.
Meanwhile... in dimension 1

- Which graphs are 1-collapsible?

A 1-dimensional elementary collapse: Remove from the graph a vertex $v$ of degree 1 and the unique edge $e$, that contains it.

Easy to verify: 1-collapsible graph = forest.

Consequently, in dimension 1: An $n$-vertex ($n-1$)-edge graph is 1-collapsible iff it is a tree.
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In dimension 1 the two properties are, in fact equivalent.
The simple 1-dimensional world

- As we saw, collapsibility implies acyclicity in all dimensions.
- In dimension 1 the two properties are, in fact equivalent.
- Does this equivalence hold in higher dimension as well?
No, we are not in Kansas anymore

$$\binom{6-1}{2} = 10$$

Figure: A triangulation of the projective plane
Can we find more examples of non-collapsible hypertrees?
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A construction: Let $n$ be prime and $d \geq 2$. Fix a subset $A \subset \mathbb{Z}_n$ of cardinality $|A| = d + 1$. The sum complex $X_A$ corresponding to $A$ has a full $(d - 1)$-dimensional skeleton and contains a $d$-face $\sigma$ iff $\sum_{x \in \sigma} x \in A$. 

Theorem (Linial, Meshulam, Rosenthal) The complex $X_A$ is always a $Q$-hypertree. It is collapsible iff $A$ forms an arithmetic progression.
The plot thickens

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**Theorem (Linial, Meshulam, Rosenthal)**

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Let $\mathcal{C} = \mathcal{C}(n, k)$ be the class of all simplicial complexes $C$ over a fixed set of $n$ vertices ($2 \leq k \leq n$) such that: (1) $C$ has a complete $(k-1)$-skeleton, (2) $C$ has precisely $\binom{n}{k-1}$ $k$-faces, (3) $H_k(C) = 0$. We prove that for $C \in \mathcal{C}$, $H_{k-1}(C)$ is a finite group, and our main result is:

$$\sum_{C \in \mathcal{C}} |H_{k-1}(C)|^2 = n \binom{n-2}{k-2}.$$
What field are we in?

Zero right kernel: over what field?
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Zero right kernel: over what field?

- For the hyperstar or any collapsible hypertree the answer is the same: It does not matter what the underlying field is
- What about the 6-point triangulation of the projective plane?
Orient all triangular faces \textit{clockwise}. The sum of all the columns of the boundary operator matrix is

$$-2(e_{12} + e_{23} + e_{31})$$
We conclude that unlike the 1-dimensional case of graphs, the notion of a $d$-dimensional hypertree depends on the underlying field.
The field matters

We conclude that unlike the 1-dimensional case of graphs, the notion of a $d$-dimensional hypertree depends on the underlying field.

Specifically: The 6-point triangulation of the projective plane is a $\mathbb{Q}$-hypertree, but not a $\mathbb{F}_2$-hypertree.
More collapsibility vs. acyclicity

As already posed by Kalai:

Conjecture
For every $d \geq 2$ and for every field $F$ and $n \to \infty$ almost none of the $n$-vertex $d$-dimensional $F$-hypertrees are collapsible.

This is supported by rigorous numerical experiments, but remains open.

More evidence - in our analysis of random complexes (coming up)
More collapsibility vs. acyclicity

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For every $d \geq 2$ and for every field $\mathbb{F}$ and $n \to \infty$ almost none of the $n$-vertex $d$-dimensional $\mathbb{F}$-hypertrees are collapsible.

This is supported by rigorous numerical experiments, but remains open.

More evidence - in our analysis of random complexes (coming up)
Three conditions define when $X$ is a hypertree

- Full $(d - 1)$-skeleton.
- Right number of $d$-faces.
- Empty cycle space.

The latter condition can be replaced by:

- The $d$-faces of $X$ span all $\binom{n}{d+1}$ faces of dimension $d$.

and this requirement makes sense not only over a field but also over $\mathbb{Z}$. 

