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Cheating husbands and other stories: A case study of knowledge, action, and communication

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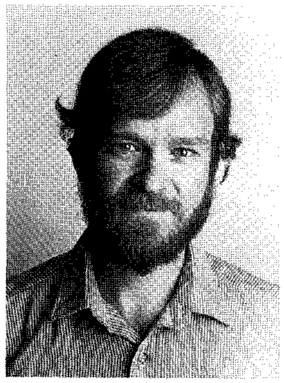

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Abstract. The relationship between knowledge and action is a fundamental one: a processor in a computer network (or a robot or a person, for that matter) should base its actions on the knowledge (or information) it has. One of the main uses of communication is passing around information that may eventually be required by the receiver in order to decide upon subsequent actions. Understanding the relationship between knowledge, action, and communication is fundamental to the design of computer network protocols, intelligent robots, etc. By looking at a number of variants of the *cheating husbands* puzzle, we illustrate the subtle relationship between knowledge, communication, and action in a distributed environment.

Key words: Common knowledge – Cheating wires – Distributed protocols

1 Introduction

The relationship between knowledge and action is a fundamental one: a processor in a computer network (or a robot or a person, for that matter) should base its actions on the knowledge (or information) it has. One of the main uses of communication is passing around information that may eventually be required by the receiver in order to decide upon subsequent actions. Understanding the relationship between knowledge, action, and communication is fundamental to the design of computer network protocols, intelligent robots, etc.

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Halpern and Moses (1984/86) show that the success of certain cooperative actions in a distributed environment may depend on the attainment of various states of knowledge by the group of agents involved. In particular, the state of *common knowledge*, corresponding to "public information", is of primary importance. A group has *common knowledge* of a fact p, denoted Cp, if they all know p, they all know that they all know p, they all know that they all know ... and so on, *ad infinitum*. Halpern and Moses further show that common knowledge is not attainable in many practical systems. For each type of communication channel they present a corresponding approximation of common knowledge that captures the state of knowledge resulting from a broadcast using such a channel.

The "cheating wives" puzzle, a well known puzzle from the folklore (cf. Gamow and Stern 1958), has long been one of the primary examples of the subtle interdependence between knowledge and action. It involves an initial step in which a set of facts is announced publicly, thereby becoming common knowledge. In this paper we reveal the contents of recently discovered scrolls, allegedly written by the great scholar Josephine of the lost continent of Atlantis. These scrolls describe how modernizing the means of communication in Atlantis over the generations affected the resolution of the recurring problem of unfaithful husbands there. A close analysis of her account provides a better understanding of the issues involved in the interaction between knowledge, action and communication. In particular, it illustrates how an agent that knows something about how other individuals' actions are related to the facts they know, can obtain knowledge by observing the other individuals' actions.

The original cheating husbands problem is introduced in Sect. 2.[1] Section 3 describes what happens when an asynchronous communication channel is used to communicate the protocol to be followed. Section 4 involves different types of synchronous communication, and includes a discussion of the conditions under which a "cheating husbands"-like protocol can tolerate "faults" (disobedient wives). Section 5 deals with ring-based communication. Section 6 treats the question of how allowing wives to communicate a small amount of extra information allows a substantially faster solution to the problem. Some conclusions are presented in Sect. 7.

2 The cheating husbands puzzle

Josephine's account of the history of a major city in Atlantis starts with the following incident:

The queens of the matriarchal city-state of Mamajorca, on the continent of Atlantis, have a long record of opposing and actively fighting the male infidelity problem. Ever since the technologically-primitive days of queen Henrietta I, women in Mamajorca have been required to be in perfect health and pass an extensive logic and puzzle-solving exam before being allowed to take a husband. The queens of Mamajorca, however, were not required to show such competence.

It has always been common knowledge among the women of Mamajorca that their queens are truthful and that the women are obedient to the queens. It was also common knowledge that all women hear every shot fired in Mamajorca. Queen Henrietta I awoke one morning with a firm resolution to do away with the male infidelity problem in Mamajorca. She summoned all of the women heads-of-households to the town square and read them the following statement:

There are (one or more) unfaithful husbands in our community. Although none of you knew before this gathering whether your own husband was faithful, each of you knows which of the other husbands are unfaithful. I forbid you to discuss the matter of your husband's fidelity with anyone. However, should you discover that your husband is unfaithful, you must shoot him on the midnight of the day you find out about it.

Thirty nine silent nights went by, and on the fortieth night, shots were heard.

Josephine does not explicitly say how many unfaithful husbands were shot, how many unfaithful husbands were in Mamajorca at the time, how some cheated wives learned of their husbands' infidelity after thirty nine nights in which *nothing* happened, or whether any more husbands were shot on later nights. The interested reader should stop at this point and try to answer these questions based on Josephine's account.

