Simplicial Complexes - Combinatorial and Probabilistic Challenges

Nati Linial
Hebrew University

RS&A Meeting
August 2022

Simplicial Complexes - A quick reminder/primer

A simplicial complex X on vertex set V is a collection of subsets of V that is closed down i.e.,

$$A \in X \text{ and } B \subseteq A \Rightarrow B \in X$$

Simplicial Complexes - A quick reminder/primer

A simplicial complex X on vertex set V is a collection of subsets of V that is closed down i.e.,

$$A \in X \text{ and } B \subseteq A \Rightarrow B \in X$$

A member $A \in X$ is called a simplex or a face of dimension $\dim(A) := |A| - 1$

Simplicial Complexes - A quick reminder/primer

A simplicial complex X on vertex set V is a collection of subsets of V that is closed down i.e.,

$$A \in X \text{ and } B \subseteq A \Rightarrow B \in X$$

A member $A \in X$ is called a simplex or a face of dimension $\dim(A) := |A| - 1$ and $\dim(X) := \max\{\dim(A) | A \in X\}$

The elephant outside the room

The habitat of simplicial complexes in topology.

However, we will not even mention here the T word

Nor the H and C words - Homology and Cohomology.

Familiar objects with new names

We see that a graph is synonymous with a one-dimensional simplicial complex. It has zero-dimensional faces (aka vertices)

Familiar objects with new names

We see that a graph is synonymous with a one-dimensional simplicial complex. It has zero-dimensional faces (aka vertices) and one dimensional faces = edges

What is expressible in this language?

How do you say a tree in the language of simplicial complexes?

What is expressible in this language?

How do you say a tree in the language of simplicial complexes?

Of course, a tree is a connected and acyclic graph

What is expressible in this language?

How do you say a tree in the language of simplicial complexes?

Of course, a tree is a connected and acyclic graph

Does the language of simplicial complexes provide analogous terms?

Gil Kalai 1983

ISRAEL JOURNAL OF MATHEMATICS, Vol 45, No. 4, 1983

ENUMERATION OF Q-ACYCLIC SIMPLICIAL COMPLEXES

BY

GIL KALAI

ABSTRACT

Let $\mathscr{C} = \mathscr{C}(n, k)$ be the class of all simplicial complexes C over a fixed set of n vertices $(2 \leqq k \leqq n)$ such that: (1) C has a complete $(k-1)$-skeleton, (2) C has precisely $\binom{n-1}{k}$ k-faces, (3) $H_k(C) = 0$. We prove that for $C \in \mathscr{C}$, $H_{k-1}(C)$ is a finite group, and our main result is:

$$\sum_C |H_{k-1}(C)|^2 = n^{\binom{n-2}{k}}.$$

Recall the incidence matrix of a graph

$V \times E$ Vertices vs. edges.

$$A_G = \begin{array}{c} \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \dots & ij & \dots & \dots & \dots \end{array} \\ \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & +1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & -1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}$$

Incidence matrices tell many things

- G is connected iff A_G has a trivial left kernel.

Incidence matrices tell many things

- G is connected iff A_G has a trivial left kernel.
  - Because A_G's left kernel is the linear span of the indicator vectors of G's connected components.

Incidence matrices tell many things

- G is connected iff A_G has a trivial left kernel.
  - Because A_G's left kernel is the linear span of the indicator vectors of G's connected components.
- The cycle space of G is the right kernel of A_G.

Incidence matrices tell many things

- G is connected iff A_G has a trivial left kernel.
  - Because A_G's left kernel is the linear span of the indicator vectors of G's connected components.
- The cycle space of G is the right kernel of A_G.
  - Because A_G's right kernel is the linear span of the indicator vectors of G's cycle.

Recall: Equivalent descriptions of trees

Theorem

If $G = (V, E)$ is a graph with n vertices and $n - 1$ edges, then TFAE

1. *G is connected.*
2. *G is acyclic.*

Why G is connected iff it is acyclic

For every G $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

Why G is connected iff it is acyclic

For every G $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

The only linear dependence among the rows is

$$1A_{K_n} = 0.$$

Why G is connected iff it is acyclic

For every G $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

The only linear dependence among the rows is

$$1A_{K_n} = 0.$$

1. G is connected $\Leftrightarrow$

Why G is connected iff it is acyclic

For every G $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

The only linear dependence among the rows is

$$1A_{K_n} = 0.$$

1. G is connected $\Leftrightarrow$ A_G has a trivial left kernel.

Why G is connected iff it is acyclic

For every G $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

The only linear dependence among the rows is

$$1A_{K_n} = 0.$$

1. G is connected $\Leftrightarrow$ A_G has a trivial left kernel.
2. G is acyclic $\Leftrightarrow$

Why G is connected iff it is acyclic

For every G $\quad \text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

The only linear dependence among the rows is

$$1A_{K_n} = 0.$$

1. G is connected $\Leftrightarrow$ A_G has a trivial left kernel.
2. G is acyclic $\Leftrightarrow$ A_G has a zero right kernel.

