

The Joint Gaussian CEO/MAC Problem and Relaying*

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Abstract—We consider the problem of agents which observe a noisy version of a source, and communicate with a central decoder (“CEO”) through a multiple access channel (MAC). We assume that the source, the observation noises and the channel noise are all Gaussian, and we are interested in reconstruction under a mean squared error distortion measure. The solution by separation (combination of CEO and MAC coding) is sub-optimal. When the source and channel bandwidth (BW) are equal, analog transmission is optimal. We present a scheme for non-equal BW which becomes asymptotically optimal in the limit of low observation noise. We use this scheme to improve our recently presented Rematch and Forward approach for the parallel relay network, and thus extend the achievable rate.

I. INTRODUCTION

The emerging field of communication over networks witnesses the collapse of the traditional distinction between channel, source and joint source/channel problems. Specifically, consider relay-type problems, in which a message source node wishes to pass information to a destination node, while other nodes act as relays, whose sole purpose is to help in this data transfer. Though this is a channel problem, the techniques used to solve it are diverse. Consider the best known relaying techniques (see e.g. [3]), where each one is known to be optimal under different conditions of network topology and signal-to-noise ratios:

1. A channel coding approach: *Decode and Forward* (D&F), where a relay decodes the message, and then re-encodes it.
2. A source and channel coding approach: *Compress and Forward* (C&F), where a relay treats its input as a source, compresses it, and then uses a channel code to forward it.
3. A joint source/channel coding (JSCC) approach: *Amplify and Forward* (A&F), where a relay simply forwards its input, only applying power adjustment.

The last is indeed a JSCC approach, since it does not opt to decode the input, thus it treats it as a source, and then the analog treatment of this source, reminiscent of analog transmission in Gaussian point-to-point communications [7], relies upon matching between the statistics of that “source” and of the channel which initiates at the relay.

As a simple test-case, consider the Gaussian parallel relay network, first introduced by Schein and Gallager [18]. In this network, all the relays are ordered in a parallel manner; The source is connected to the relays by a Gaussian broadcast channel (BC), while the relays are connected to the destination

by a Gaussian multiple access channel (MAC). In the original setting, all noises are white and the channels all have the same bandwidth (BW). In the limit of high signal to noise ratio (SNR) in the MAC section, as well as in the limit of many relays [6], the A&F approach is optimal. A recent work [8] extended the view to networks where the noises are colored, and specifically to the important case of BW mismatch between the BC and MAC sections, by introducing a new relaying strategy, a JSCC approach named *Rematch and Forward* (R&F): The encoder uses a codebook of the MAC section BW; Between the encoder and the relays, a JSCC scheme suitable for BW mismatch translates the BC into an equivalent BC with the MAC BW and a “mutual-information-preserving” SNR.

In this paper we further enhance the R&F approach by considering the dual strategy, where the encoder uses a codebook of the BC section BW, thus the task of BW conversion shifts to the MAC section.

In the quadratic-Gaussian CEO problem [19], agents observe a Gaussian source contaminated by Gaussian noise, independent between agents. These agents communicate with a central decoder through rate-limited channels, and the goal is to enable that decoder to estimate the source with minimum mean squared error (MMSE). As noted by Gastpar and Vetterli [6], the joint source/channel problem of the CEO and MAC is connected with the parallel relay network. Namely, this approach treats the transmitted codeword as a source, the relays as agents which receive noisy versions of that source, and the final decoder as a central processor which needs to achieve a good estimation of the source in the MMSE sense (in order to decode the codeword). Remaining in the equal-BW case, [6] shows how the superiority of analog transmission over separate CEO and MAC coding leads to asymptotic optimality of the A&F strategy. In [12], Nazer and Gastpar consider this JSCC problem with unequal BW, and consider a scheme which outperforms any separation-based scheme. Their scheme builds upon the Modulo-Lattice Modulation (MLM) approach [10], using two of its features: The analog nature of the signal which is transmitted modulo-lattice, and the structure of the lattice. In this paper we use a similar approach but show that in many interesting cases, an even better performance may be obtained, approaching the cooperation outer bound for the problem.

