Joint Wyner-Ziv/Dirty-Paper Coding by Modulo-Lattice Modulation†

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Abstract

The combination of source coding with decoder side-information (Wyner-Ziv problem) and channel coding with encoder side-information (Gel'fand-Pinsker problem) can be optimally solved using the separation principle. In this work we show an alternative scheme for the quadratic-Gaussian case, which merges source and channel coding. This scheme achieves the optimal performance by a applying modulo-lattice modulation to the analog source. Thus it saves the complexity of quantization and channel decoding, and remains with the task of "shaping" only. Furthermore, for high signal-to-noise ratio (SNR), the scheme approaches the optimal performance using an SNR-independent encoder, thus it is robust to unknown SNR at the encoder.

keywords: joint source/channel coding, analog transmission, Wyner-Ziv problem, writing on dirty paper, modulo lattice modulation, MMSE estimation, unknown SNR, broadcast channel.

I. Introduction

Consider the quadratic-Gaussian joint source/channel coding problem for the Wyner-Ziv (WZ) source [1] and Gel'fand-Pinsker channel [2], as depicted in Figure 1. In the Wyner-Ziv setup, the source is jointly distributed with some side information (SI) known at the decoder. In the Gaussian case, the WZ-source sequence S_k is given by:

$$S_k = Q_k + J_k \quad , \tag{1}$$

where the unknown source part, Q_k, is Gaussian i.i.d. with variance σ_Q^2, while J_k is an arbitrary SI sequence known at the decoder. In the Gel'fand-Pinsker setup, the channel transition distribution depends

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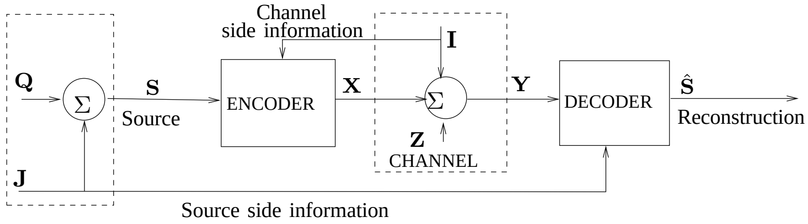


Figure 1: The Wyner-Ziv / dirty-paper coding problem

on a state that serves as encoder SI. In the Gaussian case, known as the dirty paper channel (DPC) [3], the DPC output, Y_k, is given by:

$$Y_k = X_k + Z_k + I_k \quad , \tag{2}$$

where X_k is the channel input, the unknown channel noise, Z_k, is Gaussian i.i.d. with variance N, while I_k is an arbitrary interference, known at the encoder. When referring to I_k and J_k, we use the terms interference and SI interchangeably, since they may be seen either as external components added to the source and to the channel noise, or as known parts of these entities. From here onward we use the bold notation to denote K-dimensional vectors, i.e.

$$\mathbf{X} = [X_1, \cdots, X_k, \cdots, X_K] \quad .$$

The sequences $\mathbf{Q}, \mathbf{J}, \mathbf{Z}$ and $\mathbf{I}$ are all mutually independent, hence the channel noise $\mathbf{Z}$ is independent of the channel input sequence $\mathbf{X}$. The encoder is some function of the source vector that may depend on the channel SI vector as well:

$$\mathbf{X} = f(\mathbf{S}, \mathbf{I}) \quad , \tag{3}$$

and must obey the power constraint

$$\frac{1}{K} E\{\|\mathbf{X}\|^2\} \leq P, \tag{4}$$

where $\|\cdot\|$ denotes the Euclidean norm. The decoder is some function of the channel output vector that may depend on the source SI vector as well:

$$\hat{\mathbf{S}} = g(\mathbf{Y}, \mathbf{J}) \quad , \tag{5}$$

and the reconstruction quality performance criterion is the mean-squared error (MSE):

$$D = \frac{1}{K} E\{\|\hat{\mathbf{S}} - \mathbf{S}\|^2\} \quad . \tag{6}$$

The setup of Figure 1 described above is a special case of the joint WZ-source and Gel'fand-Pinsker channel setting. Thus, by Merhave and Shamai [4], Shannon's separation principle holds. So a combination of optimal source and channel codes can approach the optimum distortion D^{opt}, satisfying:

$$R_{\text{WZ}}(D^{opt}) = C_{\text{DPC}} \tag{7}$$

where $R_{\text{WZ}}(D)$ is the WZ-source rate-distortion function and C_{DPC} is the dirty paper channel capacity. However, the optimality of "digital" separation-based schemes comes at the price of large delay and complexity. Moreover, they suffer from lack of robustness: if the channel signal-to-noise ratio (SNR) turns out to be lower than expected, the resulting distortion may be very large, while if the SNR is higher than expected, there is no improvement in the distortion [6], [7].

In the special case of white Gaussian source and channel without side information ($\mathbf{I} = \mathbf{J} = \mathbf{0}$), it is well known that analog transmission is optimal [8]. In that case, the encoding and decoding functions

$$\begin{aligned} X_k &= \beta S_k \quad , \\ \hat{S}_k &= \frac{\alpha}{\beta} Y_k \end{aligned} \tag{8}$$

are mere scalar factors, where β is a "zoom in" factor chosen to satisfy the channel power constraint and α is the channel MMSE (Wiener) coefficient. This scheme achieves the optimal distortion (7) while having low complexity (two multiplications per sample), zero delay and *full robustness*: only the receiver needs to know the channel SNR, while the transmitter is completely ignorant of that. Such a perfect matching of the source to the channel, which allows *single-letter coding*, only occurs under very special conditions [9].

In the quadratic-Gaussian setting in the presence of side information, these conditions do not hold [4]. It is interesting to note that in this case, $R_{\text{WZ}}(D)$ is just the Gaussian rate-distortion function for the unknown source part $\mathbf{Q}$ [5], while C_{DPC} is just the AWGN capacity for the channel noise $\mathbf{Z}$ [3], i.e. the SI components $\mathbf{I}$ and $\mathbf{J}$ are "eliminated" as would be done had they been known to both the encoder and the decoder. We see, then, that this perfect interference cancelation is not achievable by single-letter coding.

In this work we propose a scheme for the joint Wyner-Ziv/dirty-paper problem that takes a middle path, i.e., a "semi-analog" solution which partially gains the complexity and robustness advantages of analog transmission: It can be made optimal (in the sense of (7)) for any fixed SNR, with reduced complexity. Moreover, it allows a good compromise between the performance at different SNRs, and becomes SNR-independent at the limit of high SNR.

The scheme we present subtracts the channel interference $\mathbf{I}$ at the encoder modulo-lattice, then uses again subtraction of the source known part $\mathbf{J}$ in conjunction with modulo-lattice arithmetic at the decoder. Thus it achieves an *equivalent single-letter channel* with $\mathbf{I} = \mathbf{J} = \mathbf{0}$. Since the processing is applied to the analog signal, without using any information-bearing code, we call this approach *modulo-lattice modulation* (MLM).

Modulo-lattice codes were suggested as a tool for side information source and channel problems; see [10], [11], where a lattice is used for shaping of a digital code (which may itself have a lattice structure as well, yielding a nested lattice structure). Modulo-lattice transmission of an analog signal in the WZ setting was first introduced in [12], in the context of joint source/channel coding with bandwidth expansion, i.e. when there are several channel uses per each source sample. Here we generalize and formalize this approach, and apply it to SI problems. In a preliminary version of this work [13], we used the MLM scheme as a building block in *Analog Matching* of colored sources to colored channels. Later, Wilson et al. [14], [15] used transmission of an analog signal modulo a *random* code to arrive at similar results. Recently, MLM was used in network settings for computation over the Gaussian MAC [16] or for coding for the colored Gaussian relay network [17].