Why G is connected iff it is acyclic

For every G $\text{rank}(A_G) \leq \text{rank}(A_{K_n}) = n - 1$

The only linear dependence among the rows is

$$1 A_{K_n} = 0.$$

1. G is connected $\Leftrightarrow$ A_G has a trivial left kernel.
2. G is acyclic $\Leftrightarrow$ A_G has a zero right kernel.

A_G is an $n \times (n-1)$ matrix, hence this basic theorem is just an exercise in linear algebra...

Setting the scene

We need a high-dimensional analog of the incidence matrix.

Boundary operator of a simplicial complex

$(d-1)$-dimensional faces vs. d-dimensional faces.

$$\partial_2 = \begin{array}{c} \\ \vdots \\ ij \\ \vdots \\ ik \\ \vdots \\ \vdots \\ jk \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \dots & \dots & ijk & \dots & \dots & \dots & \dots & \dots \end{array} \\ \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & +1 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -1 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & +1 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}$$

Does this suggest what a hypertree is ?

We know where to start:

Does this suggest what a hypertree is ?

We know where to start:

Q: What is the rank of $\partial_d^{K_n}$?

Does this suggest what a hypertree is ?

We know where to start:

Q: What is the rank of $\partial_d^{K_n}$?

A: $\binom{n-1}{d}$,

Does this suggest what a hypertree is ?

We know where to start:

Q: What is the rank of $\partial_d^{K_n}$?

A: $\binom{n-1}{d}$, let's prove it

That $\text{rank}(\partial_d) \leq \binom{n-1}{d}$ follows from $\partial_{d-1}\partial_d = 0$.

To show that $\text{rank}(\partial_d) \geq \binom{n-1}{d}$ we exhibit an explicit acyclic set of $\binom{n-1}{d}$ columns, i.e., with zero right kernel. This is the set of d-faces of a d-dimensional hypertree.

So, what is a d-dimensional hypertree?

It is a d-dimensional simplicial complex with

So, what is a d-dimensional hypertree?

It is a d-dimensional simplicial complex with

- A full $(d-1)$-dimensional skeleton.

So, what is a d-dimensional hypertree?

It is a d-dimensional simplicial complex with

- A full $(d-1)$-dimensional skeleton.
- It has $\binom{n-1}{d}$ d-dimensional faces.

So, what is a d-dimensional hypertree?

It is a d-dimensional simplicial complex with

- A full $(d-1)$-dimensional skeleton.
- It has $\binom{n-1}{d}$ d-dimensional faces.

Whose boundary operator ∂_d has

- a trivial left kernel.

So, what is a d-dimensional hypertree?

It is a d-dimensional simplicial complex with

- A full $(d-1)$-dimensional skeleton.
- It has $\binom{n-1}{d}$ d-dimensional faces.

Whose boundary operator ∂_d has

- a trivial left kernel.
- zero right kernel.

The simplest example of a hypertree

Arguably the simplest one-dimensional (=graphic) tree is a star, i.e., all 1-dimensional faces (=edges) that contain, say, vertex n.

The simplest example of a hypertree

Arguably the simplest one-dimensional (=graphic) tree is a star, i.e., all 1-dimensional faces (=edges) that contain, say, vertex n.
The same works in every dimension: Take all d-faces (=sets of size $d+1$) which contain the vertex n.
Let's see how this works.

The d-dimensional hyperstar

- The boundary operator ∂_d^X of a d-dimensional simplicial complex X is the signed inclusion matrix of its $(d-1)$ vs. d-dimensional faces.
- Rows are indexed by X's faces of dimension $d-1$ and columns by its d-faces.
- The signing reflects the chosen orientation of the faces, but here we skip this issue.
- In the case of the hyperstar the rows naturally fall in two categories:

Boundary operator of the hyperstar

- The top rows are the d-sets (=($d-1$)-faces) that do not contain the element n
- The bottom rows are the d-sets that contain the element n
- Columns: All $(d+1)$-sets that contain the element n.

Therefore:

$$\partial_d^{S_n} = \left(\frac{I}{\partial_{d-1}^{S_{n-1}}} \right)$$

The d-dimensional hyperstar (contd.)

- This complex, the d-dimensional hyperstar is acyclic, i.e., has zero right kernel due to the identity matrix part of its boundary operator.
- and since the number of its d-faces is $\binom{n-1}{d}$ it is a hypertree.