The rest of this paper is organized as follows: In Section II we address the joint CEO/MAC problem in the equal BW case,

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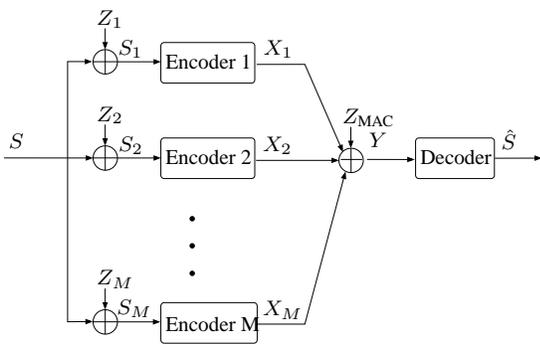


Figure 1: The joint CEO/MAC problem.

and discuss the reasons for the sub-optimality of separation. In Section III we present our scheme for BW mismatch. Finally in Section IV we apply these results to the parallel relay network, to arrive at new achievable rates.

II. CEO AND MAC: SEPARATION VS. ANALOG TRANSMISSION

The CEO and MAC problems are both defined with M encoders and a single decoder. We use subscript for the index $m = 1, \dots, M$, and bold for M -dimensional vectors. Superscript denotes time indexes, which will be omitted when possible, for simplicity.

The quadratic-Gaussian CEO problem [19] is defined as follows: A Gaussian i.i.d. source S is observed by M encoders. These observations are contaminated by mutually-independent i.i.d. Gaussian noise sequences:

$$S_m = S + Z_m \quad , \quad m = 1, \dots, M \quad . \quad (1)$$

We denote the signal to observation-noise ratios as:

$$\text{SOR}_m \triangleq \frac{\text{Var}\{S\}}{\text{Var}\{Z_m\}} \quad . \quad (2)$$

Each of these encoders (“agents”) translates a source observation block of length N_{CEO} to digital messages of rates $\{R_m\}$. These messages are sent to a central decoder (“CEO”), so that the decoder should produce a source reconstruction $\hat{S}^{(n)}$ with minimum mean-squared error (MSE). We measure the performance by the signal to distortion ratio:

$$\text{SDR} \triangleq \frac{\text{Var}\{S\}}{\frac{1}{N_{\text{CEO}}} \sum_{n=1}^{N_{\text{CEO}}} \text{Var}\{S^{(n)} - \hat{S}^{(n)}\}} \quad . \quad (3)$$

The Gaussian MAC problem [2] is a channel coding problem where the decoder observes the sum of the decoder outputs and AWGN:

$$Y = \sum_{m=1}^M X_m + Z_{\text{MAC}} \quad . \quad (4)$$

The encoders translate independent messages of rates $\{R_m\}$ to channel input blocks of length N_{MAC} . We choose to consider

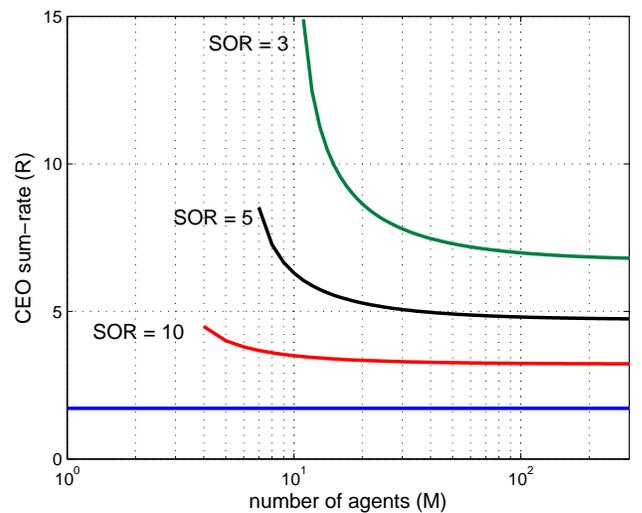


Figure 2: The CEO sum-rate as a function of the number of agents. In this example, $\text{SDR} = 31$. The different curves correspond to $\text{SOR} = 3$, $\text{SOR} = 5$ and $\text{SOR} = 10$, resulting in $M_0 = 10, 6$ and 3 , respectively. The lower line corresponds with “clean” measurements ($\text{SOR} \rightarrow \infty$, thus $\Delta R(M) = 0$).

an input sum-power constraint

$$P = \sum_{m=1}^M \frac{1}{N_{\text{MAC}}} \sum_{n=1}^{N_{\text{MAC}}} \text{Var}\{X_m^{(n)}\} \quad ,$$

and define the channel signal to noise ratio as:

$$\text{SNR} = \frac{P}{\text{Var}\{Z_{\text{MAC}}\}} \quad . \quad (5)$$

The joint CEO/MAC problem is depicted in Fig. 1. We assume in this section that one channel use is allowed per source input, i.e. the block lengths N_{CEO} and N_{MAC} are equal; For this setting, we compare the performance of separation-based schemes and simple analog transmission.