Over $\mathbb{Z}$ rather than over a field?
**Z-hypertrees**

**Definition**
A $d$-dimensional hypertree $X$ on vertex set $[n]$ is called a $\mathbb{Z}$-hypertree if the $d$-faces of $X$ span over the integers all $(\binom{n}{d+1})$ faces of dimension $d$.

**Example**
The 6-point triangulation of the projective plane is not a $\mathbb{Z}$-hypertree. If you want to generate the triple $(1, 2, 3)$, you need to divide by 2.

**Example**
Hyperstars are $\mathbb{Z}$-hypertrees. Likewise for all collapsible hypertrees.
Problem

Do there exist non-collapsible $\mathbb{Z}$-hypertrees? 
Even $\mathbb{Z}$-hypertrees that afford no collapses?

Theorem (Even-Zohar, Linial, Nowik, Peled; Work in progress)

We constructed infinitely many 2-dimensional $\mathbb{Z}$-hypertrees in which no elementary collapse is possible. (i.e., every edge is covered at least twice).
Problem

*Do there exist non-collapsible $\mathbb{Z}$-hypertrees? Even $\mathbb{Z}$-hypertrees that afford no collapses?*

Theorem (Even-Zohar, Linial, Nowik, Peled; Work in progress)

*We constructed infinitely many 2-dimensional $\mathbb{Z}$-hypertrees in which no elementary collapse is possible. (i.e., every edge is covered at least twice). This area is still poorly understood*
A random process

We consider the random process that starts with a full \((d - 1)\)-dimensional skeleton. At each step pick a random \(d\)-dimensional face \(\sigma \notin X\). If possible, we add \(\sigma\) to \(X\). Otherwise, we discard \(\sigma\).
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We consider the random process that starts with a full \((d - 1)\)-dimensional skeleton. At each step pick a random \(d\)-dimensional face \(\sigma \notin X\). If possible, we add \(\sigma\) to \(X\). Otherwise, we discard \(\sigma\).

We cannot add \(\sigma\) to \(X\) iff this creates a new cycle. In this case we say that \(\sigma\) is in the shade of \(X\).

To wit: At each step we add to the current complex a random \(d\)-face \(\sigma\) whose addition creates no new cycle (”\(\sigma\) is not in the shade of \(X\”).
Let $G = (V, E)$ be a disconnected graph, and let $ij \not\in E$. We say that $ij$ is in $G$’s shadow if $i$ and $j$ belong to the same connected component of $G$. 
Let \( G = (V, E) \) be a disconnected graph, and let \( ij \notin E \). We say that \( ij \) is in \( G \)'s shadow if \( i \) and \( j \) belong to the same connected component of \( G \). In other words \( ij \) is in \( G \)'s shadow iff the column corresponding to the edge \( ij \) is in the linear span of the columns of \( A_G \).
Outside the shadow of an evolving graph
Shadow of an evolving 2-complex
The quick jump in the probability of being in the shade is very intriguing. The complex at the transition point is very curious "the dream-catcher". It is acyclic, has $\Theta(n^2)$ faces, yet 70% of the $\binom{n}{3}$ faces are at its shadow.

Problem

Provide a description of the dream-catcher.
In dimension $d = 1$, the above process yields the random minimum spanning tree MST. (Equivalently, pick random edge weights in $(0, 1)$ and construct the resulting MST).

**Problem**

It is known that uniform trees and random MST differ. E.g., their diameters are $n^{1/2}$ resp. $n^{1/3}$. Can we develop the analogous theory in higher dimensions?
Can we run the same process also over $\mathbb{Z}$?

Yes, but linear algebra does not suffice. You must bring the relevant matrices to their Smith Normal Form.

Things are harder outside of linear algebra.

Namely, at least experimentally this process ends one short of the mark. I.e., a complex with $\binom{n}{2} - 1$ two-dimensional faces that cannot be extended to a 2-dimensional $\mathbb{Z}$-hypertree.

Much remains unknown here.
Inspiration from $G(n, p)$

No need to explain to this audience what $G(n, p)$ is.

Or what the evolution of random graphs is.