The d-dimensional hyperstar (contd.)

- This complex, the d-dimensional hyperstar is acyclic, i.e., has zero right kernel due to the identity matrix part of its boundary operator.
- and since the number of its d-faces is $\binom{n-1}{d}$ it is a hypertree.
- In some cases you can draw a similar conclusion even when no such identity matrix sits there and without any linear-algebraic calculations.

Collapsibility

Let X be a d-dimensional complex.

Collapsibility

Let X be a d-dimensional complex.
If some $(d-1)$-dimensional face τ is contained in a unique d-dimensional face σ, then in the corresponding elementary collapse we remove both τ and σ from X.

Collapsibility

Let X be a d-dimensional complex.
If some $(d-1)$-dimensional face τ is contained in a unique d-dimensional face σ, then in the corresponding elementary collapse we remove both τ and σ from X.
X is d-collapsible if it is possible to eliminate all its d-faces by a series of elementary collapses.

Collapsibility

Let X be a d-dimensional complex.
If some $(d-1)$-dimensional face τ is contained in a unique d-dimensional face σ, then in the corresponding elementary collapse we remove both τ and σ from X.
X is d-collapsible if it is possible to eliminate all its d-faces by a series of elementary collapses.
A simple linear algebra argument shows that collapsibility implies acyclicity.

Collapsing - a linear algebra perspective

Let ∂^X be the (matrix representing the) boundary operator of the complex X. If σ is the one and only d-face that contains the $(d-1)$-face τ, then (τ, σ) is the only nonzero entry in row τ. This yields the elementary collapse which erases row τ and column σ of ∂^X.

Collapsing - a linear algebra perspective

Let ∂^X be the (matrix representing the) boundary operator of the complex X. If σ is the one and only d-face that contains the $(d-1)$-face τ, then (τ, σ) is the only nonzero entry in row τ. This yields the elementary collapse which erases row τ and column σ of ∂^X.

But column σ cannot participate in any linear dependency of the columns, since no other column can help us zero out the τ-th coordinate.

Collapsing - a linear algebra perspective

Therefore, whether X is acyclic or not, does not change due an elementary collapse.

X is d-collapsible if it is possible to eliminate all columns of A_X by a series of elementary collapses.

Collapsing - a linear algebra perspective

Therefore, whether X is acyclic or not, does not change due an elementary collapse.

X is *d-collapsible* if it is possible to eliminate all columns of A_X by a series of elementary collapses.

Consequently, if X is collapsible then it is acyclic. Thus, collapsing is a way to prove that the right kernel is empty.

Meanwhile... in dimension 1

- Which graphs are 1-collapsible?

Meanwhile... in dimension 1

- Which graphs are 1-collapsible?
- A 1-dimensional elementary collapse: Remove from the graph a vertex v of degree 1 and the unique edge e, that contains it.

Meanwhile... in dimension 1

- Which graphs are 1-collapsible?
- A 1-dimensional elementary collapse: Remove from the graph a vertex v of degree 1 and the unique edge e, that contains it.
- Easy to verify: 1-collapsible graph=forest.
- Consequently, in dimension 1: An n-vertex $(n-1)$-edge graph is 1-collapsible iff it is a tree

The simple 1-dimensional world

- As we saw, collapsibility implies acyclicity in all dimensions

The simple 1-dimensional world

- As we saw, collapsibility implies acyclicity in all dimensions
- In dimension 1 the two properties are, in fact equivalent

The simple 1-dimensional world

- As we saw, collapsibility implies acyclicity in all dimensions
- In dimension 1 the two properties are, in fact equivalent
- Does this equivalence hold in higher dimension as well?

No, we are not in Kensas anymore

$\binom{6-1}{2} = 10$

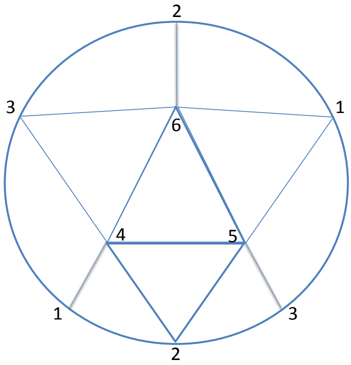


Figure: A triangulation of the projective plane

The plot thickens

Can we find more examples of non-collapsible hypertrees?

The plot thickens

Can we find more examples of non-collapsible hypertrees?

A construction: Let n be prime and $d \geq 2$. Fix a subset $A \subset \mathbb{Z}_n$ of cardinality $|A| = d + 1$. The sum complex X_A corresponding to A has a full $(d-1)$-dimensional skeleton and contains a d-face σ iff $\sum_{x \in \sigma} x \in A$.