For any separation-based scheme, the CEO agent rates $\{R_m\}$ must be within the MAC rate-region, and specifically the CEO sum-rate is bounded by the MAC sum-rate. For independent messages and a fixed total power constraint, the MAC sum-rate does not depend upon the number of terminals M . The resulting necessary condition is:

$$R \triangleq \sum_{m=1}^M R_m < C_{\text{MAC}} = \frac{1}{2} \log(1 + \text{SNR}) \quad . \quad (6)$$

Here and onward, logarithms are taken to the natural base, and rates are in nats. We turn, then, to consider the CEO sum-rate. In the symmetric case ($\text{SOR}_1 = \dots = \text{SOR}_M \triangleq \text{SOR}$), this

sum-rate is given by [1], [14]:

$$R = \frac{1}{2} \log(\text{SDR}) + \frac{M}{2} \log \left(\frac{1}{1 - \frac{[\text{SDR}-1]^+}{M \cdot \text{SOR}}} \right) \quad (7)$$

$$\triangleq \frac{1}{2} \log(\text{SDR}) + \Delta R(M) .$$

In this expression $[\cdot]^+$ means $\max(\cdot, 0)$, and in the sequel we assume non-trivial distortion ($\text{SDR} > 1$). The rate is only defined for $M > M_0$ agents, where

$$M_0 \triangleq \frac{\text{SDR} - 1}{\text{SOR}} , \quad (8)$$

since M_0 agents would be needed in order to achieve the desired SDR even if the CEO could see the agents observations directly, i.e. $R \rightarrow \infty$. The second term in (7) reflects excess rate due to the observation noises and the distributed encoding setting. This term does not vanish even when the number of agents is large. In fact, we have that

$$\Delta R \triangleq \lim_{M \rightarrow \infty} \Delta R(M) = \frac{\text{SDR} - 1}{2 \cdot \text{SOR}} , \quad (9)$$

resulting in the longer-known rate for infinite number of agents [15]. See Fig. II for an example of the behavior of the sum-rate as a function of the number of agents.

Substituting (7) and (9) in (6), we find a condition for the SDR of a separation-based scheme, even with an unlimited number of terminals:

$$\log \left(\frac{1 + \text{SNR}}{\text{SDR}_{\text{sep}}} \right) > \frac{\text{SDR}_{\text{sep}} - 1}{\text{SOR}} . \quad (10)$$

The resulting SDR can be analyzed using known results about Lambert's W-function, but for our purposes two immediate upper bounds suffice:

$$\text{SDR}_{\text{sep}} < 1 + \text{SNR} \quad (11a)$$

$$\text{SDR}_{\text{sep}} < \text{SOR} \cdot \log(1 + \text{SNR}) . \quad (11b)$$

The first bound becomes tight when $\text{SOR} \gg \text{SNR}$, i.e. when the performance is limited by the channel noise, and it is the same as if a *single* agent observed the source. The second bound is significant under the opposite condition, $\text{SOR} \ll \text{SNR}$. These results are particularly disappointing when compared with the performance of a simple analog scheme, which only applies scalar operations: power adjustment at the encoders and MMSE estimation at the decoder. Such a scheme is optimal, as proven by Gastpar in [5], following a technique of Lapidoth and Tinguely [11]. This optimal performance is given by:

$$\begin{aligned} \text{SDR}_{\text{analog}} &= 1 + M \cdot (\text{SNR} \parallel \text{SOR}) \\ &\geq 1 + \frac{M}{2} [\min(\text{SNR}, \text{SOR}) - 1] , \end{aligned} \quad (12)$$