What about their $d$-dimensional counterparts?
A $d$-dimensional analog of $G(n, p)$

In 2006, Roy Meshulam and I introduced a model of a random $d$-dimensional $n$-vertex complex $X_d(n, p)$. The $X_1(n, p)$ model coincides with $G(n, p)$.
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Start with a full $(d - 1)$-dimensional skeleton. (For graphs, we start with $n$ vertices.)
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Start with a full $(d-1)$-dimensional skeleton. (For graphs, we start with $n$ vertices.)

Each $d$-dimensional face (For graphs - each edge), is placed in $X$ independently and with probability $p$. 
Recall basic facts in $G(n, p)$ theory

Theorem (Erdős and Rényi '60)

The threshold for graph connectivity in $G(n, p)$ is

$$p = \frac{\ln n}{n}$$
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**Theorem (L. - Meshulam - Wallach)**

In $X_d(n, p)$ the threshold for the event that $\partial_d$ has a trivial left kernel is

$$p = \frac{d \ln n}{n}.$$
Collapsibity vs. acyclicity in $X_d(n, p)$

Theorem (Aronshtam, Linial, Łuczak, Meshulam, Peled)

- The collapsibility threshold in $X_d(n, p)$ is

$$ (1 + o_d(1)) \frac{\log d}{n}. $$
Collapsibility vs. acyclicity in $X_d(n, p)$

Theorem (Aronshtam, Linial, Łuczak, Meshulam, Peled)

- The collapsibility threshold in $X_d(n, p)$ is

$$\left(1 + o_d(1)\right) \frac{\log d}{n}.$$

- The threshold for having a cycle whp is

$$\frac{d + 1 - o_d(1)}{n}.$$
Phase transition in the evolution of random complexes?

The evolution of random graphs, starts with isolated edges. Connected components appear later and are

- small = cardinality $O(\log n)$. 

- simple = a tree.
- Plus a Poisson number of unicyclic graphs with $O(\log n)$ vertices.

Around step $n^{2/3}$ (i.e., $p = 1/n$) a giant component quickly emerges. giant = linear size
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- Around step $\frac{n}{2}$ (i.e., $p = \frac{1}{n}$) a giant component quickly emerges. **giant**=linear size
More events around the phase transition

For $p < \frac{1-\epsilon}{n}$, the probability that the graph is acyclic is bounded away from both zero and one.
For $p < \frac{1-\epsilon}{n}$, the probability that the graph is **acyclic** is bounded away from both zero and one.

But if $p > \frac{1+\epsilon}{n}$, $G$ almost surely contains a cycle.
High-dimensional phase transition

Connected component in dimension $d \geq 2$?

Theorem (Linial, Peled)

Exactly at the same $p = \frac{c}{n}$ where $X_d(n, p)$ almost surely acquires a cycle, the shadow of the complex becomes gigantic (i.e., has $\Omega(n^{d+1})$ faces).
Connected component in dimension $d \geq 2$?

**Theorem (Linial, Peled)**

Exactly at the same $p = \frac{c}{n}$ where $X_d(n,p)$ almost surely acquires a cycle, the *shadow* of the complex becomes *gigantic* (i.e., has $\Omega(n^{d+1})$ faces).

This statement applies in all dimensions, including $d = 1$. However, when $d = 1$ the limit distribution is continuous but not smooth, while for $d \geq 2$ the limit distribution is discontinuous.
A view of phase transition in $G(n, p)$
Phase transition in $X_2(n, p)$ complexes
The number of trees with vertex set \([n]\) is \(n^{n-2}\).

Theorem (Kalai 1983)

\[ \sum_{T} |H_{d-1}(T)|^2 = n \binom{n-2}{d} \]

where the sum is over all \(n\)-vertex \(d\)-dimensional \(\mathbb{Q}\)-hypertrees \(T\).
But how many $d$-hypertrees are there?

