Approaching $R(D) = C$ in Colored Joint Source/Channel Broadcasting by Prediction

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***Abstract*— We consider transmission of a colored Gaussian source through a power constrained colored Gaussian broadcast channel subject to a mean-squared error distortion measure. It is well known that separation of source and channel coding cannot achieve the point $R(D) = C$ simultaneously for more than one receiver. We characterize the distortion region achieved by the recently proposed joint source/channel "analog matching" coding scheme. In the special case of equal bandwidth (but arbitrary source and channel spectra) and in the limit of high signal to noise ratio (SNR), we prove that full robustness is asymptotically possible, i.e., the encoder becomes SNR-independent and each decoder approaches the ideal performance $R(D) = C$. This result extends the well known optimality of analog transmission in the white source / white channel case. Our results are based upon an encoder which employs modulo-lattice arithmetics, i.e. the transmitted signal is the residue of an analog signal with respect to a lattice.**

I. Introduction

We consider the problem of transmitting a Gaussian source over a Gaussian broadcast channel. The source is defined by a spectrum $S_S(f)$ limited to bandwidth B_S. The channel has an input power constraint $\overline{E\{X_n^2\}} \leq P$, where $\overline{(\ )}$ denotes time average. The channel to each of the M receivers is a bandlimited additive Gaussian noise channel:

$$Y_n^{(m)} = g_n * X_n + Z_n^{(m)} \quad , \quad m = 1, \ldots, M \tag{1}$$

where $*$ denotes convolution, g_n is the impulse response of an ideal low pass filter of bandwidth B_C and the noise sequence $Z_n^{(m)}$ seen by the the receiver m is Gaussian with spectrum $S_{Z^{(m)}}(f)$, bandlimited to B_C (here and onward we use f to denote frequency for discrete-time process, i.e. $S(f) \triangleq S(z)|_{z=e^{j2\pi f}}$). We define the channel signal to noise ratio (SNR) for the m-th user as:

$$SNR^{(m)} \triangleq \frac{P}{N^{(m)}} \triangleq \frac{P}{\mathrm{Var}\{Z_n^{(m)}\}} \quad . \tag{2}$$

The reconstruction distortion is measured by the average mean square error (MSE):

$$D^{(m)} = \overline{E\{(\hat{S}_n^{(m)} - S_n)^2\}} \triangleq \overline{E\{{E_n^{(m)}}^2\}} \quad . \tag{3}$$

An equivalent representation of the channels (1) consists of a filter ("inter-symbol interference" (ISI)) followed by additive *white* Gaussian noise (AWGN):

$$Y_n^{(m)} = X_n^{(m)} * h_n^{(m)} + W_n^{(m)}, \tag{4}$$

where $\mathrm{Var}\{W_n^{(m)}\} = N_0$ for all m. The filter $H(f)$ is bandlimited to B_C, and inside this band it satisfies the connection with the model (1):

$$S_Z(f) = \frac{N_0}{|H(f)|^2} \quad . \tag{5}$$

From Shannon, we know that the distortion at each receiver must satisfy:

$$R(D^{(m)}) \leq C^{(m)} \quad , \tag{6}$$

where $R(D)$ is the source rate-distortion function and $C^{(m)}$ is the capacity of the m-th channel (1). Ideally, each receiver m would achieve the optimal point to point distortion $D^{*(m)}$ associated with equality in (6). However, this is generally not possible simultaneously for more than one receiver. The question of what *can* be achieved is still open. It is well known, that a separation-principle based solution (i.e. successive refinements source coding followed by broadcast channel coding) does not yield the optimal trade-off between the distortions $D^{(m)}$[1]. On the other hand, an analog solution is not generally optimal even for the point to point problem.