The rest of the paper is organized as follows: In Section II we bring preliminaries about multi-dimensional lattices, and discuss the existence of lattices that are asymptotically suitable for joint WZ/DPC coding. In Section III we present the joint WZ/DPC scheme and prove its optimality. In Section IV we examine the scheme in an unknown SNR setting and show its asymptotic robustness. Finally, Section V discusses complexity reduction issues.

II. Background: Good Shaping Lattices for Analog Transmission

Before we present the scheme, we need some definitions and results concerning multi-dimensional lattices. Let Λ be a K-dimensional lattice, defined by the generator matrix $G \in \mathbb{R}^{K\times K}$. The lattice includes all points $\{\mathbf{l} = G \cdot \mathbf{i} : \mathbf{i} \in \mathbb{Z}^K\}$ where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. The nearest neighbor quantizer associated with Λ is defined by

$$Q(\mathbf{x}) = \arg\min_{\mathbf{l}\in\Lambda} \|\mathbf{x} - \mathbf{l}\| \quad ,$$

where $\|\cdot\|$ denotes the Euclidian norm and ties are broken in a systematic manner. Let the basic Voronoi cell of Λ be

$$\mathcal{V}_0 = \{\mathbf{x} : Q(\mathbf{x}) = \mathbf{0}\} \quad .$$

The second moment of a lattice is given by the variance of a uniform distribution over the basic Voronoi cell:

$$\sigma^2(\Lambda) = \frac{1}{K}\int_{\mathcal{V}_0} \|\mathbf{x}\|^2 d\mathbf{x} \quad . \tag{9}$$

The modulo-lattice operation is defined by:

$$\mathbf{x} \bmod \Lambda = \mathbf{x} - Q(\mathbf{x}) \quad .$$

By definition, this operation satisfies the "distributive law":

$$[\mathbf{x} \bmod \Lambda + \mathbf{y}] \bmod \Lambda = [\mathbf{x} + \mathbf{y}] \bmod \Lambda \quad . \tag{10}$$

The covering radius of a lattice is given by

$$r(\Lambda) = \max_{\mathbf{x}\in\mathcal{V}_0} \|\mathbf{x}\| \quad . \tag{11}$$

For a dither vector $\mathbf{d}$, the dithered modulo-lattice operation is:

$$\mathbf{y} = [\mathbf{x} + \mathbf{d}] \bmod \Lambda \quad .$$

If the dither vector $\mathbf{D}$ is independent of $\mathbf{x}$ and uniformly distributed over the basic Voronoi cell $\mathcal{V}_0$, then $\mathbf{Y} = [\mathbf{x} + \mathbf{D}] \bmod \Lambda$ is uniformly distributed over $\mathcal{V}_0$ as well, and independent of $\mathbf{x}$ [18]. Consequently, the second moment of $\mathbf{Y}$ per element is $\sigma^2(\Lambda)$.

The loss factor $L(\Lambda, p_e)$ of a lattice w.r.t. Gaussian noise at error probability p_e is defined as follows. Let $\mathbf{Z}$ be Gaussian i.i.d. vector with element variance equal to the lattice second moment $\sigma^2(\Lambda)$. Then

$$L(\Lambda, p_e) = \min\left\{ l : \ \Pr\left\{ \frac{\mathbf{Z}}{\sqrt{l}} \notin \mathcal{V}_0 \right\} \le p_e \right\} \quad . \tag{12}$$

For small enough p_e this factor is at least one. By [19, Theorem 5], there exists a sequence of lattices which possesses a vanishing loss at the limit of high dimension[1], i.e.:

$$\lim_{p_e\to 0} \lim_{K\to\infty} L(\Lambda_K, p_e) = 1 \quad . \tag{13}$$

Moreover, there exists a sequence of such lattices that is also *good for covering*, i.e. defining:

$$\tilde{L}(\Lambda) = \frac{r^2(\Lambda)}{K \cdot \sigma^2(\Lambda)} \quad , \tag{14}$$

where $r(\Lambda)$ was defined in (11), the sequence also satisfies[2]: $\lim_{K\to\infty} \tilde{L}(\Lambda_K) = 1$. However, for this work we need a slightly modified result, which allows to replace the Gaussian noise by a combination

[1] These lattices are simultaneously good for source and channel coding; see more on this in Appendix I.

[2] Note that by definition, $\tilde{L}(\Lambda_K) \ge 1$ always.

of Gaussian and "self-noise" components. To that end, we define for any $0 \leq \alpha \leq 1$ the α-mixture noise as:

$$\mathbf{Z}_\alpha = \sqrt{1-(1-\alpha)^2}\mathbf{W} - (1-\alpha)\mathbf{D} \quad ,$$

where $\mathbf{W}$ is Gaussian i.i.d. with element variance $\sigma^2(\Lambda)$, and $\mathbf{D}$ is uniform over $\mathcal{V}_0$ and independent of $\mathbf{W}$. Note that since $\frac{1}{K}\|\mathbf{D}\|^2 = \sigma^2(\Lambda)$, the resulting mixture also has average per-element variance $\sigma^2(\Lambda)$. We re-define the loss factor w.r.t. this mixture noise as

$$L(\Lambda, p_e, \alpha) = \min\left\{l : \ \Pr\left\{\frac{\mathbf{Z}_\alpha}{\sqrt{l}} \notin \mathcal{V}_0\right\} \leq p_e\right\} \quad . \tag{15}$$

Note that this definition reduces to (12) for $\alpha = 1$. Using this definition, we have the following, which is a direct consequence of [20].

Proposition 1: **(Existence of good lattices)** For any error probability $p_e > 0$, and for any $0 \leq \alpha \leq 1$, there exists a sequence of K-dimensional lattices Λ_K satisfying:

$$\lim_{p_e \to 0} \lim_{K \to \infty} L(\Lambda_K, p_e, \alpha) = 1 \quad , \tag{16}$$

and

$$\lim_{K \to \infty} \tilde{L}(\Lambda_K) = 1 \quad . \tag{17}$$

Note that since by definition, $L(\Lambda_K, p_e, \alpha)$ is non-increasing in p_e, it follows that for *any* $p_e > 0$ this sequence of lattices satisfies:

$$\limsup_{K \to \infty} L(\Lambda_K, p_e, \alpha) \leq 1 \quad . \tag{18}$$

In Appendix I we elaborate more on the significance of this result, and on its connection to more commonly used measures of goodness of lattices.