Let us consider the questions Josephine leaves unanswered. Since Henrietta I was truthful, there must have been at least one unfaithful husband in Mamajorca. How would events have evolved if there was exactly one unfaithful husband? His wife, upon hearing the queen's statement, would have concluded that her own husband was unfaithful, and would have shot him on the midnight of the first night. Clearly, there must have been more than one unfaithful husband. (Recall that the wives are all perfect logicians).[2] If there had been exactly

1 The cheating husbands puzzle is essentially the cheating wives puzzle of Gamow and Stern (1958), and equivalent to the "muddy children" puzzle of Barwise (1981) and Halpern and Moses (1984). Martin Gardner independently presented the puzzle in terms of "cheating husbands" in the thoroughly amusing Gardner (1984)

2 The fact that the wives are perfect reasoners plays a crucial role in all of the cases we treat. The nature of the situation changes substantially if we relax this assumption, since wives must then reason about the logical capabilities of other wives. Some preliminary steps towards dealing with such a situation

two unfaithful husbands, then every cheated wife would have initially known of exactly one unfaithful husband, and would have reasoned as follows: "If the unfaithful husband I know of is the only unfaithful husband, then his wife will shoot him on the first night." Therefore, neither one of the cheated wives would shoot on the first night. On the morning of the second day each cheated wife would realize that the unfaithful husband she knew about was not the only one, and that therefore her own husband must be unfaithful. The unfaithful husbands would thus both be shot on the second night. In fact, similar reasoning is used by the wives in general, and the following theorem, well known in the folklore, resolves our doubts regarding Josephine's presentation of the facts:

Theorem 1. *If there had been n unfaithful husbands in Mamajorca at the time Henrietta I announced her ruling, they would all have been shot on the midnight of the* n^{th} *day.*

Proof. The discussion above shows the claim for $n=1$. Assume that the claim holds for $n=k$. Thus, if there were k unfaithful husbands they would be shot on the k^{th} night. We wish to show that if there were $n=k+1$ unfaithful husbands they would have been shot on the $(k+1)^{\text{st}}$ night. Assume therefore that there were $k+1$ unfaithful husbands. Every cheated wife knows of exactly k unfaithful husbands. Because of the wives' logical competence, they know that if there are exactly k unfaithful husbands then those husbands will all be shot on the k^{th} night. Before the k^{th} night, a cheated wife cannot determine that her husband is unfaithful, and therefore no shots are fired in any of the first k nights. Since the k^{th} night is silent, every cheated wife concludes that there must be more than k unfaithful husbands and that her own husband is unfaithful. The unfaithful husbands are shot on the $(k+1)^{\text{st}}$ night. The theorem follows by induction. □

Notice the subtlety of the situation: On the first day, immediately after the queen delivers her statement, a wife who knows of k unfaithful husbands knows that every cheated wife knows of at least $k-1$ unfaithful husbands, and knows that their wives know of at least $k-2$ unfaithful husbands, and that their wives know of at least $k-3$ unfaithful husbands .... It follows that every wife thinks that it is possible that a cheated wife thinks that it is possible that a cheated wife thinks it is possible ... that a cheated wife knows of no unfaithful husbands other than her own. Thus, for all $k>1$, it is not common knowledge that there are at least k unfaithful husbands. The queen's statement, however, *is* common knowledge. This follows from the fact that the queen announced it publicly, thereby making it common knowledge that all of the wives heard her announcement.[3] It follows that after the queen speaks, it is common knowledge that there is at least one unfaithful husband. Given the wives' famous logical capabilities, it is common knowledge that if there is only one unfaithful husband then he will be shot on the first night. Therefore, once the first night is silent it becomes common knowledge that there are at least two unfaithful husbands. Similarly, after k silent nights (but not earlier!), it is common knowledge that there are at least $k+1$ unfaithful husbands and that every wife knows of at least k unfaithful husbands other than her own. So although a wife that knows of k unfaithful husbands knows that there will be no shots before the k^{th} night, her state of knowledge changes following every silent night, even though there is no "communication" at all!

3 Asynchronous communication

Josephine's description of Mamajorca continues with the following account:

Queen Henrietta I was highly regarded by her subjects for her wisdom in running the monarchy. She ordered her daughters to continue her moral fight against male infidelity.

Her daughter, Henrietta II, succeeded her. In order to facilitate communication with her subjects, Henrietta II installed a mail system from her court to all of the households in Mamajorca. Her first letter to her subjects told them about the properties of the new mail system: every letter she sends her subjects is guaranteed to eventually reach each one of them. Thus, she will not need to gather them in the town square for announcements any more. Eager to fulfill her mother's wish, Henrietta II's second letter to her subjects was an exact copy of her mother's original statement.

Henrietta II suffered great disgrace and died in despair. She ordered her daughters not to repeat her mistake.