It turns out, that the ratio of the source and channel bandwidths plays a major role in the analysis of the achievable distortion region. We define the *bandwidth expansion factor*:

$$\rho = \frac{B_C}{B_S}. \tag{7}$$

We call the situation where $\rho > 1$ and $\rho < 1$ generalized bandwidth expansion (GBE) and generalized bandwidth compression (GBC), respectively. In between these cases we have the equal bandwidth case of $\rho = 1$. The distinction between these three cases is well known in the case where the source and the channel are white, where we have:

1. **Matching bandwidth, white spectra:** For the case of white source and channel with $\rho = 1$, analog transmission achieves the ideal performance (6) with an encoder which is independent of the noise variance [3], and therefore is simultaneously optimal for all the decoders. In such a system, the encoder and the decoders will each consist only of one multiplication by a constant factor.

2. **Bandwidth expansion:** In the white setting with $\rho > 1$ channel uses per each source sample, (6) amounts to:

$$\frac{\mathrm{Var}\{S_n\}}{D^{(m)}} \leq \left(1 + SNR^{(m)}\right)^{\rho} \quad . \tag{8}$$

Various broadcasting strategies have been offered for this case [1], [7], [10], [9]. They most share the *hybrid digital-analog* (HDA) approach, transmitting a combination of digital and analog signals, trying to combine the point to point optimality of digital transmission with the simplicity and robustness of analog transmission. [10] gives an outer bound for two receivers, showing that if the encoder is designed for optimality with respect to the bad receiver, than the distortion in the good receiver is at most inversely proportional to its SNR, i.e. far from (8).

3. **Bandwidth compression:** This varies from the previous case, in that we are allowed less than one channel use per source sample. This case has been treated in [7], [8], [9], where HDA system are suggested, but there is no known non-trivial outer bound.

In this paper we give a unified treatment to any source and channel spectra. In the case of equal bandwidth ($\rho = 1$) and for high SNR, we show a scheme which is robust in the sense that the ideal performance (equality in (6)) is approached for all noise variance (but otherwise fixed spectral shape) using a single encoder. Thus, in the high SNR limit this scheme generalizes the perfect matching property of white sources and channels [3] to the colored case. For the GBE and GBC cases ($\rho \neq 1$), we derive achievable distortions regions, and analyze the limitations of our method. We conjecture that no scheme can do better asymptotically, when the encoder is optimized for one receiver.

Our results are based upon a scheme presented in [6], called the *analog matching* scheme. This scheme can "match" any given source spectrum to any given channel spectrum optimally, by treating the source and the channel in the *time domain*. The underlying principle is inspired by precoding [12] and differential pulse code modulation (DPCM) [5], and it is based on prediction and on solving side-information problems using modulo lattice operations [13]. the solution is analog in the sense that it does not contain analog-to-digital (A/D) conversion; the only non-linear components are a single modulo-lattice operation at the encoder and another modulo-lattice operation at the decoder.

The rest of this paper is organized as follows: We start in Section II by giving some preliminaries. Then in Section III we prove our basic robustness result using a zero-forcing scheme which is asymptotically optimal at high SNR. In Section IV we analyze the performance of a minimum mean-squared error scheme which is suitable for any ρ and optimal for a given noise.

II. Preliminaries: Spectral Decomposition and Shannon Bounds

The Paley-Wiener condition for a spectrum $S(f)$ is:

$$\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\Big(S(f)\Big) df \right| < \infty \quad .$$

This condition holds for example if the spectrum $S(f)$ is bounded away from zero. Whenever the Paley-Wiener condition holds, the spectrum has a spectral decomposition:

$$S(f) = B(z)B^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j2\pi f}} P_e\Big(S(f)\Big) \quad , \tag{9}$$

where $B(z)$ is a monic causal filter, and the entropy-power of the spectrum $P_e\Big(S(f)\Big)$ is defined by:

$$P_e\Big(S(f)\Big) = \exp \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\Big(S(f)\Big) df \quad . \tag{10}$$