III. Modulo-Lattice WZ/DPC Coding

We now present the joint source/channel scheme for the SI problem of Figure 1. As explained in the Introduction, the quadratic-Gaussian rate-distortion function (RDF) of the WZ source (1) is equal to the RDF of the source Q_k (without the known part J_k), given by:

$$R_{\text{WZ}}(D) = \frac{1}{2} \log \frac{\sigma_Q^2}{D} \quad . \tag{19}$$

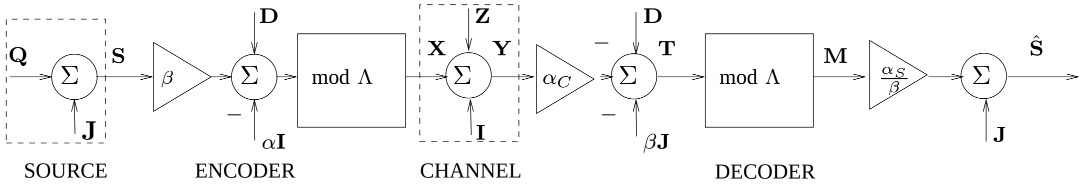


Figure 2: Analog Wyner-Ziv / dirty-paper coding scheme: $\mathbf{S}$ = source, $\hat{\mathbf{S}}$ = reconstruction, $\mathbf{Z}$ = channel noise, $\mathbf{I}$ = interference known at the encoder, $\mathbf{J}$ = source component known at the decoder, $\mathbf{D}$ = dither

Similarly, the capacity of the Gaussian DPC (2) is equal to the AWGN capacity (without the interference I_k):

$$C_{\text{DPC}} = \frac{1}{2}\log\left(1 + \frac{P}{N}\right) \quad . \tag{20}$$

Recalling that the separation principle holds for this problem [4], the optimum distortion (7) is thus given by:

$$D^{opt} = \frac{N}{P+N}\sigma_Q^2 \quad . \tag{21}$$

We show how to approach D^{opt} using the joint source/channel coding scheme depicted in Figure 2. In this scheme, the K-dimensional encoding and decoding functions (3),(5) are given by:

$$\mathbf{X} = [\beta\mathbf{S} + \mathbf{D} - \alpha\mathbf{I}] \bmod \Lambda \tag{22a}$$

$$\hat{\mathbf{S}} = \frac{\alpha_S}{\beta}\Big\{[\alpha_C\mathbf{Y} - \mathbf{D} - \beta\mathbf{J}] \bmod \Lambda\Big\} + \mathbf{J} \quad , \tag{22b}$$

respectively, where the second moment (9) of the lattice is $\sigma^2(\Lambda) = P$, and the dither vector $\mathbf{D}$ is uniformly distributed over $\mathcal{V}_0$ and independent of the source and of the channel. The channel power constraint is satisfied automatically by the properties of dithered lattice quantization discussed in Section II. The factors α_S, α_C and β will be chosen in the sequel. For optimum performance, β which is used at the encoder will depend upon the variance of the source unknown part, while α_C used at the decoder will depend upon the channel SNR. It is assumed, then, that both the encoder and the decoder have full knowledge of the source and channel statistics; we will break with this assumption in the next section.

The following theorem gives the performance of the scheme, in terms of the lattice parameters $L(\cdot,\cdot,\cdot)$ in (15) and in $\tilde{L}(\cdot)$ (14), and the quantities:

$$\alpha_0 \triangleq \frac{P}{P+N}, \tag{23a}$$

$$\tilde{\alpha} \triangleq \max\left(\alpha_0 - \frac{L(\Lambda, p_e, \alpha_0) - 1}{L(\Lambda, p_e, \alpha_0)}, 0\right) \quad . \tag{23b}$$

We will also use these quantities in the sequel to specify the choice of factors α_S, α_C and β.

Theorem 1: **(Performance of the MLM scheme with any lattice)** For any lattice Λ and any error probability $p_e > 0$, there exists a choice of factors $\alpha_C, \alpha_S, \beta$ such that the system of (22) (depicted in Figure 2) satisfies:

$$D \leq L(\Lambda, p_e, \alpha_0) D^{opt} + p_e D^{max} \quad ,$$

where the optimum distortion D^{opt} was defined in (21), and

$$D^{max} = 4\sigma_Q^2 \left(1 + \frac{\tilde{L}(\Lambda)}{\tilde{\alpha}}\right) \quad . \tag{24}$$

We prove this theorem in the sequel. As a direct corollary from it, taking p_e to be an arbitrarily small probability and using the properties of good lattices (17) and (18), we have the following asymptotic optimality result[3]

Theorem 2: **(Optimality of the MLM scheme)** Let $D(\Lambda_K)$ be the distortion achievable by the system of (22) with a lattice from a sequence $\{\Lambda_K\}$ that is simultaneously good for source and channel coding in the sense of Proposition 1. Then for any $\epsilon > 0$, there exists a choice of factors α_C, α_S and β, such that

$$\limsup_{K \to \infty} D\left(\Lambda_K\right) \leq D^{opt} + \epsilon \quad .$$

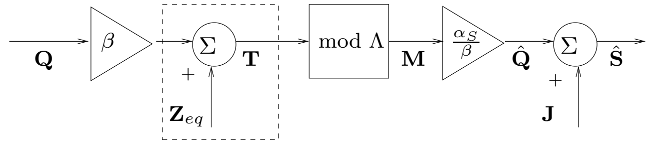
For proving Theorem 1 we start with a lemma, showing equivalence in probability to a real-additive noise channel (see Figure 3b). The equivalent additive noise is:

$$\mathbf{Z}_{eq} = \alpha_C \mathbf{Z} - (1 - \alpha_C)\mathbf{X} \quad , \tag{25}$$

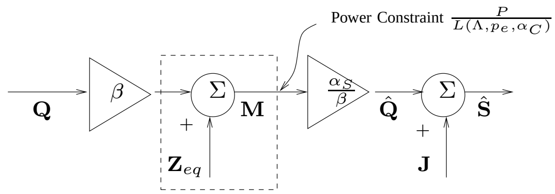
where $\mathbf{Z}$ and $\mathbf{X}$ are the physical channel input and AWGN, respectively. By the properties of the dithered modulo-lattice operation, the physical channel input $\mathbf{X}$ is uniformly distributed over $\mathcal{V}_0$ and independent of the source. Thus, $\mathbf{Z}_{eq}$ is indeed additive and has per-element variance:

$$\sigma_{eq}^2 = \alpha_C^2 N + (1 - \alpha_C)^2 P \quad . \tag{26}$$

[3]The explicit derivation of D^{max} is not necessary for proving Theorem 2; see Appendix II-B.



(a) Equivalent modulo-lattice channel.



(b) Equivalent real-additive noise channel w.p. $(1-p_e)$.

Figure 3: Equivalent channels for the WZ/WDP coding scheme

Lemma 1: **(Equivalent additive noise channel)** Fix some $p_e > 0$. In the system defined by (1),(2) and (22), the decoder modulo output $\mathbf{M}$ (see Figure 2) satisfies:

$$\mathbf{M} = \beta\mathbf{Q} + \mathbf{Z}_{eq} \;\; \text{w.p. } (1-p_e), \tag{27}$$

provided that

$$\beta^2\sigma_Q^2 + \sigma_{eq}^2 \leq \frac{P}{L(\Lambda, p_e, \alpha_C)} \quad , \tag{28}$$

where $\mathbf{Z}_{eq}$, defined in (25), is independent of $\mathbf{Q}$ and $\mathbf{J}$ and has per-element variance σ_{eq}^2 (26), and $L(\cdot,\cdot,\cdot)$ was defined in (15).