where

$$A \parallel B \triangleq \frac{A \cdot B}{1 + A + B} \quad (13)$$

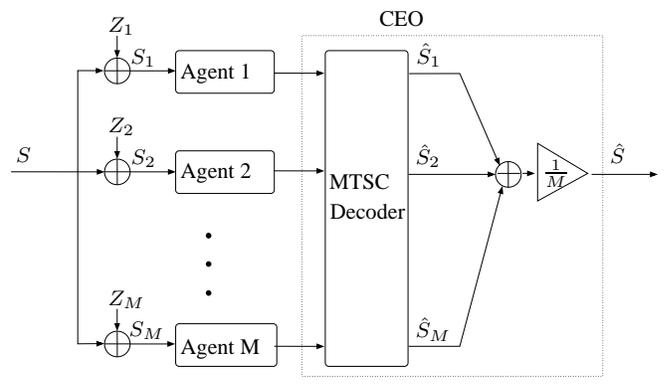


Figure 3: The CEO (indirect source coding) problem as a direct multiterminal source coding problem for the observations.

is the equivalent SNR in transmitting a signal through the concatenation of additive noise channels of SNRs A and B ¹. For any $M > 1$, the performance of the analog scheme is strictly better than that of the digital scheme; For example, if $\text{SOR} > \text{SNR} + 1$ then the lower bound in (12) is already higher than the upper bound (11a) for $M = 2$ agents. In the rest of this section we will try to gain some insight regarding the origins of this phenomenon, previously discussed in [6].

The fundamental limitation of the digital (CEO) approach, can be explained by the fact that the optimum rate (7) can be achieved by solving the direct multiterminal source coding problem for the sources S_1, \dots, S_M . That is, an optimal scheme for the CEO problem should also bear optimal reproduction of each of the signals observed by the agents (under a sum-distortion measure), although they are of no direct interest in the final reconstruction (see Fig. 3). In particular, such a scheme “wastes” rate resources on describing the observation noises to the CEO. In contrast, the analog approach uses the MAC summation to average the observation noises; The resulting decoder input already contains the sum of the observation noises.

Specifically, we demonstrate the “noise transmission” effect in terms of an optimum successive-encoding implementation. Consider the test channel depicted in Fig. 4. In this test channel, the quantizers are represented by the additive noise channels $\{W_m\}$, all of which have the same noise variance, and we define the signal to quantization-noise ratio as:

$$\text{SQR} \triangleq \frac{\text{Var}\{S_m\}}{\text{Var}\{W_m\}} . \quad (14)$$

The multipliers $\{\gamma_{m_1, m_2}\}$ are chosen such that

$$\bar{S}_m \triangleq \sum_{j=1}^{m-1} \gamma_{m,j} \hat{S}_j \quad (15)$$

¹Here for each channel, the SNR is defined as the ratio of its input to noise power, i.e. for the second channel the “signal” includes the noise of the first channel. This causes the additional term “1” in the denominator, without which this would have been a harmonic mean of the SNRs.

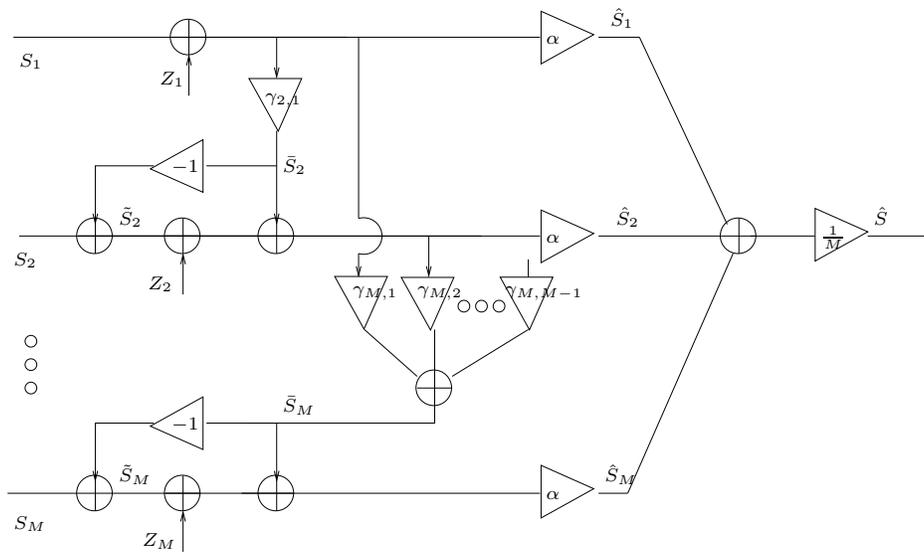


Figure 4: Sequential-encoding test channel for the CEO problem.

