Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 4

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Lecturer: Guy Kindler | Scribe by: Ofira Burstein | Updated: April 6, 2008

At a previous lecture there was shown that for each function over a discrete cube there exists a single way to represent it as polynomial.

That's: ∀f:$\{\pm 1\}^n \rightarrow \Re$ ∃f(x)=$\sum \hat{f} \prod_{x \in S}$x That means that: I$i$(f) = $\left\| \frac{f - \sigma i f}{2} \right\|_2^2$ - directly from computation Looking at the above formula makes a lot of sense. It generalizes the Influence formula of boolean function. In case of boolean function it is probability that f(x)≠f(xe$_i$)

$$\sigma_i x_s = \underbrace{x_s i \notin S, -x_s i \in S} \tag{1}$$

σ_i is linear operator. Thus,

$$Ii(f) = \left\| \frac{f - \sigma i f}{2} \right\|_2^2 = \left\| \sum_{s:i \in S} f(s) \chi_s \right\|_2^2 = \sum_{i \in S} \hat{f}(s)^2 \tag{2}$$

Thus,

$$Ii(f) = \sum_{S \subseteq [n]} |s| \, \hat{f}(s)^2 \tag{3}$$

Thus, if V(f)=I(f) then f is linear:

$$f = \hat{f}(\phi) + \sum \hat{f}(i) \chi_i \tag{4}$$

From here it is understandable:

$$f(x) = a_0 + \sum_{n}^{i=1} a_i x_i \tag{5}$$

Claim 1 *If **f** is Boolean linear function, then **f** is a dictatorship. This keeps also for almost linear case, and the proof will lead us to it.*

Remark

$$\chi_\phi = \prod_{x \in \phi} x_i = 1 \tag{6}$$

Proof Write

$$f = a_0 + \sum_{\chi_i(x) = x_i} a_i x_i \tag{7}$$

We want to pick one **i** item from it and prove that all other are zeros. We will choose for it **i** with the highest weight. Assume without loss of generality that:

$$|a_1| > |a_2| > \ldots \geq |a_n| \tag{8}$$

We should prove that $a_2 = 0$, and then $a_3, a_4, \ldots = 0$ also. If $a_2 \neq 0$ (otherwise we are done) then $|a_2| \leq \frac{1}{\sqrt{2}}$ That's because:

$$\|f\|_2^2 = \sum a_i^2 + a_0^2 = 1 \tag{9}$$

(Because it's 2-norm of **f** and **f** is boolean. So according to Parseval's theorem the above equality keeps). In particular $a_1^2 + a_2^2 \leq 1$ and $a_1^2 > a_2^2$. Thus, $a_2^2 \leq \frac{1}{2}$. Thus, $\|a_2\| \leq \frac{1}{\sqrt{2}}$. Hence, for every $x \in \{\pm 1\}^n$ either f(x) is non-boolean, or f(x) is boolean, but then: $f(xe_2) = f(x) - 2x_2a_2$ is not boolean. **Remark** f(x) is some a_ix_i For every x either f(x) or $f(xe_i)$ is "'far"' from Boolean. The best hope is that a_2 is small number, so that $f(xe_2) \approx f(x)$ $\|f(x) - sign(f(x))\| \geq c\|a_2\|$ or $\|f(xe_2) - sign(f(x)e_2)\| \geq c\|a_2\|$. This is a contradiction $\Rightarrow a_2 = \ldots = a_n = 0 \Rightarrow \chi_\phi = \prod_{x\in\phi} x_i = 1$

Corollary 2 *Thus, if $a_2 \neq 0$, then a distance from boolean function is at least a_2.*

Remark Since $f = \sum \hat{f}(s)\chi_s$, we define $f^{\leq k} = \sum_{s,|s|\leq k} \hat{f}(s)\chi_s$ - k-head of f and $f^{>k} = \sum_{s,|s|>k} \hat{f}(s)\chi_s$ - k-tail of f. Hence, $f = f^{\leq k} + f^{>k}$

$$\left\|f^{>k}\right\|_2^2 + \left\|f^{\leq k}\right\|_2^2 = \|f\|_2^2 \tag{10}$$

This holds because low-degree part and high-degree part are orthogonal. To verify it we can write Parseval's equality and check it. If k-tail is small, then most of the weight of **f** is in k-head. **f** is 'close' to deg-k if $\left\|f^{>k}\right\|_2^2$ is small. ■

Theorem 3 *The FKN(Friedqut, Kalai, Naor) theorem: There exists global const C_0 with the following property: If $f : \{\pm 1\}^n \rightarrow \{\pm 1\}$ has $\left\|f^{>1}\right\|^2 < \epsilon < 0.000001$ (i.e. **f** is almost linear) then **f** is close to dicatatorship $\left\|f - (\hat{f}(\phi) + \hat{f}(i^*\chi_{i^*}))\right\|_2^2 \leq C_0\epsilon$, when i^* is selected such that $\left|\hat{f}(i_*)\right| = max_i \left|\hat{f}(i)\right|$*

Remark The question is if there is another dictatorship to which **f** is close. The answer is negative, and it can be proved using Parseval. On the other hand, $\|f\|_2^2 = 1$.If we have 2 dictatorships close to **f**, they will weight both close to 1. Thus, we have 2 different functions with weight close to 1. It can happen only if the constant coefficient (a_0) is close to 1. But it means that both dictatorships are the same.

Remark Question: Is there a transitive boolean function **f** with $\left\|f^{=1}\right\|_2^2 > const > 0$? Is there really a threshold here ? Answer: **Yes!** $\left\|Maj^{=1}\right\|_2^2 = const > 0 \Rightarrow \left\|Maj^{>1}\right\|_2^2 = 1 - const$.