Consequently, as long as (28) holds, the whole system is equivalent with probability $(1-p_e)$ to the channel depicted in Figure 3b:

$$\begin{aligned}\hat{\mathbf{S}} &= \mathbf{J} + \frac{\alpha_S}{\beta}\mathbf{Z}_{eq} + \alpha_S\mathbf{Q} \\ &= \mathbf{S} + \frac{\alpha_S}{\beta}\mathbf{Z}_{eq} - (1-\alpha_S)\mathbf{Q} \quad .\end{aligned} \tag{29}$$

Proof: We will first prove equivalence to the channel of Figure 3a:

$$\mathbf{M} = [\beta\mathbf{Q} + \mathbf{Z}_{eq}] \bmod \Lambda \quad , \tag{30}$$

where $\mathbf{Z}_{eq}$ was defined in (25). To that end, let $\mathbf{T} = \alpha_C\mathbf{Y} - \mathbf{D} - \beta\mathbf{J}$ denote the input of the decoder modulo operation (see (22b) and Figure 2). Combine (2) and (22a) to assert:

$$\begin{aligned}\mathbf{T} &= \alpha_C(\mathbf{X}+\mathbf{Z}+\mathbf{I}) - \mathbf{D} - \beta\mathbf{J} \\ &= [\beta\mathbf{S}+\mathbf{D}-\alpha_C\mathbf{I}] \bmod \Lambda + \mathbf{Z}_{eq} + \alpha_C\mathbf{I} - \mathbf{D} - \beta\mathbf{J} \quad .\end{aligned}$$

Now, using (1) and the "distributive law" (10):

$$\mathbf{T} \bmod \Lambda = [\beta\mathbf{Q}+\mathbf{Z}_{eq}] \bmod \Lambda \quad ,$$

and since $\mathbf{T} = \mathbf{M} \bmod \Lambda$, we establish (30). Now we note that

$$\beta\mathbf{Q}+\mathbf{Z}_{eq} = \beta\mathbf{Q} + \alpha_C\mathbf{Z} - (1-\alpha_C)\mathbf{X} \triangleq \sqrt{1-(1-\alpha_C)^2}\mathbf{W} - (1-\alpha_C)\mathbf{X} \quad ,$$

where $\mathbf{W}$ is Gaussian i.i.d., $\mathbf{X}$ is uniform over the basic cell $\mathcal{V}_0$ of the lattice Λ, and the total variance (per element) is given by the l.h.s. of (28). By the definition of $L(\cdot,\cdot,\cdot)$, we have that

$$\mathbf{T} = \beta\mathbf{Q}+\mathbf{Z}_{eq} \in \mathcal{V}_0 \tag{31}$$

w.p. at least $(1-p_e)$. Substituting this in (30), we get (27). ■

This channel equivalence holds for any choice of dimension K, lattice Λ and factors α_C, α_S and β, as long as (28) holds. For the proof of Theorem 1 we make the following choice (using the parameters of (23)):

$$\alpha_C = \alpha_0 \quad , \tag{32a}$$

$$\beta^2 = \tilde{\alpha}\frac{P}{\sigma_Q^2} \quad , \tag{32b}$$

$$\alpha_S = \frac{\tilde{\alpha}P}{\tilde{\alpha}P+\alpha_0 N} \quad . \tag{32c}$$

It will become evident in the sequel, that α_C and α_S are the MMSE (Wiener) coefficients for estimating $\mathbf{X}$ from $\mathbf{X}+\mathbf{Z}$ and $\mathbf{Q}$ from $\mathbf{Q}+\frac{\mathbf{Z}_{eq}}{\beta}$, respectively, while β is the maximum zooming factor that allows to satisfy (28) with equality, whenever possible.

Proof of Theorem 1: For calculating the achievable distortion, first note that by the properties of MMSE estimation,

$$\sigma_{eq}^2 = \alpha_C N = \alpha_0 N \quad .$$

Using this, it can be verified that our choice of β satisfies (28), thus (29) holds with probability $(1-p_e)$. Denoting by $D^{correct}$ and $D^{incorrect}$ the distortions when (29) holds or does not hold, respectively, we have:

$$\begin{aligned} D &= (1-p_e)D^{correct} + p_e D^{incorrect} \\ &\leq D^{correct} + p_e D^{incorrect} \quad . \end{aligned} \tag{33}$$

We shall now bound both conditional distortions. For the first one, we have:

$$\begin{aligned} D^{correct} &= \frac{1}{K} E\left\{ \left\| \frac{\alpha_S}{\beta_K} \mathbf{Z}_{eq} - (1-\alpha_S)\mathbf{Q} \right\|^2 \right\} \\ &\overset{(a)}{=} \alpha_S \frac{\sigma_{eq}^2}{\beta^2} = \frac{\sigma_Q^2 \sigma_{eq}^2}{\beta^2 \sigma_Q^2 + \sigma_{eq}^2} \\ &= \frac{D^{opt}}{1-\alpha_0+\tilde{\alpha}} \\ &= \min\left(L(\Lambda, p_e, \alpha_C) D^{opt}, \sigma_Q^2 \right) \leq L(\Lambda, p_e, \alpha_C) D^{opt} \quad , \end{aligned}$$

where (a) stems from the properties of MMSE estimation. It remains to show that $D^{incorrect} \leq D^{max}$, which is established in Appendix II-A. □

As mentioned in the Introduction, a recent work [15] derives a similar asymptotic result, replacing the shaping lattice of our scheme by a *random* shaping code. Such a choice is less restrictive since it is not tied to the properties of good Euclidean lattices, though it leads to higher complexity due to the lack of structure. The use of lattices also allows analysis in finite dimension as in Theorem 1 and in Section V. Furthermore, structure is essential in network joint source/channel settings; see e.g. [16]. Lastly, the dithered lattice formulation allows to treat any interference signals, see Remark 2 in the sequel.

We conclude this section by the following remarks, intended to shed more light on the significance of the results above.

1. **Optimal decoding.** The decoder we described is *not* the MMSE estimator of $\mathbf{S}$ from $\mathbf{Y}$. This is for two reasons: First, the decoder ignores the probability of incorrect lattice decoding. Second, since $\mathbf{Z}_{eq}$ is not Gaussian, the modulo-lattice operation w.r.t. the lattice Voronoi cells is not equivalent to maximum-likelihood estimation of the lattice point (see [20] for a similar discussion in the context of channel coding). Consequently, for any finite dimension the decoder can be improved. We shall discuss further the issue of working with finite-dimension lattices in Section V.

2. **Universality w.r.t. $\mathbf{I}$ and $\mathbf{J}$.** None of the scheme parameters depend upon the nature of the channel interference $\mathbf{I}$ and source known part $\mathbf{J}$. Consequently, the scheme is adequate for arbitrary (individual) sequences. This has no effect on the asymptotic performance of Theorem 2, but for finite-dimensional lattices the scheme may be improved, e.g. if the interference signals are known to be Gaussian with low

enough variance. A similar argument also holds when the source or channel statistics is not perfectly known, see Section IV in the sequel.

3. **Non-Gaussian Setting.** If the source unknown part $\mathbf{Q}$ or the channel noise $\mathbf{Z}$ are not Gaussian, the optimum quadratic-Gaussian distortion D^{opt} may still be approached using the MLM scheme, though it is no longer the optimum performance for the given source and channel.

4. **Asymptotic choice of parameters.** In the limiting case where $L(\Lambda, p_e, \alpha_0) \to 1$, we have that $\alpha_S = \tilde{\alpha} = \alpha_0$ in (32), i.e. the choice of parameters approaches:

$$\alpha_C = \alpha_S = \frac{P}{P+N} = \alpha_0 \quad , \tag{34a}$$

$$\beta^2 = \alpha_0 \frac{P}{\sigma_Q^2} \quad . \tag{34b}$$

5. **Properties of the equivalent additive-noise channel.** With high probability, we have the equivalent real-additive noise channel of (29) and Figure 3b. This differs from the modulo-additivity of the lattice strategies of [20], [21]: Closeness of point under a modulo arithmetic does not mean closeness under a difference distortion measure. The condition (28) forms an output-power constraint: No matter what the noise level of the channel is, its output must have a power of no more than P; this replaces the input-power constraint of the physical channel. Furthermore, by the lattice quantization noise properties [18], the "self noise" component $(1-\alpha_C)\mathbf{X}$ in (25) is asymptotically Gaussian i.i.d., and consequently so is the equivalent noise $\mathbf{Z}_{eq}$. Thus the additive equivalent channel (29) is asymptotically an *output-power constrained AWGN channel*.