Josephine suggests that despite the fact that Henrietta II gave the wives of Mamajorca exactly the same instructions as her mother, her mother was honored, whereas she was disgraced. Again, Josephine refrains from explicitly stating why this happened. Let us consider the possible outcomes of

are presented in Konolige (1984), where he considers a version of the *wise men* puzzle – a well known puzzle that is a special case of the cheating husbands problem – which he calls the *not-so-wise men puzzle*, in which the knowers are not perfect logicians

3 For a discussion of this point, see Halpern and Moses (1984)

Henrietta II's action. Had there been exactly one unfaithful husband at the time, his wife would have shot him on the first night after receiving the queen's letter, and the queen would have been saved from disgrace. If there had been exactly two unfaithful husbands, however, each one of their wives would know about the existence of one unfaithful husband, and that if the husband she knows about is the only unfaithful one, then his wife will shoot him on the day she receives the letter. Because the mail system is asynchronous, with messages only guaranteed to be delivered *eventually*, neither wife would ever know that the other had already received the queen's letter. Thus, neither wife would know that her husband is unfaithful: she would always consider it possible that her own husband is faithful and that the cheated wife she knows about has not shot yet because the queen's letter has yet to reach her. An immediate consequence of the above argument is:

Theorem 2. *If there is more than one unfaithful husband, and the original instructions are broadcast over an asynchronous channel, then no unfaithful husbands are shot.* □

Because the letter is broadcast using an asynchronous channel, the queen's letter becomes *eventual common knowledge:* once the queen sends it, every wife will eventually receive the letter, and when she does she'll know that all wives will eventually receive the letter, and know ... (cf. Halpern and Moses 1984). However, at no time does a wife know that all other wives have received the letter. Thus, a wife can never determine whether the silent nights are a result of other wives' reaction to receiving the letter or a result of the fact that they have yet to receive the letter. This property of asynchronous communication comes up in a similar fashion in the analysis of the Byzantine agreement problem in asynchronous networks (cf. Fisher et al. 1983). There, the asynchronous nature of the system prevents a processor from ever determining whether it has not received messages from another processor because the other processor did not send any (and thus is faulty), or because the messages are still on their way.

Notice that even if all of the wives happened to receive the queen's letter simultaneously, this would not help. The fact that a wife must always consider it *possible* that other wives have not yet received the queen's letter is sufficient to prevent her from being able to figure out whether her own husband is unfaithful.

4 Synchronous communication

Josephine proceeds to describe the controversial actions that ensued:

Henrietta III succeeded her mother, Henrietta II. She decided to upgrade the mail system that her mother had installed in order to avoid her mother's problem. Thus, she improved the mail system so that any letter sent by the queen was guaranteed to reach all of her subjects not later than one day after it was sent.

Henrietta III knew that unless her subjects were aware of the improvement in the mail system, she would repeat her mother's mistake. Thus, Henrietta III's first letter to her subjects announced the new advances in the mail delivery system, and her second one was an exact copy of Henrietta I's statement.

Henrietta III was considered a more effective monarch than her mother, but she will always be remembered for the great injustice she brought upon Mamajorca. If only she had told her subjects to wait a few days before shooting, however, she could have attained her grandmother's fame!

A mail system that guarantees that every letter sent is delivered no more than $b-1$ days after it is sent is called *weakly synchronous with bound b.* If we call the sending day the first day, then such a letter is delivered to all wives no later than on day b. Before we continue, we remark that in Henrietta III's days no calendar had been established in Mamajorca.

Let Ep denote "everyone knows p", and

$$E^{m+1}p \overset{\text{def}}{=} E(E^m p), \quad \text{for } m>0.$$

Notice that an easy proof by induction shows that if there are n unfaithful husbands, and E^n ("the queen sent the letter") becomes true at some point, then at least one cheated wife will shoot her husband, and the first shot will be fired at most n days after E^n ("the queen sent the letter") first holds. In our case, a letter sent by the queen is guaranteed to be delivered to all of the wives in less than b days. Thus, once the letter is sent its contents become *b-common knowledge:* within b days every wife receives the letter and knows that within b days every wife will receive the letter and know that within b days ... every wife will know the contents of the letter (cf. Halpern and Moses 1984/86). Thus, kb days after the queen sends the letter, E^k ("the queen sent the letter") holds, so it is certain that at least one unfaithful husband will be eliminated.