The plot thickens

Can we find more examples of non-collapsible hypertrees?

A construction: Let n be prime and $d \geq 2$. Fix a subset $A \subset \mathbb{Z}_n$ of cardinality $|A| = d + 1$. The sum complex X_A corresponding to A has a full $(d-1)$-dimensional skeleton and contains a d-face σ iff $\sum_{x \in \sigma} x \in A$.

Theorem (Linial, Meshulam, Rosenthal)

The complex X_A is always a $\mathbb{Q}$-hypertree.

The plot thickens

Can we find more examples of non-collapsible hypertrees?

A construction: Let n be prime and $d \geq 2$. Fix a subset $A \subset \mathbb{Z}_n$ of cardinality $|A| = d + 1$. The sum complex X_A corresponding to A has a full $(d-1)$-dimensional skeleton and contains a d-face σ iff $\sum_{x \in \sigma} x \in A$.

Theorem (Linial, Meshulam, Rosenthal)

The complex X_A is always a $\mathbb{Q}$-hypertree. It is collapsible iff A forms an arithmetic progression.

What field are we in?

ISRAEL JOURNAL OF MATHEMATICS, Vol 45, No. 4, 1983

ENUMERATION OF Q-ACYCLIC SIMPLICIAL COMPLEXES

BY

GIL KALAI

ABSTRACT

Let $\mathscr{C} = \mathscr{C}(n, k)$ be the class of all simplicial complexes C over a fixed set of n vertices $(2 \leqq k \leqq n)$ such that: (1) C has a complete $(k-1)$-skeleton, (2) C has precisely $\binom{n-1}{k}$ k-faces, (3) $H_k(C) = 0$. We prove that for $C \in \mathscr{C}$, $H_{k-1}(C)$ is a finite group, and our main result is:

$$\sum |H_{k-1}(C)|^2 = n^{\binom{n-2}{k}}.$$

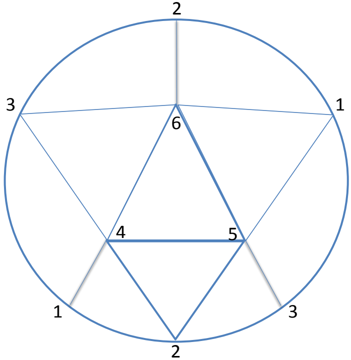
What field are we in?

Zero right kernel: over what field?

What field are we in?

Zero right kernel: over what field?

- For the hyperstar or any collapsible hypertree the answer is the same: It does not matter what the underlying field is
- What about the 6-point triangulation of the projective plane?



Orient all triangular faces clockwise. The sum of all the columns of the boundary operator matrix is

$$-2(e_{12} + e_{23} + e_{31})$$

The field matters

We conclude that unlike the 1-dimensional case of graphs, the notion of a d-dimensional hypertree depends on the underlying field.

The field matters

We conclude that unlike the 1-dimensional case of graphs, the notion of a d-dimensional hypertree depends on the underlying field.

Specifically: The 6-point triangulation of the projective plane is a $\mathbb{Q}$-hypertree, but not a $\mathbb{F}_2$-hypertree.

More collapsibility vs. acyclicity

As already posed by Kalai:

More collapsibility vs. acyclicity

As already posed by Kalai:

Conjecture

For every $d \geq 2$ and for every field $\mathbb{F}$ and $n \to \infty$ almost none of the n-vertex d-dimensional $\mathbb{F}$-hypertrees are collapsible.

This is supported by rigorous numerical experiments, but remains open.

More evidence - in our analysis of random complexes (coming up)

Over $\mathbb{Z}$ rather than over a field?

Three conditions define when X is a hypertree

- Full $(d-1)$-skeleton.
- Right number of d-faces.
- Empty cycle space.

The latter condition can be replaced by:

- The d-faces of X span all $\binom{n}{d+1}$ faces of dimension d.

and this requirement makes sense not only over a field but also over $\mathbb{Z}$.

$\mathbb{Z}$-hypertrees

Definition

A d-dimensional hypertree X on vertex set $[n]$ is called a $\mathbb{Z}$-hypertree if the d-faces of X span over the integers all $\binom{n}{d+1}$ faces of dimension d.

Example

The 6-point triangulation of the projective plane is not a $\mathbb{Z}$-hypertree. If you want to generate the triple $(1, 2, 3)$, you need to divide by 2.

Example

Hyperstars are $\mathbb{Z}$-hypertrees. Likewise for all collapsible hypertrees

Is that all?