6. **Noise margin.** The additivity in (29) is achieved through leaving a "noise margin". The condition (28) means that the sum of the (scaled) unknown source part and equivalent noise should "fit into" the lattice cell (see (31)). Consequently, the unknown source part $\mathbf{Q}$ is inflated to a power strictly smaller than the lattice power P. In the limit of infinite dimension, when the choice of parameters becomes (34), this power becomes $\beta^2\sigma_Q^2 = \alpha_0 P$. In comparison, it is shown in [21] that in a lattice solution to a digital SI problem, if the information-bearing code (fine lattice) occupies a portion of power γP with any $\alpha_0 \leq \gamma \leq 1$, capacity is achieved[4]. This freedom, however, has to do with the modulo-additivity of the equivalent channel; in our joint source/channel setting, necessarily $\gamma = \alpha_0$.

7. **Comparison with analog transmission.** Lastly, consider the similarity between our asymptotic AWGN channel and the optimal analog transmission scheme without SI (8): Since we have "eliminated

[4]In [22] a similar observation is made, and a code of power $\alpha_0 P$ is presented as a preferred choice, since it allows easy iterative decoding between the information-bearing code and the coarse lattice.

from the picture" the SI components $\mathbf{I}$ and $\mathbf{J}$, we are left with the transmission of the source unknown component through an equivalent additive noise channel. As mentioned above, the unknown source part $\mathbf{Q}$ is only adjusted to power $\alpha_0 P$ (in the limit of high dimension), while in (8) the source $\mathbf{S}$ is adjusted to power P; but since the equivalent noise $\mathbf{Z}_{eq}$ has variance $\alpha_0 N$, the equivalent channel has signal-to-noise ratio of P/N, just as the physical channel.

IV. Transmission under Uncertainty Conditions

We now turn to case where either the variance of the channel noise N, or the variance of the source unknown part σ_Q^2, are unknown at the encoder[5]. In Section IV-A we assume that σ_Q^2 is known at both sides, but the channel SNR is unknown at the encoder. We show that in the limit of high SNR, optimality can still be approached. In Section IV-B, we address the general SNR case, as well as the case of unknown σ_Q^2; for that, we adopt an alternative broadcast-channel point of view.

For convenience, we present our results in terms of the channel signal-to-noise ratio

$$\text{SNR} \triangleq \frac{P}{N} \tag{35}$$

and the achieved signal-to-distortion ratio

$$\text{SDR} \triangleq \frac{\sigma_Q^2}{D} \quad . \tag{36}$$

Denoting the theoretically optimal SDR as SDR^{opt}, (21) becomes:

$$\text{SDR}^{opt} = 1 + \text{SNR} \quad . \tag{37}$$

Our achievability results in this section are based upon application of the MLM scheme, generally with a sub-optimal choice of parameters due to the uncertainty. We only bring asymptotic results, using high-dimensional "good" lattices. We present, then, the following lemma, using the definition:

$$\beta_0^2 = \frac{P}{\sigma_Q^2} \quad . \tag{38}$$

Lemma 2: Let $\text{SDR}(\Lambda_K)$ be the distortion achievable by the system of (22) with a lattice from a sequence $\{\Lambda_K\}$ that is good in the sense of Proposition 1. For any choice of factors α_C, α_S and β,

$$\liminf_{K\to\infty} \text{SDR}\left(\Lambda_K\right) \geq \frac{\beta^2}{(1-\alpha_S)^2\beta^2 + \alpha_S^2\left[\frac{\alpha_C^2}{\text{SNR}} + (1-\alpha_C)^2\right]\beta_0^2} \quad , \tag{39}$$

[5]We do not treat uncertainty at the decoder, since N can be learnt, while the major insight into the matter of unknown σ_Q^2 is gained already by assuming uncertainty at the encoder.

provided that

$$\frac{\beta^2}{\beta_0^2} + \frac{\alpha_C^2}{\text{SNR}} + (1-\alpha_C)^2 < 1 \quad . \tag{40}$$

Proof: This is a direct application of Lemma 1 and of (18). First we fix some $p_e > 0$, and note that (40) is equivalent to (28). The SDR of the equivalent channel (29), at the limit $L(\Lambda_K, p_e, \alpha_C) \to 1$ is then given by (39). Then for $p_e \to 0$ the effect of decoding errors vanishes, as shown in Appendix II-B ∎

Note, that by substituting the asymptotically optimal choice of parameters (34) in (39), the limit becomes SDR^{opt}.

A. Asymptotic Robustness for Unknown SNR

Imagine that we know that SNR $\geq$ SNR_0, for some specific SNR_0, and that σ_Q^2 is known. Suppose that we set the scheme parameters such that the correct decoding condition (40) holds for SNR_0. Since the variance of the equivalent noise can only decrease with the SNR, correct lattice decoding will hold for any SNR $\geq$ SNR_0, and we are left with the equivalent additive-noise channel where the resulting SDR is a strictly decreasing function of the SNR. We use this observation to derive an asymptotic result, showing that for high SNR a *single* encoder can approach optimality simultaneously for all actual SNR. To that end, we replace the choice given in (32), which leads to optimality at one SNR, by the high-SNR choice $\alpha_C = \alpha_S = 1$, where β is chosen to ensure correct decoding even at the minimal SNR_0.

Theorem 3: **(Robustness at high SNR)** Let the source and channel be given by (1) and (2), respectively. Then for any $\epsilon > 0$, there exists an <u>SNR-independent</u> sequence of encoding-decoding schemes (each one achieving SDR_K) that satisfies:

$$\liminf_{K\to\infty} \text{SDR}_K \geq (1-\epsilon)\text{SDR}^{opt} \quad , \tag{41}$$

for all sufficiently large (but finite) SNR. I.e., (41) holds for all SNR $\geq$ $\text{SNR}_0(\epsilon)$, where $\text{SNR}_0(\epsilon)$ is finite for all $\epsilon > 0$.

A limit of a *sequence* of schemes is needed in the theorem, rather than a single scheme, since for any single scheme we have $p_e > 0$, thus the effect of incorrect decoding cannot be neglected in the limit SNR $\to \infty$ (meaning that the convergence in Lemma 2 in not uniform). If we restricted our attention to SNRs bounded by some arbitrarily high value, a single scheme would be sufficient.

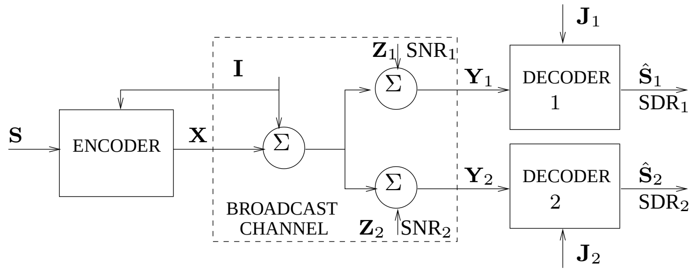


Figure 4: A broadcast presentation of the uncertainty problem.