Although Henrietta III was probably not familiar with the concept of b-common knowledge, apocryphal records indicate that she was able to prove the following proposition:

Proposition 3. *In the weakly synchronous case with the bound on delivery being b, a wife that knows* *of exactly k unfaithful husbands will know that her own husband is unfaithful once kb silent nights pass after the day she receives the queen's letter.*

Proof. A wife knowing of $k=0$ unfaithful husbands requires $kb=0$ silent nights to conclude that her own husband is unfaithful. By the queen's statement, that wife does not know that her husband is unfaithful any earlier than that. Assume inductively that a wife knowing of k unfaithful husbands requires kb silent nights to conclude that her own husband is unfaithful, and suppose Mary knows of $k+1$ unfaithful husbands. Mary knows that if her own husband is faithful, then every cheated wife knows of exactly k unfaithful husbands, and, by the induction hypothesis, will shoot her husband on the following night should kb silent nights go by after the cheated wife receives the letter. For all Mary knows, it is initially possible that her husband is faithful, and the letter may reach the first cheated wife to receive it $b-1$ days after Mary receives it. Thus, she must consider it possible that no shots will be fired before the $(k+1)b^{\text{th}}$ night after she receives the queen's letter. However, should that night be silent, Mary will know that her husband is unfaithful. The lemma follows by induction. □

Thus, Henrietta III was guaranteed not to suffer her mother's disgrace. However, what she didn't realize was that noisy nights might confuse some of the wives. Consider, for example, the following scenario: The queen's letters are guaranteed to arrive in less than 2 days (i.e., $b=2$), and Susan knows that Mary's husband is unfaithful. Suppose Susan receives the queen's letter on a Monday, and hears Mary shoot her own husband at midnight on Tuesday night. Unfortunately, now Susan will not be able to figure out whether or not her own husband is faithful. Susan does not know whether the queen originally sent the letter on Sunday or on Monday, and thus considers it possible that Mary received the queen's letter on either Sunday, Monday or Tuesday. In particular, Susan considers both of the following scenarios possible:

- Mary received the letter on Tuesday and, knowing that Susan's husband is faithful, shot her own husband on Tuesday night.
- Mary received the letter on Sunday and, knowing that Susan's husband is unfaithful, waited to see if Susan would shoot her husband on Sunday or Monday night. Since Susan did not shoot, on Tuesday Mary concluded that her own husband was unfaithful, and shot him.

Thus, Susan cannot determine whether her own husband is faithful based on Mary's actions. Furthermore, she will never obtain any more information on the subject and will remain in doubt forever.

We call the first day on which the queen's letter is delivered to a cheated wife the first *significant* day. Given Proposition 3, it is easy to see that cheated wives that receive the queen's letter on the first significant day will be the first to shoot their husbands. Do any other cheated wives shoot their husbands?

Every wife has an interval of $b-1$ days in which a noisy night would leave her in doubt regarding her husband's fidelity. To see this, recall that a wife knowing of, say, $k>0$ unfaithful husbands does not initially know whether there are k or $k+1$ unfaithful husbands in all. Furthermore, for all she knows the first significant day may happen anywhere between $b-1$ days before she receives queen's letter and $b-1$ days after she receives it. "If there are k unfaithful husbands," she reasons, "then at least one of them will be shot on the $((k-1)b+1)^{\text{th}}$ night after the day his wife receives the letter, that is, between the $((k-2)b+2)^{\text{nd}}$ and the kb^{th} night after the day I receive the letter. If, however, there are $k+1$ unfaithful husbands, one of them will be shot between the $((k-1)b+2)^{\text{nd}}$ and the $(kb+1)^{\text{st}}$ night after the day I receive the letter." Thus, if the first shot occurs between the $((k-1)b+2)^{\text{nd}}$ and the kb^{th} night after the day she receives the queen's letter, a wife initially knowing of exactly k unfaithful husbands will be left in doubt regarding her husband's fidelity. Since a cheated wife that receives the queen's letter after the first significant day will hear a shot in her interval of uncertainty, we have:

Theorem 4. *Using weakly synchronous broadcast, cheated wives that receive the queen's letter on the first significant day shoot their husbands $((n-1)b$ days after the first significant day, where n is the number of unfaithful husbands). All other cheated wives remain forever in doubt about their husbands' fidelity.* □

How could Henrietta III have changed the instructions slightly and avoided the problem? Josephine seems to suggest that this could have been done by requiring a cheated wife to wait a few days after learning of her husband's infidelity, before shooting him. First notice that the wives' reasoning is slowed down considerably if the shooting happens only after a delay:

Proposition 5. *In a weakly synchronous mail system with bound b, if every wife is required to wait d days from the day she discovers her husband's infidelity before shooting him, then a wife that knows of exactly k unfaithful husbands will know that her own husband is unfaithful once $k(b+d)$ silent nights pass from the day she receives the queen's letter (and, as long as all preceding nights are silent, no earlier!).*