Problem

Do there exist non-collapsible $\mathbb{Z}$-hypertrees?
Even $\mathbb{Z}$-hypertrees that afford no collapses?

Theorem (Even-Zohar, Linial, Nowik, Peled; Work in progress)

We constructed infinitely many 2-dimensional $\mathbb{Z}$-hypertrees in which no elementary collapse is possible. (i.e., every edge is covered at least twice).

Is that all?

Problem

Do there exist non-collapsible $\mathbb{Z}$-hypertrees?
Even $\mathbb{Z}$-hypertrees that afford no collapses?

Theorem (Even-Zohar, Linial, Nowik, Peled; Work in progress)

We constructed infinitely many 2-dimensional $\mathbb{Z}$-hypertrees in which no elementary collapse is possible. (i.e., every edge is covered at least twice).

This area is still poorly understood

A random process

We consider the random process that starts with a full $(d-1)$-dimensional skeleton. At each step pick a random d-dimensional face $\sigma \notin X$. If possible, we add σ to X. Otherwise, we discard σ.

A random process

We consider the random process that starts with a full $(d-1)$-dimensional skeleton. At each step pick a random d-dimensional face $\sigma \notin X$. If possible, we add σ to X. Otherwise, we discard σ.

We cannot add σ to X iff this creates a new cycle. In this case we say that σ is in the shade of X.

To wit: At each step we add to the current complex a random d-face σ whose addition creates no new cycle ("σ is not in the shade of X").

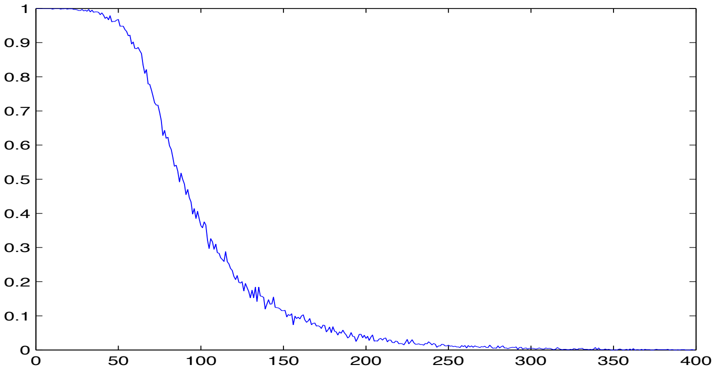
Living in the shades

Let $G = (V, E)$ be a disconnected graph, and let $ij \notin E$. We say that ij is in G's shadow if i and j belong to the same connected component of G.

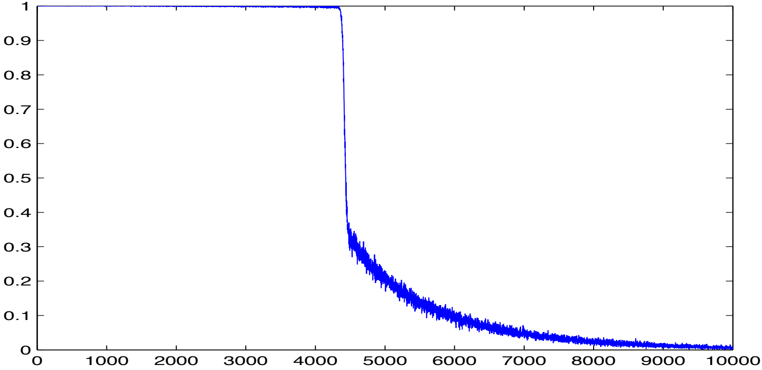
Living in the shades

Let $G = (V, E)$ be a disconnected graph, and let $ij \notin E$. We say that ij is in G's shadow if i and j belong to the same connected component of G.
In other words ij is in G's shadow iff the column corresponding to the edge ij is in the linear span of the columns of A_G.

Outside the shadow of an evolving graph



Shadow of an evolving 2-complex



Life on the edge

The quick jump in the probability of being in the shade is very intriguing.
The complex at the transition point is very curious "the dream-catcher". It is acyclic, has $\Theta(n^2)$ faces, yet 70% of the $\binom{n}{3}$ faces are at its shadow.

Problem

Provide a description of the dream-catcher.

Uniform vs. Minimum Spanning Trees

In dimension $d = 1$, the above process yields the random minimum spanning tree MST. (Equivalently, pick random edge weights in $(0, 1)$ and construct the resulting MST).

Problem

It is known that uniform trees and random MST differ. E.g., their diameters are $n^{1/2}$ resp. $n^{1/3}$. Can we develop the analogous theory in higher dimensions?

Over $\mathbb{Z}$?