The *optimal predictor* of a process having a spectrum $S(f)$ from its infinite past is $B(z) - 1$, a filter with an impulse response satisfying $b_n = 0$ for all $n \leq 0$, with the prediction mean squared error (MSE) being the entropy power. Consequently, the prediction gain of a spectrum $S(f)$ is:

$$\Gamma\Big(S(f)\Big) = \frac{\mathrm{Var}\{S_n\}}{\mathrm{Var}\{S_n | \mathbf{S}_{-\infty}^{n-1}\}} = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} S(f) df}{P_e\Big(S(f)\Big)} \quad . \tag{11}$$

Note that this gain is 1 for a white spectrum (representing an unpredictable process), and bigger than 1 otherwise, and it is finite whenever the Paley-Wiener condition holds. Equivalently, the process can be represented as generated from its innovations by the filter $B^*\left(\frac{1}{z^*}\right)$, where these innovations also constitute the minimum MSE prediction errors of the process.

We define the source and channel prediction gains Γ_S and Γ_C to be the prediction gains of the spectra $S_S(f)$ and $S_Z(f)$ respectively.

The Gaussian-Quadratic rate-distortion function (RDF) is given by:

$$R(D) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log \frac{S_S(f)}{D(f)} df \quad , \tag{12}$$

where the *distortion spectrum* $D(f)$ is given by the reverse water-filling solution: $D(f) = \min\Big(\theta_S, S(f)\Big)$ with the *water level* θ_S set by the distortion level D: $D = \int_{-1/2}^{1/2} D(f) df$. The channel capacity for the additive Gaussian channel is given by:

$$C = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + \frac{P(f)}{S_Z(f)}\right) df \quad , \tag{13}$$

where the *channel input spectrum* $P(f)$ is given by the water-filling solution: $P(f) = \max\Big(\theta_C - S_Z(f), 0\Big)$ with the *water level* θ_C set by the power constraint P: $P = \int_{-1/2}^{1/2} P(f) df$.

When the source and noise spectra satisfy the Paley-Wiener conditions, the *Shannon lower bound* (SLB) for the RDF and the *Shannon upper bound* (SUB) for the channel capacity [11, Thm. 18 and Thm. 23] are defined. The SLB is given by:

$$R(D) \geq \frac{1}{2} \log\left(\frac{1}{\Gamma_S} \frac{\mathrm{Var}\{S_n\}}{D}\right) \triangleq R_{SLB}(D) \tag{14}$$

and it is tight for a Gaussian source whenever the distortion level D is low enough such that $D(f) = D = \theta$, while the SUB is:

$$C \leq \frac{1}{2} \log\left[\Gamma_N \cdot (1 + SNR)\right] \triangleq C_{SUB} \quad , \tag{15}$$

with the SNR given by (2). The bound is tight for a Gaussian channel whenever the SNR is high enough such that $S_Z(f) + P(f) = P + N = \theta_C$. Though the Shannon bounds may never hold with equality if the spectra are not bounded away from zero, they are asymptotically tight. Using this fact, comparing the SLB and SUB leads to the following representation of the ideal performance (6):

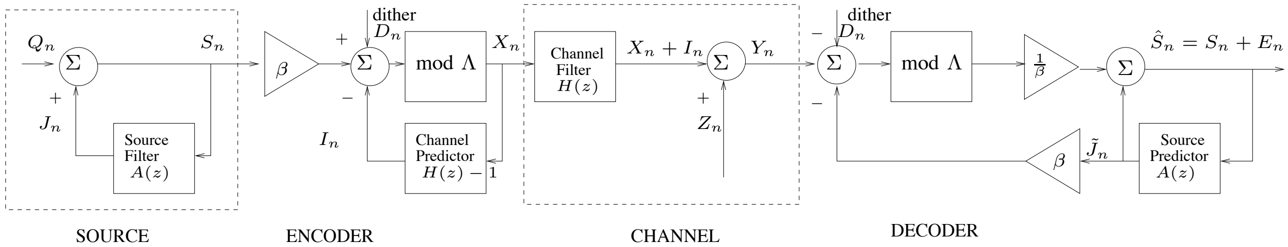


Figure 1: The Zero-Forcing Scheme

Proposition 1: Let $R(D^*) = C$, then:

$$\frac{1}{1+SNR}\frac{\mathrm{Var}\{S_n\}}{D^*} \leq \Gamma_S\Gamma_C \quad , \tag{16}$$

with equality if and only if the SLB and SUB hold with equality. Furthermore if the noise spectrum is held fixed while the allowed power P is taken to infinity:

$$\lim_{SNR\to\infty}\frac{1}{SNR}\frac{\mathrm{Var}\{S_n\}}{D^*} = \Gamma_S\Gamma_C \quad . \tag{17}$$