is the MMSE linear estimator of S_m from $\hat{S}_1, \dots, \hat{S}_{m-1}$, with estimation gains

$$\Gamma_m \triangleq \frac{\text{Var}\{S_m\}}{\text{Var}\{S_m|\hat{S}_1, \dots, \hat{S}_{m-1}\}}. \quad (16)$$

The coefficient α is the MMSE estimation (Wiener) factor for S from the average of $\hat{S}_1, \dots, \hat{S}_M$. The resulting signal to distortion ratio (3) is given by:

$$\text{SDR} = 1 + M \cdot (\text{SQR}||\text{SOR}), \quad (17)$$

while the sum of mutual information over the AWGNs $\{Z_m\}$ is:

$$\begin{aligned} R &= \sum_{m=1}^M \frac{1}{2} \log \left(1 + \frac{\text{Var}\{S_m|\hat{S}_1, \dots, \hat{S}_{m-1}\}}{\text{Var}\{W_m\}} \right) \\ &= \sum_{m=1}^M \frac{1}{2} \log \left(1 + \frac{\text{SQR}}{\Gamma_m} \right). \end{aligned} \quad (18)$$

By using arguments from [21], [20], it can be shown that, for any choice of the SQR, these SDR and rate are related according to (7). This test channel leads to an implementation of the CEO encoders, where the AWGN channels represent quantizers². Now consider the signal \tilde{S}_m , the “new” information quantized by the m -th encoder. Since the optimal estimator of S_m is also optimal for S , we have that:

$$\tilde{S}_m = E_m + Z_m, \quad (19)$$

²The AWGN channels are materialized by entropy-constrained dithered quantization (ECDQ) elements, while the subtraction of the estimation is done at the decoder; In this Gaussian setting, using a Wyner-Ziv (WZ) side-information approach bears no loss, thus the desired distortion and mutual informations may be achieved by WZ-ECDQ quantizers (for example by a nested lattice structure, see [22]). It is worth mentioning that a sequential approach to the CEO problem was first suggested by Draper [4]

where E_m is the estimation error of S , satisfying:

$$\begin{aligned} \text{Var}\{E_m\} &= \text{Var}\{S|S + Z_1 + W_1, \dots, S + Z_{m-1} + W_{m-1}\} \\ &= \frac{\text{Var}\{S\}}{1 + m \cdot (\text{SQR}||\text{SOR})}. \end{aligned} \quad (19)$$

This variance is inversely proportional to m , thus eventually it will become much smaller than $\text{Var}\{Z_m\}$, in which case the encoders use almost all their rate to describe the observation noise. Interestingly, the SDR expression (17) resembles the analog-transmission expression (12), with the quantization noise playing the part of channel noise, and the SDR growing linearly with M ; However, in the sequential scheme, the introduction of each additional agents results in a non-vanishing rate penalty according to (18), while in analog transmission, no additional “rate” resources are required.

A final remark for this section regards the number of agents in the CEO problem, beyond which adding further agents does not have much effect anymore. To see what this number is, we can use Taylor expansion of (7) to assert:

$$\frac{\Delta R(M)}{\Delta R} = 1 + \frac{M_0}{M} + O\left(\frac{1}{M^2}\right), \quad (20)$$

where M_0 was defined in (8). We see, then, what Fig. II demonstrates: There is no point in using much more than the minimum number of agents, M_0 .

III. JOINT CEO/MAC CODING FOR BW MISMATCH

In this section we consider the same joint CEO/MAC problem of Section II, with the restriction of one channel use per source sample removed: We now assume that for each source block of length N_{CEO} , each encoder emits $N_{\text{MAC}} = \rho N_{\text{CEO}}$ channel inputs, where $\rho > 0$ is the *bandwidth expansion factor*. Since a separation-based scheme must obey:

$$R < \rho C_{\text{MAC}}, \quad (21)$$

we have that (10) and (11) hold, with the term $(1 + \text{SNR})$ replaced by $(1 + \text{SNR})^\rho$:

$$\text{SDR}_{\text{sep}} < (1 + \text{SNR})^\rho \quad (21a)$$

$$\text{SDR}_{\text{sep}} < \rho \cdot \text{SOR} \cdot \log(1 + \text{SNR}) \quad (21b)$$

An upper bound on the performance of a scheme for this joint source/channel setting is given by the *cooperation bound*, which assumes that the agents can all cooperate and use a point-to-point multi-antenna channel³:

$$\text{SDR} \leq 1 + \left[M \cdot \text{SOR} \right] \uparrow \left[(1 + M \cdot \text{SNR})^\rho - 1 \right] \quad (22)$$

where the noise-accumulation operation \uparrow was defined in (13). This bound reflects a gain M with respect to both SOR and SNR: The first is the gain of estimation from multiple independent observations, while the second is the coherence gain (also called “array gain”) of cooperative transmission to the MAC.