- Can we run the same process also over $\mathbb{Z}$?
- Yes, but linear algebra does not suffice. You must bring the relevant matrices to their Smith Normal Form.
- Things are harder outside of linear algebra.
- Namely, at least experimentally this process ends one short of the mark. I.e., a complex with $\binom{n}{2} - 1$ two-dimensional faces that cannot be extended to a 2-dimensional $\mathbb{Z}$-hypertree.
- Much remains unknown here.

Inspiration from $G(n, p)$

No need to explain to this audience what $G(n, p)$ is.

Or what the evolution of random graphs is.

What about their d-dimensional counterparts?

A d-dimensional analog of $G(n, p)$

In 2006, Roy Meshulam and I introduced a model of a random d-dimensional n-vertex complex $X_d(n, p)$.

The $X_1(n, p)$ model coincides with $G(n, p)$.

A d-dimensional analog of $G(n, p)$

In 2006, Roy Meshulam and I introduced a model of a random d-dimensional n-vertex complex $X_d(n, p)$.

The $X_1(n, p)$ model coincides with $G(n, p)$.

Start with a full $(d-1)$-dimensional skeleon. (For graphs, we start with n vertices.)

A d-dimensional analog of $G(n, p)$

In 2006, Roy Meshulam and I introduced a model of a random d-dimensional n-vertex complex $X_d(n, p)$.

The $X_1(n, p)$ model coincides with $G(n, p)$.

Start with a full $(d - 1)$-dimensional skeleon. (For graphs, we start with n vertices.)

Each d-dimensional face (For graphs - each edge), is placed in X independently and with probability p.

Recall basic facts in $G(n, p)$ theory

Theorem (Erdős and Rényi '60)

The threshold for graph connectivity in $G(n, p)$ is

$$p = \frac{\ln n}{n}$$

Recall basic facts in $G(n, p)$ theory

Theorem (Erdős and Rényi '60)

The threshold for graph connectivity in $G(n, p)$ is

$$p = \frac{\ln n}{n}$$

Theorem (L. - Meshulam - Wallach)

In $X_d(n, p)$ the threshold for the event that ∂_d has a trivial left kernel is

$$p = \frac{d \ln n}{n}.$$

Collapsibity vs. acyclicity in $X_d(n, p)$

Theorem (Aronshtam, Linial, Łuczak, Meshulam, Peled)

- *The collapsibility threshold in $X_d(n, p)$ is*

$$(1 + o_d(1))\frac{\log d}{n}.$$

Collapsibity vs. acyclicity in $X_d(n, p)$

Theorem (Aronshtam, Linial, Łuczak, Meshulam, Peled)

- *The collapsibility threshold in* $X_d(n, p)$ *is*

$$(1 + o_d(1))\frac{\log d}{n}.$$

- *The threshold for having a cycle whp is*

$$\frac{d + 1 - o_d(1)}{n}.$$

Phase transition in the evolution of random complexes?

The evolution of random graphs, starts with isolated edges. Connected components appear later and are

- small = cardinality $O(\log n)$.

Phase transition in the evolution of random complexes?

The evolution of random graphs, starts with isolated edges. Connected components appear later and are

- small = cardinality $O(\log n)$.
- simple = a tree.

Phase transition in the evolution of random complexes?

The evolution of random graphs, starts with isolated edges. Connected components appear later and are

- small = cardinality $O(\log n)$.
- simple = a tree.
- Plus a Poisson number of unicylic graphs with $O(\log n)$ vertices.

Phase transition in the evolution of random complexes?

The evolution of random graphs, starts with isolated edges. Connected components appear later and are

- small = cardinality $O(\log n)$.
- simple = a tree.
- Plus a Poisson number of unicylic graphs with $O(\log n)$ vertices.
- Around step $\frac{n}{2}$ (i.e., $p = \frac{1}{n}$) a giant component quickly emerges. giant=linear size

More events around the phase transition

For $p < \frac{1-\epsilon}{n}$, the probability that the graph is acyclic is bounded away from both zero and one.

More events around the phase transition

For $p < \frac{1-\epsilon}{n}$, the probability that the graph is acyclic is bounded away from both zero and one.

But if $p > \frac{1+\epsilon}{n}$, G almost surely contains a cycle.

High-dimensional phase transition

Connected component in dimension $d \geq 2$?