Proof. Analogous to the proof of Proposition 3. For $k=0$ the statement is trivially true. Assume inductively that it holds for k and that Mary knows of $k+1$ unfaithful husbands. Mary knows that if there are exactly $k+1$ unfaithful husbands, then every cheated wife knows of k unfaithful husbands. Thus, a cheated wife that receives the queen's letter on the first significant day (i.e., at least as early as any other cheated wife) will know that her husband is unfaithful once $k(b+d)$ silent nights pass from the day she receives the letter. Ordinarily she would wish to shoot on the $(k(b+d)+1)^{st}$ night, but since she must delay d days, she will shoot her husband on the $(k(b+d)+d+1)^{st}$ night after receiving the letter. Since Mary must consider it possible that the first significant day occurs as many as $b-1$ days after she receives the letter, Mary will know that her own husband is unfaithful once $k(b+d)+d+1+b-1=(k+1)(b+d)$ silent nights pass and no earlier. The lemma follows by induction. □

Josephine's claim is confirmed by the following theorem:

Theorem 6. *If the delay is sufficiently long, more precisely if $d \geq b-1$, then all cheated wives shoot their husbands and no wife remains in doubt.*

Proof. We use Proposition 5 in a fashion similar to that in which Theorem 4 uses Proposition 3. First, some notation is needed. Let F be the first significant day, let D be the day Mary receives the letter, and let S be the day preceding the night of the first shot. Notice that the proof of Proposition 5 implies that if $n \geq 1$ is the number of unfaithful husbands, then $S=F+(n-1)(b+d)+d+1$. Assume that Mary knows of exactly $k \geq 1$ unfaithful husbands. Initially, as far as Mary is concerned, there are two possibilities:

– Mary's own husband is faithful. In this case Mary knows that $D-(b-1) \leq F \leq D+(b-1)$. (Notice that Mary must consider the whole interval possible.) Since the number of unfaithful husbands is k, it follows that $S=F+(k-1)(b+d)+d+1$. Substituting h for $D+(k-1)(b+d)+d+1$, Mary has:

  $h-(b-1) \leq S \leq h+(b-1)$.

– Mary's own husband is unfaithful. In this case Mary knows that $D-(b-1) \leq F \leq D$. (We must have $F \leq D$, since otherwise, the first cheated wife to receive a letter does so after Mary does, contradicting the assumption that Mary's husband is unfaithful.) Also, $S=F+k(b+d)+d+1$, because there are $k+1$ unfaithful husbands. Substituting h as above, Mary has:

  $h+(d+1) \leq S \leq h+(b+d)$.

Therefore, if $d+1>b-1$ (i.e., $d \geq b-1$), then Mary can distinguish these possibilities (given that she knows S, h, b, and d), and thus is guaranteed to be able to determine whether her husband is unfaithful. It is easy to present scenarios that show that no smaller delay suffices. One such scenario is the example following Proposition 3 above. There $b=2$ and $d=0=b-2$. □

Josephine remarks:

> ... Of course, the shrewd residents of the Wisegal district of Mamajorca avoided any eventual doubts by bribing the mailperson.

We assume that the social attitude towards bribes in Mamajorca was quite different from the attitude towards infidelity. Consequently, (it was common knowledge that) bribery would be kept a secret between a bribing wife and her mailperson. It is also known that delivering mail was not an acceptable profession for the wives of Mamajorca. Thus, it was common knowledge that no wife knew of a wife that bribed the mailperson. Given these circumstances, the following proposition clarifies Josephine's statement:

Proposition 7. *In the weakly synchronous case, a wife that bribes the mailperson into telling her when the queen had originally sent the letter, does eventually know whether her own husband is faithful.*

Proof. Let the bound on delivery be b. Using Proposition 3, it is easy to show by a straightforward induction that if there are k unfaithful husbands then the first shot occurs between the $((k-1)b+1)^{st}$ night and the kb^{th} night after the queen sends the letter. Thus, a wife that knows of k unfaithful husbands and bribes her mailperson, knows that her husband is unfaithful if no shot is heard on or before the kb^{th} night, and knows that he is faithful otherwise. The crucial point is that a wife that bribes her mailperson knows *which* night is the kb^{th} night, and thus eventually knows whether her husband is faithful. □

Josephine continues with the reign of Henrietta IV:

Henrietta IV, who succeeded her mother as queen, concluded that the lack of a calendar was the reason behind the injustice of her mother's scheme. She summoned the women of Mamajorca to the town square and announced the initiation of a *calendar* beginning that day. "From this day on," she said, "the mail system will be *strongly synchronous:* every letter sent from the queen will bear the mailing date, and will be guaranteed to be delivered to all of her subjects within less than b days." At a later date, Henrietta IV sent her subjects a letter bearing the mailing date, and containing an exact copy of Henrietta I's original instructions. A thousand silent nights followed, and on the thousand and first day, Henrietta IV decided to send another letter. She had finally realized that as a result of Henrietta III's great injustice, the wives of Mamajorca lost much of their faith in the monarchy and its orders. It was still common knowledge that the queens were truthful, and the vast majority of her subjects were obedient, but it was no longer clear that all wives would obey the queen's orders. Henrietta IV's letter contained one line: "There is at least one obedient wife whose husband is unfaithful."