III. Broadcasting by Zero-Forcing Analog Matching for $\rho = 1$

In this section we concentrate on the case of equal source and channel bandwidth ($\rho = 1$ in (7)). For this case we present a simple scheme which is not optimal for any non-zero noise spectrum, yet is asymptotically optimal at high SNR. We call it a zero-forcing (ZF) scheme, because the filtering and prediction done assume the limit of low noise, just as a receiver employing zero-forcing equalization does. We assume that the source is an auto-regressive (AR) process of some finite order:

$$S_n = Q_n + \sum_{l=1}^{L} a_l S_{n-l} \triangleq Q_n + J_n \quad , \tag{18}$$

where a_n is the impulse response of the filter $A(z)$ and Q_n is the source i.i.d. innovations process of power $P_e\Big(S(f)\Big)$. The channel is given by the equivalent ISI form (4) with the noise variance $N_0 = \frac{N}{\Gamma_C}$, and with $H(z)$ set by (5) being a monic, minimum-phase filter of some finite order. Note that under these conditions, both the source and noise spectra satisfy the Paley-Wiener conditions.

For these source and channel we show that the ZF scheme approaches optimality in the limit of high SNR. By that, we give a constructive proof to the following achievability result, which arbitrarily approaches equality in (6):

Theorem 1: **(Robustness at unknown high SNR)** For an AR source S_n, a channel with finite impulse response $H(z)$ and any $\epsilon > 0$, there exists a sufficiently large SNR_0 such that a single encoder achieves a distortion D satisfying

$$R\Big((1-\epsilon)D\Big) \geq C$$

for all $SNR \geq SNR_0$, where the channel SNR is defined in (2).

Figure 1 illustrates the ZF analog matching scheme. Motivated by the form (17), the basic idea behind the workings of the analog matching scheme is of reducing the original modulation problem (a colored source through a colored channel) into that of transmitting the source innovations Q_n through the equivalent noise innovations channel. Thus, the analog matching scheme uses source and channel prediction to extract the corresponding innovations and enjoy the desired prediction gains. However, unlike the more common configuration, the scheme predicts the source at the *decoder* side and inverts the channel at the *encoder* side.

The encoder and decoder of the scheme are given by:

$$X_n = \Big[\beta S_n - \tilde{I}_n\Big] \mod \Lambda \tag{19}$$

and

$$\hat{S}_n = \frac{[Y_n - \beta\tilde{J}_n] \mod \Lambda}{\beta} + \tilde{J}_n \tag{20}$$

respectively, where I_n and $\tilde{J}_n$ denote the source and channel predictor outputs respectively. While the channel ISI is canceled completely, the source innovations can be only recovered up to the effect of the estimation error:

$$\tilde{J}_n = J_n + E_n * a_n \quad . \tag{21}$$

The lattice Λ is defined by the generator matrix $G \in \mathbb{R}^{K\times K}$. The lattice includes all points $\{G\cdot\mathbf{i} : \mathbf{i}\in\mathbb{Z}^K\}$ where $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$. The nearest neighbor quantizer associated with Λ is defined by

$$Q(\mathbf{x}) = \arg\min_{\mathbf{l}\in\Lambda}\|\mathbf{x}-\mathbf{l}\|$$

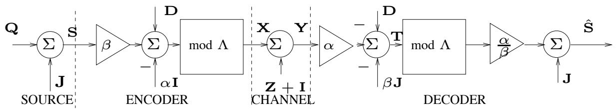
where $\|\cdot\|$ denotes Euclidean norm. The modulo-lattice operation is defined by:

$$\mathbf{x} \mod \Lambda = \mathbf{x} - Q(\mathbf{x}) \quad .$$

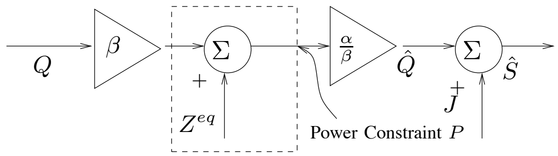
The dither vector $\mathbf{D}$ is uniformly distributed over the basic lattice cell (for which $Q(\mathbf{x}) = 0$), and is independent of the input.

The scheme depends upon *correct decoding* of the modulo-lattice operation. That is, the modulo-lattice operations in the encoder and decoder should exactly cancel each other. This happens in the limit of high lattice dimension when the signal power at the decoder lattice output does not exceed the lattice cell power (per dimension) when using *good lattices* [2], which are simultaneously good for quantization and for channel coding, leading to the following equivalence:

Proposition 2: **Equivalence to an additive output power-constrained channel** [6, Sec. III] In the K-dimensional channel depicted in Figure 2a, let $\mathbf{Q}$ and $\mathbf{Z}$ be Gaussian i.i.d. vectors, let $\mathbf{Z}^{eq} = \alpha\mathbf{Z} - (1-\alpha)\mathbf{X}$ and let the dither vector $\mathbf{D}$



(a) Original Modulo-Lattice Channel



(b) Equivalent Additive Channel

Figure 2: Basic Channel Equivalence

be uniformly distributed over the basic cell of the lattice Λ. Then exist a sequence of good lattices Λ_K such that each of the components of $\hat{\mathbf{S}}$ approaches one which is created by the channel (2b) in the mean square sense.

Using this, we are ready to state the performance of the scheme of Figure 1.

Theorem 2: Let D_K denote the distortion associated with the system above with a K-dimensional lattice Λ_K. Let $\Gamma_{S^{-1}}$ be the prediction gain of the spectrum $S_S^{-1}(f)$. For a sequence of good lattices [2] of normalized second moment P, and for $SNR > \frac{\Gamma_{S^{-1}}}{\Gamma_C}$:

$$\lim_{K\to\infty} D_K = \frac{N}{\beta^2 \Gamma_C} \quad ,$$

provided that the gain β satisfies

$$\beta^2 < \Gamma_S \left[1 - \frac{\Gamma_{S^{-1}}}{\Gamma_C SNR}\right] \frac{P}{\mathrm{Var}\{S_n\}} \quad . \tag{22}$$

The proof is left for the full paper. It is based upon identifying the ZF analog matching scheme with the scheme of Figure 2a, and then examining the condition for satisfying the output power-constraint for invoking the equivalence of Proposition 2, specifically the power of the error term in (21). Note that the predictor inputs need to be updated at each instant, while the modulo-lattice operation is done in K-blocks. This apparent contradiction is solved by an interleaving mechanism which follows the method applied in [4] to decision feedback equalizers (DFE).

Now, if we know that $SNR \geq SNR_0$, we can choose β approaching the r.h.s. of (22) for SNR_0, and then we can deduce directly from Theorem 2 and from (16):

Corollary 1: For any $SNR_0 > \frac{\Gamma_{S^{-1}}}{\Gamma_C}$ exists a single encoder, such that

$$\frac{D}{D^*} \leq \frac{1+SNR}{SNR} \frac{SNR_0}{SNR_0 - \frac{\Gamma_{S^{-1}}}{\Gamma_C}}$$

for all $SNR \geq SNR_0$, where D^* is the ideal distortion (6).

We note that the loss in distortion has two terms. The first term reflects the loss of any zero-forcing system w.r.t. an optimal MMSE system, while the second one is the loss due to "leaving space" for the noise in order to ensure correct decoding. This is not the optimal performance using this encoder, since we also used a fixed decoder instead of adjusting it to the SNR. However, we see that the two terms approach 1 for high SNR, which is sufficient for proving our main robustness result, Theorem 1.