Theorem (Linial, Peled)

Exactly at the same $p = \frac{c}{n}$ *where* $X_d(n, p)$ *almost surely acquires a cycle, the shadow of the complex becomes gigantic (i.e., has* $\Omega(n^{d+1})$ *faces).*

High-dimensional phase transition

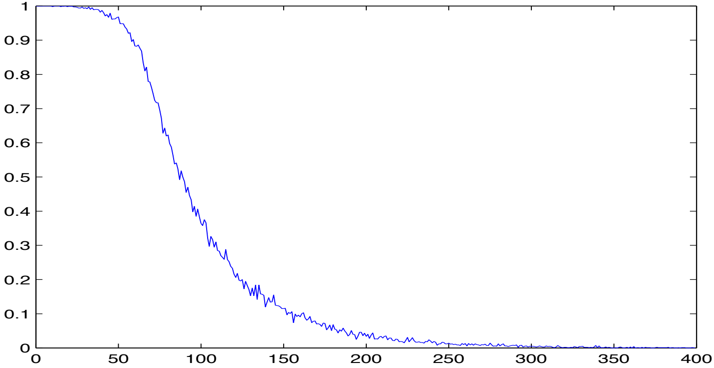
Connected component in dimension $d \geq 2$?

Theorem (Linial, Peled)

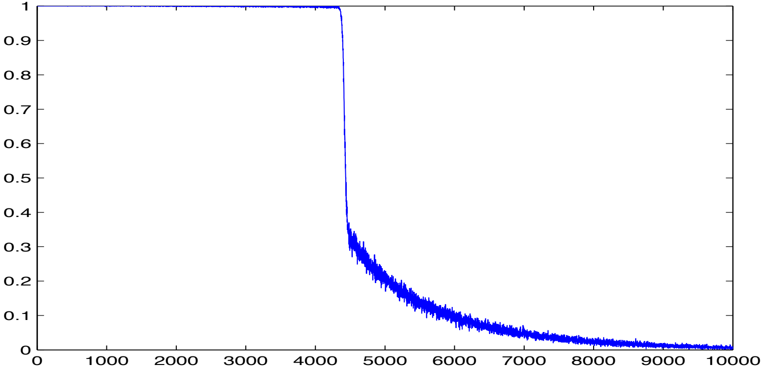
Exactly at the same $p = \frac{c}{n}$ where $X_d(n,p)$ almost surely acquires a cycle, the shadow of the complex becomes gigantic (i.e., has $\Omega(n^{d+1})$ faces).

This statement applies in all dimensions, including $d = 1$. However, when $d = 1$ the limit distribution is continuous but not smooth, while for $d \geq 2$ the limit distribution is discontinuous.

A view of phase transition in $G(n, p)$



Phase transition in $X_2(n, p)$ complexes



Cayley-Kalai Formula

Theorem (Cayley's Formula, Borchardt 1860)

The number of trees with vertex set $[n]$ is n^{n-2}.

Theorem (Kalai 1983)

$$\sum_{T} |H_{d-1}(T)|^2 = n^{\binom{n-2}{d}}$$

where the sum is over all n-vertex d-dimensional $\mathbb{Q}$-hypertrees T.

But how many d-hypertrees are there?

Open Problem

For $d \geq 2$ and large n, find (at least approximately) the number of d-dimensional n-vertex $\mathbb{Q}$-hypertrees.

Kalai's Formula yields estimates, but falls short of an asymptotic formula. Y. Peled and I have significantly improved these estimates. A full answer still eludes us.

But how many d-hypertrees are there?

Open Problem

For $d \geq 2$ and large n, find (at least approximately) the number of d-dimensional n-vertex $\mathbb{Q}$-hypertrees.

Kalai's Formula yields estimates, but falls short of an asymptotic formula. Y. Peled and I have significantly improved these estimates. A full answer still eludes us.

The estimates build on the analysis of the evolution of random complexes

More surprises in the shadows

Easy Observation

Let G be an "almost tree", i.e., an n vertex forest with $n-2$ edges (and hence with two connected components). Then at least $(1-o(1))\frac{n^2}{4}$, i.e., at least half of the remaining edges, are in G's shadow.

Surprises in the shadows (contd.)

Construction: Let X be a 2-dimensional n-vertex complex with a full 1-dimensional skeleton. The 2-faces of X are the arithmetic triples of difference $\neq 1$. Easy fact: The number of 2-faces in X is $\binom{n-1}{2} - 1$ (one less than a 2-dimensional hypertree).

[1] It actually suffices to assume the weaker Artin's conjecture

Surprises in the shadows (contd.)

Construction: Let X be a 2-dimensional n-vertex complex with a full 1-dimensional skeleton. The 2-faces of X are the arithmetic triples of difference $\neq 1$. Easy fact: The number of 2-faces in X is $\binom{n-1}{2} - 1$ (one less than a 2-dimensional hypertree).

Theorem (L., Newman, Peled, Rabinovich)

The complex X is $\mathbb{Q}$-acyclic. Assuming the Riemann hypothesis[1]*, there are infinitely many primes n for which X has an empty shadow.*

[1] It actually suffices to assume the weaker Artin's conjecture

Hyperpaths?

