The Cost of Stability in Weighted Voting Games

(Extended Abstract)

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ABSTRACT

One key question in cooperative game theory is that of coalitional stability. A coalition in such games is stable when no subset of the agents in it has a rational incentive to leave the coalition. Finding a division of the gains of the coalition (an imputation) lies at the heart of many cooperative game theory solution concepts, the most prominent of which is the core. However, some coalitional games have empty cores, and any imputation in such a game is unstable. We investigate the possibility of stabilizing the coalitional structure using external payments. In this case, a supplemental payment is offered to the grand coalition by an external party which is interested in having the members of the coalition work together. The sum of this payment plus the gains of the coalition, called the coalition's "adjusted gains", may be divided among the members of the coalition in a stable manner. We call a division of the adjusted gains a super-imputation, and define the cost of stability (CoS) as the minimal sum of payments that stabilizes the coalition.

We examine the cost of stability in weighted voting games, where each agent has a weight, and a coalition is successful if the sum of its weights exceeds a given threshold. Such games offer a simple model of decision making in political bodies, and of cooperation in multiagent settings. We show that it is coNP-complete to test whether a super-imputation is stable, but show that if either the weights or payments of agents are bounded then there exists a polynomial algorithm for this problem. We provide a polynomial approximation algorithm for computing the cost of stability.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity;
I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent Systems*;
J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*

Cite as: The Cost of Stability in Weighted Voting Games, (Extended Abstract), Yoram Bachrach, Reshef Meir, Michael Zuckerman, Jörg Rothe, Jeffrey S. Rosenschein, *Proc. of 8th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2009),* Decker, Sichman, Sierra and Castelfranchi (eds.), May, 10–15, 2009, Budapest, Hungary, pp. 1289–1290

General Terms

Algorithms, Theory, Economics

Keywords

Coalition Formation, Core, Weighted Voting Games

1. GAME-THEORETIC NOTIONS

A *transferable utility coalitional game* consists of n agents I, and a characteristic function mapping any subset (coalition) of agents to a real value $v : 2^I \to \mathbb{R}$, indicating the utility the coalition achieves. A game is *increasing* if for all coalitions $C' \subseteq C$ we have $v(C') \le v(C)$, and is *super-additive* if for all disjoint coalitions $A, B \subseteq I$ we have $v(A) + v(B) \le v(A \cup B)$. Games where v's range is $\{0, 1\}$ are called simple games, and we say $C \subseteq I$ *wins* if $v(C) = 1$, and C *loses* if $v(C) = 0$. *Weighted voting games* (WVGs) are simple games with agents $I = (a_1, \ldots, a_n)$, where each a_i has a weight w_i. Given $C \subseteq I$, we denote C's weight as $w(C) = \sum_{a_i \in C} w_i$. C wins (i.e., $v(C) = 1$) if its weight exceeds a threshold q, so $w(C) \ge q$, and loses (i.e., $v(C) = 0$) if $w(C) < q$. We denote the game with weights $w = (w_1, w_2, \ldots, w_n)$ and threshold q as $[w_1, w_2, \ldots, w_n; q]$.

An *imputation* is a division $(p_1, \ldots, p_n)$ of the grand coalition's gains among the agents, where $p_i \in \mathbb{R}$, such that $\sum_{i=1}^n p_i = v(I)$. We call p_i the *payoff of agent* a_i, and denote the *payoff of a coalition* C as $p(C) = \sum_{a_i \in C} p_i$. A coalition B *blocks the imputation* $p = (p_1, \ldots, p_n)$ if $p(B) < v(B)$. In WVGs, a coalition blocks iff it wins *and* it is paid less than 1. If a blocked payoff vector is chosen, the coalition is unstable. The core is the set of stable imputations, i.e., imputations $(p_1, \ldots, p_n)$ that are not blocked by any coalition, so for any coalition C, we have $p(C) \ge v(C)$. A folk theorem is that in a simple game G, if there are no veto agents in G, then the core is empty. Otherwise, let $a_{v_1}, \ldots, a_{v_m}$ be the veto agents in G. Then the core is the set of imputations that distribute all the gains only to veto agents: $CORE = \{(p_1, \ldots, p_n) | \sum_{i=1}^m p_{v_i} = 1\}$.