Open Problem

*For $d \geq 2$ and large $n$, find (at least approximately) the number of $d$-dimensional $n$-vertex $\mathbb{Q}$-hypertrees.*

Kalai’s Formula yields estimates, but falls short of an asymptotic formula. Y. Peled and I have significantly improved these estimates. A full answer still eludes us.
But how many $d$-hypertrees are there?

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The estimates build on the analysis of the evolution of random complexes.
Easy Observation
Let $G$ be an "almost tree", i.e., an $n$ vertex forest with $n - 2$ edges (and hence with two connected components). Then at least $(1 - o(1))\frac{n^2}{4}$, i.e., at least half of the remaining edges, are in $G$’s shadow.
Construction: Let $X$ be a 2-dimensional $n$-vertex complex with a full 1-dimensional skeleton. The 2-faces of $X$ are the arithmetic triples of difference $\neq 1$. Easy fact: The number of 2-faces in $X$ is $\binom{n-1}{2} - 1$ (one less than a 2-dimensional hypertree).

\[\text{Theorem (L., Newman, Peled, Rabinovich)}\]

The complex $X$ is $Q$-acyclic. Assuming the Riemann hypothesis, there are infinitely many primes $n$ for which $X$ has an empty shadow.

\[\text{\textsuperscript{1}It actually suffices to assume the weaker Artin’s conjecture.}\]
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**Theorem (L., Newman, Peled, Rabinovich)**

*The complex $X$ is $\mathbb{Q}$-acyclic. Assuming the Riemann hypothesis\(^1\), there are infinitely many primes $n$ for which $X$ has an empty shadow.*

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\(^1\)It actually suffices to assume the weaker Artin’s conjecture.
A path is a tree where every vertex is in two edges or fewer.

Can we construct a $d$-dimensional hypertree where every $(d - 1)$-dimensional face is contained in $d + 1$ or fewer $d$-dimensional face?

Here is what Amir Dahari and I have done on this:
Let’s concentrate on dimension $d = 2$. Although things seem to work well for higher $d$ as well, the case of $d = 2$ is already quite challenging.

Given a prime $n$ and $c \in \mathbb{F}_n$ with $c \neq 0, 1, -2$, we consider the 2-complex with vertex set $\mathbb{F}_n$, a full 1-skeleton where the triple $\{x, y, z\}$ is a 2-face iff

$$x + y + cz = 0$$
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Does it work?
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Does it work? Often it does!
Is it a Hyperpath?
Thank you for your attention
Claim: If $X$ is an acyclic $d$-complex, then the number of $d$-faces in its shadow $Y$ is $\leq \frac{n \cdot |X|}{d+1}$.
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There are exactly $(d + 1) \cdot |Y|$ pairs $(v, \sigma)$ with $v$ a vertex in $\sigma$, a $d$-face in $Y$. Let $W$ be the set of $d$-faces in $Y$ that contain $v$. 
A lower bound on the number of hypertrees: A taste of the proof

**Claim:** If $X$ is an acyclic $d$-complex, then the number of $d$-faces in its shadow $Y$ is $\leq \frac{n \cdot |X|}{d+1}$.

There are exactly $(d + 1) \cdot |Y|$ pairs $(v, \sigma)$ with $v$ a vertex in $\sigma$, a $d$-face in $Y$. Let $W$ be the set of $d$-faces in $Y$ that contain $v$. $W$ is an acyclic complex, being part of $v$’s hyperstar. So, the columns corresponding to $W$ are linearly independent and spanned by $X$. Therefore $|W| \leq |X|$, which proves our claim.
Collapsibility

An elementary collapse is a step where you remove a vertex of degree one and the single edge that contains it.
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A graph $G$ is collapsible if by repeated application of elementary collapses you can eliminate all of the edges in $G$. 
But note

As we saw, connectivity and acyclicity are *linear algebraic*. In contrast collapsibility is a *purely combinatorial* condition.
But note

As we saw, connectivity and acyclicity are linear algebraic. In contrast collapsibility is a purely combinatorial condition. Indeed we will soon see that in higher dimensions collapsibility implies connectivity and acyclicity, but the reverse implication does not hold.