Henrietta IV's wisdom was greatly appreciated throughout Atlantis, and her success restored her subjects' faith in the monarchy.

Let us see why the obedient wives could not figure out whether their husbands were faithful before receiving Henrietta's second letter:

Proposition 8. *In the strongly synchronous case, if there is exactly one cheated wife, and she is disobedient, then all of the other wives are in danger of shooting their husbands on the second night.*

Clearly, if the other wives had not suspected that the cheated wife might be disobedient, all of the faithful husbands would have been shot, whereas the unfaithful husband would have survived! Notice that once this is a possibility, even if all wives are in fact obedient they cannot shoot. To see this, consider the case in which there are exactly two cheated wives. On the second day each cheated wife cannot determine whether the first night was silent because her own husband is unfaithful or because the other cheated wife was disobedient. Thus, no shots are fired on the second night. Similarly, no shots will be fired on any later nights. It is now easy to show by induction that such is the case if there are k cheated wives, for all $k \geq 1$. So how did the queen's second letter help?

Theorem 9. *In the strongly synchronous case, if it is common knowledge that there is at least one obedient cheated wife, then all obedient cheated wives will shoot their husbands.*

Proof. The argument here is very similar to that of Theorem 1, with a slight twist. If there is only one unfaithful husband, then his wife is the only cheated wife. Since there is at least one obedient cheated wife, she must be obedient, and therefore will shoot her husband on the day she receives the second letter. If there are exactly $k=2$ cheated wives, then each obedient cheated wife reasons as follows: "If my husband is faithful then the cheated wife I know of must be obedient", and therefore will shoot her husband when she receives the letter, at most $b-1$ days after the queen sent it (on day b at the latest). Thus, if no shots are fired by day $b+1$, an obedient cheated wife knows that her own husband is unfaithful, and shoots her husband on that night. Assume inductively that if there are exactly $k \geq 2$ unfaithful husbands then all obedient cheated wives shoot their husbands on the $(b+k-1)^{\text{st}}$ night. If there are exactly $k+1$ unfaithful husbands, then each obedient cheated wife knows of k unfaithful husbands, and knows that if her own husband is faithful then at least one unfaithful husband will be shot on the $(b+k-1)^{\text{st}}$ night. Thus, once that night is silent, she knows that (even though she might be the only obedient cheated wife) her husband is unfaithful, and shoots him on the $(b+(k+1)-1)^{\text{st}}$ night. The theorem follows by induction. □

Observe the difference between the bribed dates case, described in Proposition 7, and the strongly synchronous case of Theorem 9. If all of the wives bribed the mailperson, then all of the unfaithful husbands would be shot, and no wife would remain in doubt regarding her husband's fidelity. However, it takes $(n-1)b+1$ days to eliminate $n \geq 2$ cheating husbands. Before the end of the process the wives would not necessarily know that justice would be done, and at the end it would not be known whether any wife remains in doubt regarding her own husband's fidelity. In the strongly synchronous case, it takes $b+n-1$ nights to eliminate $n \geq 2$ unfaithful husbands, and it is common knowledge that justice is done. The difference between the two cases can be best understood by noting that in the first case every wife knew on what day the queen sent the letter, but no wife knew that others knew, whereas in the strongly synchronous case the day on which the queen sent the letter was common knowledge.

5 Ring-based communication

Josephine describes the outcome of a similar approach to the male infidelity problem in the neighboring city-state of Mamaringa, in which the households were arranged in a ring:

The queens of the neighboring matriarchal city-state of Mamaringa commonly adopted customs and rules from Mamajorca. Thus, Mamaringa was similar to Mamajorca in all respects, except that its households were built in a ring around the great Mt. Rouge. The location of each household in the ring was common knowledge, as was the fact that mail was delivered in clockwise order around the ring.

The queens of Mamaringa tried to eliminate the infideltiy problem by sending Henrietta I's letter once around the ring, using the state-of-the-art mail system in every generation. None of the queens of Mamaringa suffered the disgrace of Henrietta II, and none attained the honor of Henrietta IV. They will all be forever remembered as cruel and unjust queens.

The queens of Mamaringa probably hoped that the extra knowledge of the order in which letters are delivered would be helpful in justly eliminating all unfaithful husbands. However, the asymmetry introduced by this knowledge makes a big difference, as the following theorem shows:

Theorem 10

(a) In asynchronous delivery around a ring, the last cheated wife to receive the letter will shoot her husband. All others will not.

(b) In weakly synchronous delivery around a ring, some cheated wives will shoot their husbands, but some might not.

(c) In strongly synchronous delivery around a ring, some cheated wives will shoot their husbands, but some might not.