Proof: We use a sequence of MLM schemes with good lattices in the sense of Proposition 1. If $\alpha_C = 1$, then any

$$\beta^2 < \frac{\text{SNR}_0 - 1}{\text{SNR}_0} \cdot \beta_0^2$$

satisfies the condition (40) for SNR_0, thus for any $\text{SNR} \geq \text{SNR}_0$. Here we assume that $\text{SNR}_0 > 1$, w.l.o.g. since we can always choose $\text{SNR}_0(\epsilon)$ of the theorem accordingly. With this choice and with $\alpha_S = 1$, we have by Lemma 2 that the SDR may approach (for any $\text{SNR} \geq \text{SNR}_0$):

$$\frac{\beta^2}{\beta_0^2}\text{SNR} = \frac{\text{SNR}_0 - 1}{\text{SNR}_0} \cdot \text{SNR} = \frac{\text{SNR}_0 - 1}{\text{SNR}_0} \cdot \frac{\text{SNR}}{\text{SNR}+1} \cdot \text{SDR}^{opt} \geq \frac{\text{SNR}_0 - 1}{\text{SNR}_0 + 1} \cdot \text{SDR}^{opt} \quad .$$

Now take $\epsilon = \frac{\text{SNR}_0 - 1}{\text{SNR}_0 + 1} - 1$. Since $\lim_{\text{SNR}_0 \to \infty} \epsilon = 0$, one may find SNR_0 for any $\epsilon > 0$ as required. ∎

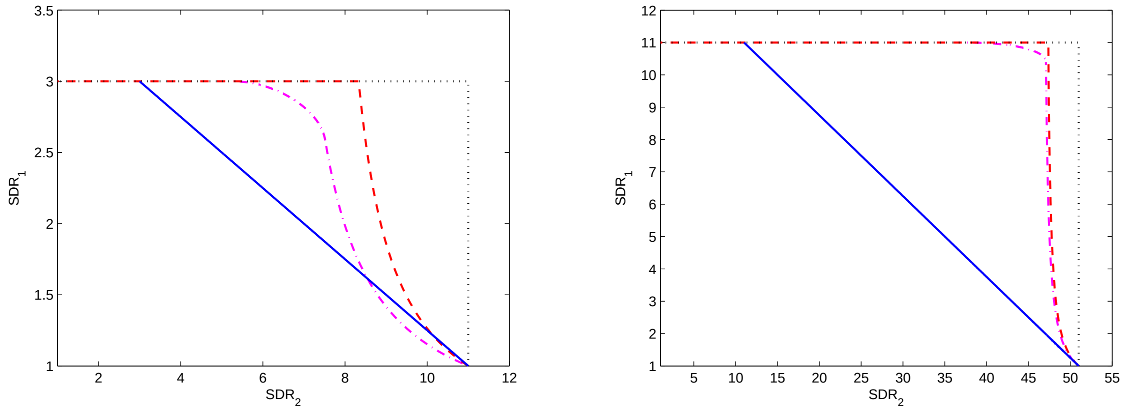
Note that we have here also a fixed decoder; if we are only interested in a fixed encoder we can adjust α_S at the decoder and reduce the margin from optimality.

B. Joint Source/Channel Broadcasting

Abandoning the high SNR assumption, we can no longer simultaneously approach the optimal performance (37) for multiple SNRs. However, in many cases we can still do better than a separation-based scheme. In order to demonstrate that, we choose to alternate our view to a *broadcast* scenario, where the same source needs to be transmitted to multiple decoders, each one with different conditions; yet all the decoders share the same channel interference $\mathbf{I}$, see Figure 4. The variation of the source SI component $\mathbf{J}$ between decoders means that the source has two decompositions:

$$\mathbf{S} = \mathbf{Q}_1 + \mathbf{J}_1 = \mathbf{Q}_2 + \mathbf{J}_2 \quad , \tag{42}$$

and we define the per-element variances of the unknown parts as σ_1^2 and σ_2^2, respectively. Note that this variation does not imply any uncertainty from the point of view of the MLM encoder, as long as



(a) $\mathrm{SNR}_1 = 2$, $\mathrm{SNR}_2 = 10$ (b) $\mathrm{SNR}_1 = 10$, $\mathrm{SNR}_2 = 50$

Figure 5: Broadcast performance. Solid line: Achievable by separation for arbitrary $\mathbf{I}$ and $\mathbf{J}$. Dash-dotted line: Achievable by MLM for arbitrary $\mathbf{I}$ and $\mathbf{J}$. Dashed line: Achievable by MLM for arbitrary $\mathbf{J}$, with $\mathbf{I} = \mathbf{0}$. Dotted line: Outer bound of ideal matching to both SNRs (achievable by analog transmission when $\mathbf{I} = \mathbf{J} = \mathbf{0}$).

$\sigma_1^2 = \sigma_2^2$; see [23] for a similar observation in the context of source coding. We denote the signal-to-noise ratios at the decoders as $\mathrm{SNR}_1 \leq \mathrm{SNR}_2$, and find achievable corresponding signal-to-distortion ratio $\{\mathrm{SDR}_1, \mathrm{SDR}_2\}$ pairs. It will become evident from the exposition, that this approach is also good for a continuum of possible SNRs.

We start from the case $\sigma_1^2 = \sigma_2^2$, for which we have the following.

Theorem 4: In the broadcast WZ/DPC channel of Figure 4 with $\sigma_1^2 = \sigma_2^2$, the signal-to-distortions pair

$$\left\{1 + \frac{\overline{\alpha} \cdot \mathrm{SNR}_1}{\alpha_C^2 + (1-\alpha_C)^2 \mathrm{SNR}_1}, 1 + \frac{\overline{\alpha} \cdot \mathrm{SNR}_2}{\alpha_C^2 + (1-\alpha_C)^2 \mathrm{SNR}_2}\right\} \quad ,$$

where

$$\overline{\alpha} = \alpha_C \left(2 - \frac{\mathrm{SNR}_1 + 1}{\mathrm{SNR}_1} \alpha_C\right) \quad , \tag{43}$$

can be approached for any $0 < \alpha_C \leq \min\left(1, \frac{2 \cdot \mathrm{SNR}_1}{1+\mathrm{SNR}_1}\right)$. In addition, if there is no channel interference ($\mathbf{I} = \mathbf{0}$), then the pair $\left\{1 + \mathrm{SNR}_1, 1 + \frac{\mathrm{SNR}_1(1+\mathrm{SNR}_2)}{1+\mathrm{SNR}_1}\right\}$ can be approached as well.

Proof: As in the proof of Theorem 3, we use Lemma 2 with a choice of β which allows correct decoding in the lower SNR. For the first part of the theorem, fix any α_C according to the theorem conditions, and choose any

$$\beta^2 < \overline{\alpha}\frac{P}{\sigma_Q^2} \quad ,$$

where $\overline{\alpha}$ was defined in (43), in order to satisfy (40). In each decoder, optimize α_S in (39) to approach the desired distortion. For the second part of the theorem, if there is no channel interference, the encoder is α_C-independent, thus each decoder may work with a different α_C value. We can therefore make the encoder and the first decoder optimal for SNR_1, while the second decoder only suffers from the choice of β at the encoder. Again we substitute in (39) to arrive at the desired result ■

By standard time-sharing arguments, the achievable SDR regions include the convex hull (in the distortions plane) defined by these points and the trivial $\{1+\text{SNR}_1, 1\}$ and $\{1, 1+\text{SNR}_2\}$ points. Figure 5 demonstrates these regions, compared to the ideal (unachievable) region of simultaneous optimality for both SNRs, and the separation-based region achieved by the concatenation of successive-refinement source code (see e.g. [24]) with broadcast channel code [25] (about the sub-optimality of this combination without SI, see e.g. [26]). It is evident, that in most cases the use of the MLM scheme significantly improves the SDR tradeoff over the performance offered by the separation principle, and that the scheme approaches simultaneous optimality where both SNRs are high, as promised by Theorem 3. Note that, unlike the separation-based approach, the MLM approach also offers reasonable SDRs for intermediate SNRs. Moreover, note that this region is achievable when no assumption is made about the statistics of $\mathbf{I}$ and $\mathbf{J}$. If these interferences are not very strong comparing to P and σ_Q^2, respectively, then one may further extend the achievable region by allowing some residual interference.