A path is a tree where every vertex is in two edges or fewer.

Can we construct d-dimensional hypertree where every $(d-1)$-dimensional face is contained in $d+1$ or fewer d-dimensional face?

Here is what Amir Dahari and I have done on this:

Hyperpaths (contd.)

Let's concentrate on dimension $d = 2$. Although things seem to work well for higher d as well, the case of $d = 2$ is already quite challenging.

Given a prime n and $c \in \mathbb{F}_n$ with $c \neq 0, 1, -2$, we consider the 2-complex with vertex set $\mathbb{F}_n$, a full 1-skeleton where the triple $\{x, y, z\}$ is a 2-face iff

$$x + y + cz = 0$$

Hyperpaths (contd.)

Let's concentrate on dimension $d = 2$. Although things seem to work well for higher d as well, the case of $d = 2$ is already quite challenging.

Given a prime n and $c \in \mathbb{F}_n$ with $c \neq 0, 1, -2$, we consider the 2-complex with vertex set $\mathbb{F}_n$, a full 1-skeleton where the triple $\{x, y, z\}$ is a 2-face iff

$$x + y + cz = 0$$

Does it work?

Hyperpaths (contd.)

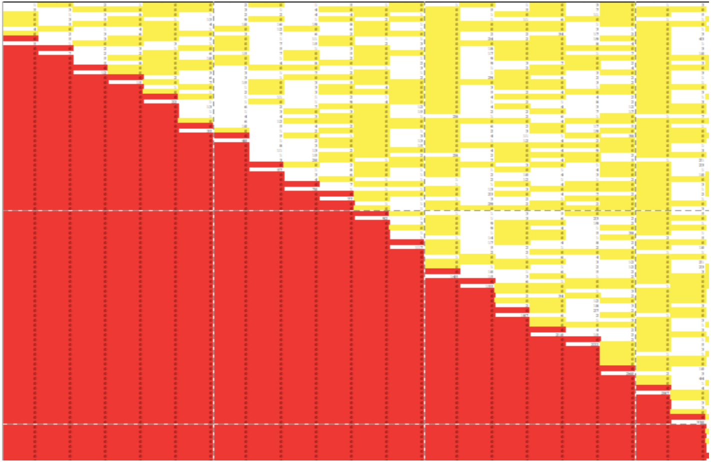
Let's concentrate on dimension $d = 2$. Although things seem to work well for higher d as well, the case of $d = 2$ is already quite challenging.

Given a prime n and $c \in \mathbb{F}_n$ with $c \neq 0, 1, -2$, we consider the 2-complex with vertex set $\mathbb{F}_n$, a full 1-skeleton where the triple $\{x, y, z\}$ is a 2-face iff

$$x + y + cz = 0$$

Does it work? Often it does!

Is it a Hyperpath?



Thank you for your attention

A lower bound on the number of hypertrees: A taste of the proof

Claim: If X is an acyclic d-complex, then the number of d-faces in its shadow Y is $\leq \frac{n \cdot |X|}{d+1}$

A lower bound on the number of hypertrees: A taste of the proof

Claim: If X is an acyclic d-complex, then the number of d-faces in its shadow Y is $\leq \frac{n \cdot |X|}{d+1}$
There are exactly $(d+1) \cdot |Y|$ pairs (v, σ) with v a vertex in σ, a d-face in Y. Let W be the set of d-faces in Y that contain v.

A lower bound on the number of hypertrees: A taste of the proof

Claim: If X is an acyclic d-complex, then the number of d-faces in its shadow Y is $\leq \frac{n\cdot|X|}{d+1}$
There are exactly $(d+1)\cdot|Y|$ pairs (v,σ) with v a vertex in σ, a d-face in Y. Let W be the set of d-faces in Y that contain v.
W is an acyclic complex, being part of v's hyperstar. So, the columns corresponding to W are linearly independent and spanned by X. Therefore $|W|\leq|X|$, which proves our claim.

Collapsibility

An elementary collapse is a step where you remove a vertex of degree one and the single edge that contains it.

Collapsibility

An elementary collapse is a step where you remove a vertex of degree one and the single edge that contains it.

A graph G is collapsible if by repeated application of elementary collapses you can eliminate all of the edges in G.

But note

As we saw, connectivity and acyclicity are linear algebraic. In contrast collapsibility is a purely combinatorial condition.

But note

As we saw, connectivity and acyclicity are linear algebraic. In contrast collapsibility is a purely combinatorial condition.

Indeed we will soon see that in higher dimensions collapsibility implies connectivity and acyclicity, but the reverse implication does not hold.