Proof. (a) We prove by induction that in the asynchronous case a cheated wife knowing of k cheated wives that are all notified before her, and knowing that no cheated wives will be notified after her, will shoot her husband k nights after she receives the queen's letter (and no earlier). For $k=0$ the claim is trivial. Assume inductively that the claim holds for k and that Mary is a cheated wife that knows of k cheated wives in the ring before her, and none after her. Thus, once she receives the letter she knows that the last of k cheated wives she knows of has received the letter no later than the same day Mary did. Thus, if Mary's husband is faithful then the last cheated wife she knows of will shoot her own husband no later than k nights after Mary received the letter. Once that fails to happen, Mary shoots her own husband on the $(k+1)^{st}$ night after receiving the letter. The claim follows by induction. To see that no other cheated wife shoots her husband, notice that because of the asynchronous nature of delivery, a wife knowing of a cheated wife later in the ring does not know when that cheated wife will receive the letter, and thus cannot deduce from the night on which a later wife shoots that her own husband is unfaithful (although in some cases she will be able to deduce that her own husband *is* faithful).

(b) The proof of Proposition 3 can be used to show that some unfaithful husbands will be shot in this case. We need to show that injustive might occur, i.e., that some unfaithful husbands might be spared. Consider the following scenario: the bound on delivery is $b=2$. Mary knows of only one cheated wife, Susan, who lives farther down the ring than Mary. Mary receives the letter on Sunday and hears Susan shoot her husband on Monday. Mary cannot distinguish between the following possibilities:

- Susan received the letter on Sunday, and knowing that Mary's husband was unfaithful, she waited to hear if Mary would shoot on Sunday night. Since Mary didn't, Susan discovered that her own husband was unfaithful, and shot him Monday night.
- Susan received the letter on Monday, and knowing that Mary's husband was faithful, discovered that her own husband was unfaithful and shot him that night.

Thus, Mary does not know whether her husband is unfaithful in the above scenario, and does not shoot her husband. If her husband is in fact unfaithful, this constitutes a case of injustice.

(c) The proof of Proposition 3 again ensures us that some husbands will be shot. To show that a case of injustice can arise with strongly synchronous delivery around a ring, consider the situation described in (b) above, with Sunday being the official sending date of the letter. Mary still considers both of the above scenarios possible, and Mary's husband is spared. Thus, if Mary's husband is unfaithful, a case of injustice occurs. □

Notice that in the asynchronous case knowing the order of delivery does help a cheated wife (in this case only the last cheated wife) discover that her husband is unfaithful. In this case the extra knowledge can be considered "helpful". However, more surprising is the fact that the wives' knowing the order of delivery allows an unjust solution in the strongly synchronous case, where none existed without such knowledge! Thus, by introducing an asymmetry in the wives' reasoning, this extra knowledge has a negative effect on the solution.

6 Quick elimination

Queen Margaret opened a new era in Mammajorca. She made the mail system an *express mail* system: All letters sent from her court were guaranteed to be delivered to all of her subjects on the day they were sent. Her first letter notified her subjects about the great advance in their communication capabilities. Margaret was an impatient queen. She knew that using her mail system she could successfully execute Henrietta I's instructions. However, knowing that there were many unfaithful husbands in Mamajorca, and not wanting to wait very long for them to be eliminated, she decided to look for a faster way to solve the problem. She did so by giving her subjects instructions that allowed wives to shoot into the air at midnight. Margaret's scheme was very successful; the unfaithful husbands were eliminated from Mamajorca in just a few days.

Notice that in Henrietta I's solution, n unfaithful husbands are eliminated on the n^{th} night following the queen's announcement. Margaret sought a solution that would requre waiting fewer than $O(n)$ nights. Given that shooting in the air at midnight is allowed, what is the minimal number of nights in which the unfaithful husbands can be eliminated? Margaret's problem can be restated as follows: Given a distributed system in which the processors share a memory consisting of a single toggle bit, each processor has a value, and it is known that the values are at most one apart, how many rounds of communication are needed for the processors with the minimal value to know it? El Gamal and Orlitsky (1984) have treated similar questions independently in a more general setting. The following theorem answers this question in Margaret's case:

Theorem 11. *There is a protocol that allows shooting in the air in which the cheating husbands are all shot by the third night. That is the best possible.*

Proof. Let us first show that a protocol in which a wife's actions depend only on the number of unfaithful husbands she initially knows of and the actual run of the protocol must require at least three nights. Such a protocol P can be viewed as a set of protocols $P(k)$, $k \geq 0$, each specifying how a wife initially knowing of exactly k unfaithful husbands should act. If for some $k \geq 1$ both $P(k-1)$ and $P(k+1)$ do not prescribe any shooting on the first night, then clearly $P(k)$ must require at least three nights, since a wife knowing of k unfaithful husbands cannot know whether her own husband is faithful after the first night. If $P(k')$ includes shooting in the air on the first night for some $k' \geq 1$, then $P(k')$ must require at least three nights when there are $k'+1$ unfaithful husbands. A wife knowing of exactly k' unfaithful husbands shoots in the air on the first night, and cannot determine whether her own husband is unfaithful before the second night. Thus, for all $k \geq 1$, one of $P(k)$, $P(k+1)$, or $P(k+2)$ must require at least three nights.