2. THE COST OF STABILITY IN WVGS

An external party can stabilize the coalitional structure when the core is empty by offering the grand coalition a supplemental payment. This payment is offered to the grand coalition as a whole, and is provided only if this coalition is formed. We call the payment Δ offered by the external party the *supplemental payoff*, and model this as an *adjusted coalitional game*, derived from the original game G. Given a game G with set I of agents and characteristic function v, an *adjusted coalitional game (with respect to G) with a supplemental payoff of* Δ is a game $G(\Delta)$ with the following characteristic function v': $v'(C) = v(C)$ if $C \neq I$, and $v'(C) = v(C) + \Delta$ if $C = I$. In the adjusted game the coalitional function v' is identical to the original function v for all coalitions except the grand coalition I, who gains in $G(\Delta)$ the original gains plus the supplemental payoff Δ. We call $v'(I) = v(I) + \Delta$ the grand coalition's *adjusted gains*. When the supplemental payment is large enough, the adjusted gains may be divided among the coalition's members in a stable manner. A super-imputation $(p_1, \dots, p_n)$ is a division of the grand coalition's adjusted gains among the agents, where $p_i \in \mathbb{R}$, such that $\sum_{i=1}^{n} p_i = v'(I) = v(I)+\Delta$. We say Δ *stabilizes* G if the adjusted game $G(\Delta)$ has a nonempty core. Some games require the supplemental payment to be high in order to stabilize the grand coalition, while other games are already inherently stable, and require no such payment to stabilize the coalition. The *cost of stability* (CoS) p^* of a game G is the *minimal* sum of payments that stabilizes the coalition:

$$p^* = \min\{v(I) + \Delta |\ \text{the adjusted game } G(\Delta) \text{ is stable}\}$$

By saying "$G(\Delta)$ is stable" we mean it has a nonempty core.

We now consider a WVG with n agents $[w, w, \dots, w; q]$ (i.e., where any agent i has weight $w_i = w$, with the threshold q). We call such a game an *identical-weight game*. Let $I_{-i} = I \setminus \{a_i\}$ denote the set of all agents except a_i.

THEOREM 1. *Let G be an identical-weight game, and p^* be the CoS of G. It can be determined in polynomial time whether $p^* = v(I)$ or $p^* > v(I)$.*

PROOF. (Sketch) We can identify veto agents in polynomial time, so we can test if the core is empty. If the core is nonempty, obviously $p^* = v(I)$, and otherwise $p^* > v(I)$. □

THEOREM 2. *Let $G = [w, w, \dots, w; q]$ be an identical-weight game with n agents, and let p^* be the CoS of G. If $p^* > v(I)$ then $p^* = n/\lceil q/w \rceil$.*

PROOF. (Sketch) We show that the imputation $p_i = 1/\lceil q/w \rceil$, where $1 \le i \le n$, is both stable and optimal. □

Despite these positive results, in *general* WVGs, verifying that a super-imputation is stable is computationally hard.

DEFINITION 1. SUPER-IMPUTATION-STABILITY (SIS): *Given a weighted voting game $G = [w; q]$, a supplemental payment Δ and a super-imputation $p = (p_1, \dots, p_n)$ in the adjusted game $G(\Delta)$, decide whether p is stable (i.e., whether there exists some blocking coalition B for p in $G(\Delta)$).*

THEOREM 3. SIS *is* coNP-*complete.*

PROOF. (Sketch) We reduce SUBSET-SUM to the complement of SIS. Given a SUBSET-SUM instance with elements $A = \{x_1, \dots, x_n\}$ and a target value B, we construct a WVG with the SUBSET-SUM weights $w_i = x_i$, and a threshold of the SUBSET-SUM target $q = B$, so $G = [x_1, x_2, \dots, x_n; B]$. We construct the super-imputation $p_i = x_i/(B+1)$, which is unstable if and only if the SUBSET-SUM instance is a "yes" instance. □