Until now we have assumed that we know the shape of $S_Z(f)$ up to a constant gain. What happens if we do not know it? Obviously, we can not use channel prediction, thus we can not scale the distortion down according to the channel prediction gain. However, by using source prediction we still enjoy the source prediction gain.

The zero-forcing approach can not be extended to cases where $\rho \neq 1$. In the GBE case, the source prediction gain is infinite. The gain β we can use is thus only limited by the noise level, and it becomes unboundedly large for low noise. Thus, fixing β so that correct decoding holds for SNR_0, causes unlimited loss as $SNR \to \infty$. In the GBC case, the channel prediction gain is infinite, and the optimal predictor is undefined. Trying to use any suboptimal predictor will again result in an unbounded loss as $SNR \to \infty$, since this predictor will not completely revert the channel filter, and the residual ISI will limit the performance when the channel noise is small.

IV. Broadcasting by Optimal MMSE Analog Matching for Any ρ

The Gaussian degraded broadcast channel problem with general spectra is still open. For the general case, nor were schemes suggested, neither is there a known non-trivial outer bound. In this section we constructively show achievable distortions, and conjecture the behavior of the outer bound in the limit of high SNR. Our analysis is based upon the minimum mean squared error (MMSE) analog matching scheme we presented in [6, Sec. IV]. This scheme approaches equality in (6) for a given source and channel. It differs from the ZF scheme described above, by replacing the predictors with MMSE predictors and adding two additional filter pairs: A source pre- and post-filter (which can be thought of as a source shaping filter and source optimal estimator, respectively) and a channel pre- and post-filters (which can be thought of as a channel shaping filter and a channel feed-forward equalizer, respectively).

Abandoning the equal BW case, we still assume that the source and the channel satisfy the Paley-Wiener condition inside the bands B_S and B_C respectively. We assume that an analog matching encoder was designed for a noise spectrum $S_{Z0}(f)$ (with associated SNR of SNR_0 according to (2)), and examine the distortion achieved when the actual noise spectrum is $S_Z(f)$, everywhere equal to or lower than $S_{Z0}(f)$. Under these degraded channel conditions, correct decoding in the decoder lattice for $S_{Z0}(f)$ also assures correct decoding for $S_Z(f)$, and the problem of finding the optimal $S_Z(f)$-dependent decoder becomes a linear estimation problem. For this worst channel $S_{Z0}(f)$ and for optimal distortion (6), we find the water-filling solutions (12),(13), resulting in the source and channel water levels θ_S and θ_C respectively, and in a *source-channel passband* $\mathcal{F}_0$, which is the intersection of the inband frequencies of the source and channel water-filling solutions. Under this notation we have the following Theorem, the proof of which will be included in the full paper to follow:

Theorem 3: For a sequence of good lattices Λ_K and for any noise spectrum $S_{Z0}(f)$, exists a single encoder (which approaches (6) for $S_{Z0}(f)$), such that for any spectrum $S_Z(f) \leq S_{Z0}(f) \forall f$, a suitable decoder can achieve:

$$\lim_{K\to\infty} D_K = \int_{-\frac{1}{2}}^{\frac{1}{2}} D(f) df \quad ,$$

where the distortion spectrum $D(f)$ satisfies:

$$D(f) = \left\{ \begin{array}{ll} \frac{S_S(f)}{1+\Phi(f)}, & \text{if } f \in \mathcal{F}_0 \\ \min\Big(S_S(f), \theta_S\Big), & \text{otherwise} \end{array} \right\} \quad , \qquad (23)$$

with

$$\Phi(f) = \left[1 - \frac{S_{Z0}(f) - S_Z(f)}{\theta_C}\right] \cdot \frac{S_S(f) - \theta_S}{\theta_S} \frac{SNR}{SNR_0} \quad . \qquad (24)$$

Remarks:

1. Outside $\mathcal{F}_0$, there is no gain where the noise spectral density is lower than expected. Inside $\mathcal{F}_0$, the distortion spectrum is strictly monotonously decreasing in $S_Z(f)$, but the dependence is never stronger than inversely proportional. It follows, that the overall distortion D is at most inversely proportional to the SNR. This is to be expected, since all the gain comes from linear estimation.