The following protocol solves the problem in three nights:

(a) A wife knowing of k_0 unfaithful husbands, with $k_0 \equiv 0 \pmod 3$, fires her gun at midnight on the first night. If $k_0 = 0$ she shoots her husband, otherwise she shoots in the air.

(b0) If there was no shot on the first night, then a wife knowing of k_1 unfaithful husbands, with $k_1 \equiv 1 \pmod 3$ should shoot her husband on the second night.

(b1) If there was a shot on the first night, then a wife knowing of k_2 unfaithful husbands, with $k_2 \equiv 2 \pmod 3$ should shoot her husband on the second night.

(c00) If both first nights were silent then all wives shoot their husbands on the third night.

(c10) If there was a shot on the first night, and no shots on the second night, then the first night shooters shoot their husbands on the third night (if he is still alive).

Let us briefly check that this protocol is correct; i.e., we now show that if there is at least one unfaithful husband, then all unfaithful husbands are shot, and no faithful husbands are shot. We first consider the case where there is at least one faithful husband. Thus, if $n=k+1$ is the number of unfaithful husbands, then some wives know of $k+1$ unfaithful husbands, and some of k. If $k \equiv 2 \pmod 3$, then the wives whose husbands are faithful will shoot in the air on the first night. The cheated wives will shoot their husbands on the second night according to step (b1). If $k=0$ then the cheated wife will shoot her husband on the first night and no other shooting occurs. If $k \equiv 0 \pmod 3$ and $k>0$, then the cheated wives shoot in the air on the first night, the other wives are silent on the second night, and by (c10) the cheated wives shoot their husbands on the third night. If $k \equiv 1 \pmod 3$ then the first night is silent, and the cheated wives shoot their husbands on the second night by (b0). We now need to show that if all wives are cheated then the husbands are shot. This is simple, since in all cases a wife that hears no shots other than on nights she shoots ends up shooting her husband (check!). □

Notice that Margaret could have appended the above protocol to Henrietta I's letter; using it, a cheated wife always shoots her husband on the midnight of the day she discovers his infidelity. In fact, a slightly more elaborate lower bound argument of a similar flavor shows that it is the only protocol Mary could have appended to Henrietta I's letter that is guaranteed to terminate in three nights. We remark that by slightly changing steps

(a) and (c10) in the above protocol it is possible to obtain a protocol that works correctly even if there are no unfaithful husbands. (Of course, in the modified protocol a wife knowing of no unfaithful husband will not shoot her husband on the first night, and thus such a protocol cannot be appended to Henrietta I's letter.) Details are left to the reader.

7 Conclusions

The cheating husbands problem is one in which communication, knowledge, and action interact in subtle ways. We have presented a case analysis of variants of this problem given different communication mediums and different degrees of clock synchronization. This problem demonstrates how sensitive the success of an operation can be to the known properties of the communication medium. It also shows how knowledge can be obtained in indirect ways by observing the actions of elements in the system, once we know something about how their actions are related to the facts they know. In fact, as we see in the bribery of the mailperson in Proposition 7, obtaining knowledge about the delivery times of a single letter can in some cases dramatically improve a wife's capability to act.

The queens' instructions in all cases can be viewed as *knowledge-based protocols* in the sense of Halpern and Fagin (1985), since the actions that a wife is required to take depend on her knowledge. The basic high-level "knowledge-based" protocol that the wives follow is:

Do not discuss the matter of your husband's fidelity with anyone. However, should you discover that your husband is unfaithful, you must shoot him on the midnight of the day you find out about it.

Consider a scenario in which the queen's letter reaches all of the wives on the day it is sent. The actual way in which the above protocol will be carried out ("implemented") will depend on the known properties of the mail system. As our analysis shows, the elimination of the $n+1$ unfaithful husbands may take $n+b$ nights, it may take nb nights, and it might never happen at all, depending on whether the mail system is commonly known to be strongly synchronous, weakly synchronous, or asynchronous, and the order of message delivery is unknown. Thus, the execution of the protocol and its success depend not only on what really happens (in this case, all letters being delivered on the same day); they also crucially depend on the wives' state of knowledge of what happens.

Another interesting point that arises here is that knowledge can in some cases be harmful. The results of Theorem 10 show that running the same knowledge-based protocol in a situation where the wives initially have strictly *more* knowledge can result in a less desirable outcome. The ignorance present in the delivery of a message that is broadcast in a strongly synchronous mail system when the order of delivery is unknown gives rise to states of knowledge that allow the wives to perform actions that they cannot perform in the ring, where the order of delivery is known.

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