Remark When $\left\|f^{=1}\right\|_2^2 \nearrow 1$ $\left\|f^{>1}\right\|_2^2 \searrow 0$. Small weight of $f^{>1}$ forces us to particular structure of the function.

Proof Let M=Maj. Then:

$$\hat{M}(1) = \langle M, \chi_1 \rangle = E_x \left[M(x)\chi_1 \right] = E_{x-1} \left[E_{x_1} \left[M(x)\chi_1 \right] \right] \tag{11}$$

Thus,

$$E_{x-1} \left[E_{x_1} \left[M(x)\chi_1 \right] \right] = E_{x-1} \left[\underbrace{\frac{1}{2} M(1, x-1)}_{case x_1 = 1} - \underbrace{\frac{1}{2} M(-1, x-1)}_{case x_1 = -1} \right] \tag{12}$$

Thus,

$$E_{x-1} \left[\underbrace{\frac{1}{2} M(1, x-1)}_{case x_1 = 1} - \underbrace{\frac{1}{2} M(-1, x-1)}_{case x_1 = -1} \right] = Pr_{x-1} \left[M(1, x=1) \neq M(-1, x-1) \right] \tag{13}$$

Thus,

$$Pr_{x-1} \left[M(1, x=1) \neq M(-1, x-1) \right] = Pr_x \left[M(x) \neq M(xe_1) \right] = I_1(M) \tag{14}$$

The above is correct because of monotoness of Maj function. Since Maj is monotone, $\underbrace{\frac{1}{2} M(1, x-1)}_{case x_1 = 1} - \underbrace{\frac{1}{2} M(-1, x-1)}_{case x_1 = -1}$ is 0 when M(1, x-1) = M(-1, x-1), and 1 otherwise.

Corollary 4 *We proved that for **f** Boolean and monotone,* $\hat{f}(i) = I_i(f)$ $\hat{M}(1) = I_1(M) = \frac{const}{\sqrt{n}}$ *- From homework Therefore:* $\sum_i \hat{M}(i)^2 = \left\|M^{=1}\right\|_2^2 = (const)^2 > 0$

Proof [Proof of closeness to boolean dictatorship] Will not be done fully in this lesson, but some parts of it will be collected. ■

Claim 5 $2 * |a| * |b| \leq a^2 + b^2$

Proof Trivial ■

$$\|f+g\|_2^2 = \langle f+g, f+g \rangle = \langle f, f \rangle + \langle g, g \rangle + 2 \langle f, g \rangle = \|f\|_2^2 + \|g\|_2^2 + 2 \langle f, g \rangle \tag{15}$$

By Cauche-Shcwarts:

$$\|f\|_2^2 + \|g\|_2^2 + 2 \langle f, g \rangle \leq \|f\|_2^2 + \|g\|_2^2 + 2 \|f\| \|g\| \tag{16}$$

By claim above:

$$\|f\|_2^2 + \|g\|_2^2 + 2 \|f\| \|g\| \leq 2 \|f\|_2^2 + \|g\|_2^2 \tag{17}$$

Corollary 6 $\|f+g\|_2^2 \leq 2\|f\|_2^2 + \|g\|_2^2$

Using this corrolary we get:

$$f - sign(\left\|\hat{f}(\phi) + \hat{f}(i^*\chi_{i^*}\right\|_2^2) \leq 2\left\|f - (\hat{f}(\phi) + \hat{f}(i^*)\chi_{i^*})\right\|_2^2 + 2\left\|\hat{f}(\phi) + \chi_{i^*} - sign(\hat{f}_{i^*}(\phi) + \hat{f}i^*\chi_{i^*})\right\|_2^2 \tag{18}$$

If f Boolean then $\forall g\ \|g - sign(g)\|_2^2 \leq \|g-f\|_2^2$

$$2\left\|f - (\hat{f}(\phi) + \hat{f}(i^*)\chi_{i^*})\right\|_2^2 + 2\left\|\hat{f}(\phi) + \chi_{i^*} - sign(\hat{f}_{i^*}(\phi) + \hat{f}i^*\chi_{i^*})\right\|_2^2 \leq 4\left\|f - \hat{f}(\phi) + \hat{f}(i^*)\chi_{i^*}\right\|_2^2 \tag{19}$$

Forgetting $f^{>1}$

$$\|f - (a_0 + a_i\chi_i)\|_2^2 \leq 2\left\|f - f^{\leq 1}\right\|_2^2 + 2\left\|f^{\leq 1} - (a_0 + a_i\chi_i)\right\|_2^2 \tag{20}$$

Since $2\left\|f - f^{\leq 1}\right\|_2^2 = 2\left\|f^{>1}\right\|_2^2 < \epsilon$, it is enough to show that $2\left\|f^{\leq 1} - (a_0 + a_i\chi_i)\right\|_2^2 \leq C_0\epsilon$

$$\|f_{\leq 1} - sign(f_{\leq 1})\|_2^2 \leq \|f_{\leq 1} - f\|_2^2 = \|f\|_2^2 < \epsilon \tag{21}$$

To prove the theorem it's enough to prove the following sub-theorem:

Theorem 7 *If* $f_{\leq 1} = a_0 + \sum a_i xi$, $\sum a_i^2 \leq 1$, *and* $\|f_{\leq 1} - sign(f_{\leq 1})\|_2^2 \leq \epsilon$, *then* $\|f_{\leq 1} - (a_0 + a_{i^*}\chi_{i^*}))\|_2^2 < \epsilon(1 + \Theta(\epsilon))$

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