THEOREM 4. SIS *is in* P *when either the weights or the agents' payments are polynomially bounded (or in unary).*

PROOF. (Sketch) We show SIS can be solved as a restricted case of KNAPSACK, which is pseudo-polynomially solvable. Given the SIS instance $G = [w; q]$ with super-imputation $p = (p_1, \dots, p_n)$, we choose ϵ small enough that for any $C \subseteq I$ we have $p(C) < 1$ iff $p(C) < 1 - \epsilon$, and construct the following KNAPSACK instance: the item values are the weights, $u_i = w_i$, the item volumes are the payoffs, $c_i = p_i$, the target total value is the threshold $A = q$, and the maximal allowed volume is $B = 1 - \epsilon$. This KNAPSACK instance has a solution iff there is a blocking coalition. □

3. APPROXIMATING THE COS IN WVGS

We present a simple approximation algorithm for computing the CoS in WVGs. Given the game $G = [w, q]$, we approximate p^*, the CoS of G. If $w(I) < q$, no coalition wins, so no coalition can block. We thus assume that $w(I) \ge q$. Our approximation algorithm simply uses the following super-imputation: $p_i = \min(1, w_i/q)$. We denote by $p^*(a_i)$ the payoff of a_i in an optimal super-imputation, and by $p^*(C)$ the total payoff of the coalition $C \subseteq I$ in such an optimal super-imputation, so $p^*(C) = \sum_{a_i \in C} p^*(a_i)$. Given a super-imputation p, we denote the total payoff under it as $V_p = p(I)$. We show that this is a 2-approximation.

THEOREM 5. *For V_p and p^* we have $V_p/p^* \le 2$.*

PROOF. (Sketch) Let K denote all agents whose weight is at least q, and let $k = |K|$. Clearly, if $a_i \in K$ then the payoff p_i in any stable super-imputation must be $p^*(a_i) = 1$, as otherwise $\{a_i\}$ would block. Sort all *other* agents by decreasing weights, and partition them into sets $C_1, \dots, C_m$ the following way: (1) Set $j = 0$; (2) while there are unallocated agents do: (2a) set $j = j + 1$; (2b) add agents to C_j until $w(C_j) \ge q$ or until there are no more agents; (3) $m = j$; (4) if $w(C_j) \ge q$ and no unallocated agents remain, then set $m = j + 1$ and $C_m = \emptyset$.

This guarantees that the weight of the last coalition C_m is *strictly* less than the threshold: $w(C_m) < q$. If $m = 1$ then the threshold is not reached, so $k \ge 1$ (since $\sum_{a_i \in I} w_i \ge q$), and then $p^* = k$, and $V_p = k + \sum_{a_i \in C_1} w_i/q < k + q/q = k+1$, so $V_p/p^* < (k+1)/k \le (1+1)/1 = 2$. We now assume $m > 1$, and denote the index of the coalition in the partition whose weight is maximal by $j' = argmax_{j \le m} w(C_j)$. We show $V_p/p^* \le 2$ by considering two cases: when $w(C_{j'}) + w(C_m) \le 2q$ and when $w(C_{j'}) + w(C_m) > 2q$. The second case requires considering three agent sets, where the weight of any two of them is at least q (so they must be paid at least 1): A_1 with only the last agent in $C_{j'}$, $A_2 = C_{j'} \setminus A_1$, $A_3 = C_m$. □

Note that the bound $V_p/p^* \le 2$ is tight, since for any $\epsilon > 0$, there is a game for which $V_p/p^* \ge 2 - \epsilon$: $[1 - \epsilon/3, 1 - \epsilon/3; 1]$.

Acknowledgment

Yoram Bachrach, Reshef Meir, Michael Zuckerman and Jeffrey S. Rosenschein were partially supported by Israel Science Foundation grant #898/05. Jörg Rothe supported in part by DFG grants RO 1202/{11-1,12-1}, the ESF program LogICCC, and the Humboldt Foundation's TransCoop program; work done in part while visiting Hebrew University.