2. In the unmatched case modulation may change performance. That is, swapping source frequency bands before the analog matching encoder will change $\mathcal{F}_0$ and $\Phi(f)$, resulting in different performance as $S_Z(f)$ varies. It can be shown that the best robustness is achieved when $S_S(f)$ is monotonously decreasing in $S_Z(f)$.

3. The degraded channel condition is not necessary. The exact condition can be stated in terms of $S_S(f)$, $S_{Z0}(f)$ and $S_Z(f)$, though it is cumbersome.

Bandwidth Expansion and Compression:

For bandwidth expansion and compression, the scheme above achieves:

$$\frac{D}{\text{Var}\{S_n\}} = \frac{1 - \min(\rho, 1)}{\Big(1 + SNR_0\Big)^{\rho}} + \frac{\min(\rho, 1)}{1 + \Phi_\rho(SNR, SNR_0)} \quad , \qquad (25)$$

where

$$\Phi_\rho(SNR, SNR_0) \triangleq \frac{1 + SNR}{1 + SNR_0} \Big[(1 + SNR_0)^{\rho} - 1\Big] \quad .$$

This is worse than the performance reported in [7], [10] for these cases, although the difference vanishes for high SNR.

The basic drawback of analog matching compared to methods developed specifically for these special cases seems to be, that these methods apply different ”zooming” to different source or channel frequency bands, analog matching uses the same ”zooming factor” β for all bands. Enhancements to the scheme, such as the combination of analog matching with pure analog transmission, may improve these results.

On the Asymptotic Behavior for High SNR:

Now we fix the shape of the noise spectrum $S_Z(f)$ up to a constant gain, and examine the distortion as a function of the SNR. Fundamentally, the asymptotic behavior of the distortion at high SNR is only a function of the bandwidth expansion factor ρ (7). We define the *asymptotic distortion slope* for a continuum of schemes operating on spectra with BW expansion factor ρ as:

$$\lambda(\rho) \triangleq \lim_{SNR_0 \to \infty} \log\Big(\frac{D}{D_0} \frac{SNR}{SNR_0}\Big) \qquad (26)$$

where D is the distortion level attained at SNR and D_0 is achieved at SNR_0, wherever the limit exists and is fixed for all $SNR \geq SNR_0$. By Proposition 1, a continuum of schemes achieving the ideal performance (6) satisfies $\lambda(\rho) = \rho$.

For the analog matching scheme, we know by Theorem 3 that the distortion spectrum is asymptotically linear with noise variance inside $\mathcal{F}_0$, but constant outside $\mathcal{F}_0$. Since the distortion outside $\mathcal{F}_0$ is zero for GBE ($\rho > 1$) but non-zero for GBC($\rho < 1$), we have the following:

Corollary 2: For any source and channel spectra of BW ratio ρ, a single MMSE analog-matching encoder optimal at SNR_0 can achieve the asymptotic distortion slope:

$$\lambda(\rho) = \left\{ \begin{array}{ll} 1, & \text{if } \rho \geq 1 \\ 0, & \text{otherwise} \end{array} \right\} \quad .$$

This asymptotic slope agrees with the outer bound of [10] for the (white) bandwidth expansion problem. For the bandwidth compression problem, no outer bound is known, but we are not aware of any proposed scheme with a non-zero asymptotic slope. We believe this to be true for all spectra:

Conjucture 1: For any source and channel spectra of BW ratio ρ, no single encoder which satisfies $R(D_0) = C$ at SNR_0 can have a better slope than that of Corollary 2.

By this conjecture, analog matching is asymptotically optimal among all encoders ideally matched to one SNR.

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