To conclude, we briefly discuss the case where $\sigma_1^2 \neq \sigma_2^2$. We define the SDR of each decoder relative to its own variance, and ask what are the achievable SDRs for a pair of SNRs, which may be equal or different. Assume here the simple case, where there is no channel interference, i.e. $\mathbf{I} = \mathbf{0}$. In this case, the encoder only needs to agree upon β with the decoders, thus (by Lemma 2) we may approach for $n = 1, 2$:

$$\text{SDR}_n = 1 + \frac{\beta^2}{\beta_{opt,n}^2}\text{SNR}_n \quad , \tag{44}$$

where $\beta_{opt,n}$ is the optimum choice of β for SNR_n according to (34). It follows, that if the two decoders require the same value of β, they may be both approach the theoretically optimal distortion. This translates to the optimality condition:

$$\sigma_1^2\frac{1+\text{SNR}_1}{\text{SNR}_1} = \sigma_2^2\frac{1+\text{SNR}_2}{\text{SNR}_2} \quad .$$

This scenario was presented in [27], where simultaneous optimality using hybrid digital/analog schemes was proven under a different condition:

$$\frac{\sigma_1^2}{\mathrm{SNR}_1} = \frac{\sigma_2^2}{\mathrm{SNR}_2} \quad .$$

Both conditions reflect the fact that better source conditions (lower σ_Q^2) can compensate for worse channel conditions (lower SNR). It follows from the difference between the conditions, that for some parameter values the MLM scheme outperforms the approach of [27], thus extending the achievable SDRs region.

V. Discussion: Delay and Complexity

We have presented the joint source/channel MLM scheme, proven its optimality for joint WZ/DPC setting with known SNR and shown its improved robustness over a separation-based scheme. We now discuss the potential complexity and delay advantages of our approach relative to separation-based schemes, first considering the complexity at high dimension and then suggesting a scalar variant.

Consider a separation-based solution, with source and channel encoder/decoder pairs. An optimal channel coding scheme typically consists of two codes: an information-bearing code and a shaping code, both of which require a nearest-neighbor search at the decoder. An optimal source coding scheme also consists of both a quantization code and a shaping code in order to achieve the full vector quantization gain (see e.g. [28]), thus two nearest-neighbor searches are needed at the encoder. The MLM approach omits the information-bearing channel code and the quantization code, and merges the channel and source shaping codes into one. It is convenient to compare this approach with the nested lattices approach to channel and source coding with SI [10], since in that approach both the channel and source information bearing/shaping code pairs are materialized by nested lattices. In comparison, our scheme require only a single lattice (parallel to the coarse lattice of nested schemes), and in addition the source and channel lattices collapse into a single one.

There is a price to pay, however: For the WZ problem, the coarse lattice should be good for channel coding, while for the WDP problem the coarse lattice should be good for source coding [10]. The lattice used for MLM needs to be simultaneously good for source *and* channel coding (see Appendix I). While the existence of such lattices in the high dimension limit is assured by [19], in finite dimension the lattice that is best in one sense is not necessarily best in the other sense [29], resulting in a larger implementation loss. Quantitively, whereas for source coding the lattice should have a low normalized second moment, and for channel coding it should have a low volume-to-noise ratio, for joint source

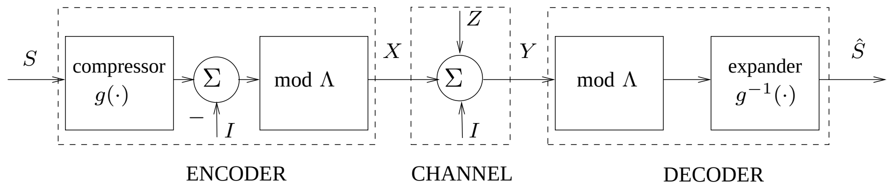


Figure 6: Scalar MLM/companding scheme for joint source/channel coding over a high-SNR dirty-paper channel: S = source, $\hat{S}$ = reconstruction, Z = channel noise, I = interference known at the encoder, $g(\cdot)$ = companding function.

channel coding the *product* $L(\Lambda, p_e)$ (12) should be low[6] (see Appendix I). The study of such lattices is currently under research. Exact comparison of schemes in high dimension will involve studying the achieved *joint source/channel excess distortion exponent* (see [30] for a recent work about this exponent in the Gaussian setting).

From the practical point of view, the question of a low-dimensional scheme is very important, since it implies both low complexity and low delay. One may ask then, what can be achieved using low-dimensional lattices, e.g. a scalar lattice? The difficulty, however, is that in low dimensions a low probability of incorrect decoding p_e implies a high loss factor $L(\Lambda, p_e)$, thus the distortion promised by Theorem 1 grows. Some improvement may be achieved by using an optimal decoder rather than the one described in this work (see Remark 1 at the end of Section III), an issue which is left for further research. A recent work [31] suggests an alternative, for the case of channel interference only ($\mathbf{J} = \mathbf{0}$), by also changing the encoder: The scalar zooming factor β of the MLM scheme is replaced by non-linear companding of the signal; see Figure 6. At high SNR, the distortion loss of such a scalar MLM scheme with optimal companding comparing to (7) is shown to be

$$\frac{D^{companding}}{D^{opt}} = \frac{\sqrt{3}\pi}{2} \cong 4.3dB \quad .$$

In comparison, the loss of a separation-based scalar scheme, consisting of a scalar quantizer and a scalar (uncoded) channel constellation, is *unbounded* in the limit SNR $\to \infty$. This is since in a separation-based scheme the mapping of quantized source values to channel inputs is arbitrary; consequently, keeping the

[6] In Theorem 1 we show that the figure of merit is $L(\Lambda, p_e, \alpha)$ (15), but for reasonably high SNR it seems that the effect of self noise should not be too dominant, so we can set $\alpha = 1$.

loss bounded implies that the error probability must go to zero in the high-SNR limit, and the gap of a scalar constellation from capacity grows.

ACKNOWLEDGEMENT

We thank Uri Erez for helping to make some of the connections which led to this work.

APPENDIX I
MEASURES OF GOODNESS OF LATTICES

In this appendix we discuss measures of goodness of lattices for source and channel coding, and their connection with the loss factor relevant to our joint source/channel scheme.

When a lattice is used as a quantization codebook in the quadratic Gaussian setting, the figure of merit is the lattice *normalized second moment*:

$$G(\Lambda) \triangleq \frac{\sigma^2(\Lambda)}{V(\Lambda)^{\frac{2}{K}}} \quad , \tag{45}$$

where the cell volume is $V(\Lambda) = \int_{\mathcal{V}_0} d\mathbf{x}$. By the isoperimetric inequality, $G(\Lambda) \geq G^*_K$, where G^*_K is the normalized second moment of a ball with the same dimension K as the lattice. This quantity satisfies $G^*_K \geq \frac{1}{2\pi e}$, with asymptotic equality in the limit of large dimension. A sequence of K-dimensional lattices is said to be *good for MSE quantization* if

$$\lim_{K\to\infty} G(\Lambda_K) = \frac{1}{2\pi e} \quad , \tag{46}$$

thus it asymptotically achieves the minimum possible lattice second moment for a given volume.

When a lattice is used as an AWGN channel codebook, the figure of merit is the lattice *volume-to-noise ratio* at a given error probability $1 > p_e > 0$ (see e.g. [32], [20]):

$$\mu(\Lambda, p_e) \triangleq \frac{V(\Lambda)^{\frac{2}{K}}}{\sigma_Z^2} \quad , \tag{47}$$

where σ_Z^2 is the maximum variance (per element) of a white Gaussian vector $\mathbf{Z}$ having an error probability

$$\Pr\{\mathbf{Z} \notin \mathcal{V}_0\} \leq p_e \quad .$$

For any lattice, $\mu(\Lambda, p_e) \geq \mu^*_K(p_e)$, where $\mu^*_K(p_e)$ is the volume-to-noise ratio of a ball with the same dimension K as the lattice. For any $1 > p_e > 0$, $\mu^*_K(p_e) \geq 2\pi e$, with asymptotic equality in the limit of large dimension. A sequence of K-dimensional lattices is *good for AWGN channel coding* if

$$\lim_{p_e\to 0} \lim_{K\to\infty} \mu(\Lambda_K, p_e) = 2\pi e \quad , \tag{48}$$

thus it possesses the property of having a minimum possible cell volume such that the probability of an i.i.d. Gaussian vector of a given power to fall outside the cell vanishes.

Combining the definitions (45) and (47), we see that the loss factor $L(\Lambda, p_e)$ (12) satisfies:

$$L(\Lambda, p_e) = G(\Lambda) \cdot \mu(\Lambda, p_e) \quad .$$

Furthermore, the existence of a good sequence of lattices in the sense of (13) is assured by the existence of a sequence that simultaneously satisfies (46) *and* (48), which was shown in [19, Theorem 5].

Proposition 1 is implicit in the proof of [20, Theorem 5]. It is based upon the existence of lattices that are simultaneously good for AWGN channel coding and for covering [19], where goodness for covering also implies goodness for MSE quantization; for such lattices, it is shown that the mixture noise cannot be much worse than a Gaussian noise of the same variance. Later, it was shown in [33] that, for such lattices, for small enough error probability p_e, the introduction of self noise actually reduces the loss factor, i.e. $L(\Lambda, p_e, \alpha) \leq L(\Lambda, p_e, 1)$.

APPENDIX II

THE EFFECT OF DECODING FAILURE ON THE DISTORTION

With probability p_e, correct lattice decoding fails, i.e. (31) does not hold. These events contribute to the total distortion a portion of

$$\tilde{D} \triangleq p_e \cdot D^{incorrect} \quad , \tag{49}$$

where $D^{incorrect}$ is the distortion given a decoding failure, as in the proof of Theorem 1. In this Appendix we quantify this effect: In the first part we show that D^{max} of (24) is a (rather loose) bound on $D^{incorrect}$, thus completing the proof of Theorem 1. In the second part, we show directly that $\tilde{D}$ must vanish in the limit of small p_e, without resorting to an explicit bound on $D^{incorrect}$.

In both parts we use the observation that

$$\hat{\mathbf{S}} - \mathbf{S} \; = \hat{\mathbf{Q}} - \mathbf{Q} \quad , \tag{50}$$

where $\hat{\mathbf{Q}} \triangleq \frac{\alpha_S}{\beta}[\beta\mathbf{Q} + \mathbf{Z}_{eq}] \bmod \Lambda$, see also Figure 3b. We note that although $\mathbf{Q}$ is unbounded, we always have that

$$\hat{\mathbf{Q}} \in \frac{\alpha_S}{\beta}\mathcal{V}_0 \quad . \tag{51}$$

A. A Bound on the Conditional Distortion for Any Lattice

In order to complete the proof of Theorem 1, we now bound $D^{incorrect}$ of (33).

$$
\begin{aligned}
D^{incorrect} &= \frac{1}{K} E\{\|\hat{\mathbf{S}} - \mathbf{S}\|^2 \ |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} \\
&= \frac{1}{K} E\{\|\hat{\mathbf{Q}} - \mathbf{Q}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} \\
&\leq \frac{2}{K}\Big(E\{\|\hat{\mathbf{Q}}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} + E\{\|\mathbf{Q}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\}\Big) \quad , \qquad (52)
\end{aligned}
$$

where the inequality follows from assuming maximizing (-1) correlation coefficient and then applying the Cauchy-Schwartz inequality. We shall now bound these two terms. For the first one, recalling the definition of the covering radius (11), we bound the conditional expectation by the maximum possible value:

$$
E\{\|\hat{\mathbf{Q}}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} \leq \max(\|\hat{\mathbf{Q}}\|^2) = \frac{\alpha_S^2 \cdot r^2(\Lambda)}{\beta^2} \leq \frac{r^2(\Lambda)}{\beta^2} \quad . \qquad (53)
$$

For the second term, we have:

$$
\begin{aligned}
E\{\|\mathbf{Q}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} &\leq E\{\|\mathbf{Q}\|^2 |\beta\mathbf{Q} \notin \mathcal{V}_0\} \\
&\leq E\{\|\mathbf{Q}\|^2 |\beta\mathbf{Q} \notin \mathcal{B}_0\} \quad ,
\end{aligned}
$$

where $\mathcal{B}_0$ is the circumsphere of $\mathcal{V}_0$, of radius $r(\Lambda)$. It follows that

$$
E\{\|\mathbf{Q}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} \leq \sigma_Q^2 E\{V | V > v_0\} \quad ,
$$

where $V \sim \mathcal{X}_K^2$ and $v_0 \triangleq \frac{r^2(\Lambda)}{\beta^2 \sigma_Q^2}$. This conditional expectation is given by:

$$
E\{V | V > v_0\} = \frac{\mathcal{Q}(\frac{K}{2} + 1, v_0 \frac{K}{2})}{\mathcal{Q}(\frac{K}{2}, v_0 \frac{K}{2})} \leq v_0 + 2 \quad ,
$$

where $\mathcal{Q}(\cdot, \cdot)$ is the regularized incomplete Gamma function, and the inequality can be shown by means of calculus. This gives the bound on the second term:

$$
E\{\|\mathbf{Q}\|^2 |\beta\mathbf{Q} + \mathbf{Z}_{eq} \notin \mathcal{V}_0\} \leq \left(\frac{r^2(\Lambda)}{\beta^2} + 2K\sigma_Q^2\right) \quad .
$$

Substituting this and (53) in (52), we have that:

$$
D^{incorrect} \leq 4\left(\frac{r^2(\Lambda)}{K\beta^2} + \sigma_Q^2\right) \quad .
$$

Recalling the choice of β in (32b) and the definition of $\tilde{L}(\cdot, \cdot)$ in (14), the bound follows.

B. Asymptotic Effect of Decoding Failures

In this part we follow the claims used by Wyner in the source coding context to establish [5, (5.2)], to see that $\lim_{p_e \to 0} \tilde{D} = 0$, where $\tilde{D}$ was defined in (49), without using the explicit bound derived in Appendix II-A. This serves as a simpler proof of Theorem 2; moreover, it also applies to a non-optimal choice of parameters, thus it serves in the analysis of performance under uncertainty conditions.

Denoting the decoding failure event by ε and its indicator by I_ε, and recalling (50), we re-write the contribution to the distortion as:

$$\tilde{D} = E\{I_\varepsilon \cdot (\hat{\mathbf{Q}} - \mathbf{Q})^2\} \quad .$$

For any value of the source unknown part $\mathbf{Q}$, the distortion is bounded by:

$$d(\mathbf{Q}) \stackrel{\Delta}{=} \sup_{\hat{\mathbf{Q}}} (\hat{\mathbf{Q}} - \mathbf{Q})^2 \quad .$$

The expectation $E\{d(\mathbf{Q})\}$ is finite, since $\mathbf{Q}$ is Gaussian and $\hat{\mathbf{Q}}$ is bounded (see (51)). We now have that

$$\tilde{D} \leq E\{I_\varepsilon \cdot d(\mathbf{Q})\} \quad .$$

Using a simple lemma of Probability Theory [5, Lemma 5.1], since $E\{d(\mathbf{Q})\}$ is finite, this expectation approaches zero as $p(\varepsilon) = p_e \to 0$.

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