Now we come to the task of closing the gap between this bound and the performance of a separation-based scheme. First note, that even a simple linear scheme, which cannot exploit the full bandwidth of both the source and the channel, may perform better than a separation-based scheme. To see this, assume that $\rho > 1$. Since the above bounds do not depend on M , even a scheme which transmits analogically only the first N_{MAC} channel uses and performs according to (12), is better than the bounds (21) for large enough M . However, non-linear joint source/channel coding schemes can also exploit the full bandwidth of the source and the channel. For example, a scheme by Nazer and Gastpar based on modulo-lattice modulation (MLM), achieves for integer ρ :

$$\text{SDR} \cong [M \cdot \text{SOR}] \uparrow \left[M \cdot (1 + \text{SNR}) \cdot \left(\frac{1 + \text{SNR}}{M} \right)^{\rho-1} \right] \quad (23)$$

While this strategy fully exploits the bandwidth, the effective SNR may deteriorate when increasing the number of agents M , instead of improving. We now present a scheme which approaches the cooperation bound (22) in the high-SOR regime, thus exploits the full bandwidth while achieving the full coherence gain.

For the sake of simplicity, we assume integer ρ , though the concept may be extended to any $\rho > 0$. We then follow the application of modulo-lattice modulation (MLM) to BW expansion as done by Reznic et al. [17] for joint source/channel broadcasting, and by Nazer and Gastpar for transmission of the sum of uncorrelated sources over a MAC [13] and for the joint CEO/MAC problem [12]: In the first N_{CEO} channel uses, the encoders employ analog transmission. In the next uses, the decoder can use the estimate it obtained as source side-information (SI), and the encoders “zoom in” on the analog data using MLM with a lattice dimension N_{CEO} . This procedure repeats in an iterative manner, until all ρN_{CEO} channel uses are exhausted.

³A tighter bound can be derived by a straightforward extension of the equal-bandwidth bound (12) of [5], see [12]. However, these bounds approach coincide at the high-SOR region in which we will be interested

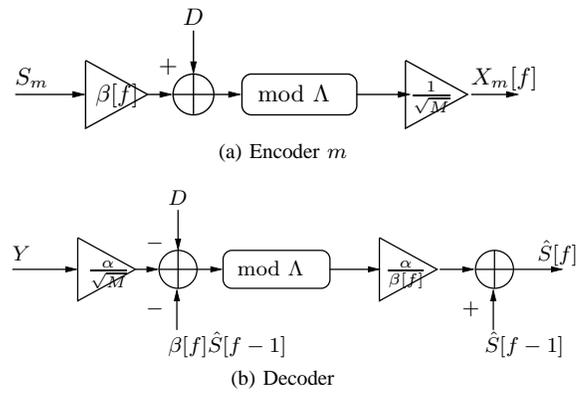


Figure 5: Components of the MLM scheme at iteration f .

Specifically, at each iteration $1 < f \leq \rho$ the relation between the channel inputs $X_m[f]$ and output $Y[f]$ is given by (4). We denote as $\hat{S}[f]$ the source estimation at the decoder using $Y[1], \dots, Y[f]$. The m -th encoder and the decoder are then given by (see (5)):

$$X_m[f] = \frac{[\beta[f]S_m + D] \bmod \Lambda}{\sqrt{M}} \quad (24a)$$

$$\hat{S}[f] = \frac{\alpha}{\beta[f]} \left[\frac{\alpha}{\sqrt{M}} Y[f] - D - \beta[f] \hat{S}[f-1] \right] \bmod \Lambda + \hat{S}[f-1] \quad (24b)$$

where we assume that the lattice Λ has a normalized cell power P , and that the dither vector D is uniformly distributed over the basic lattice cell (for the relevant properties of lattices see e.g. [10]). The MMSE factor α and the zooming factors $\{\beta[f]\}$ are set for optimizing performance; Roughly speaking, $\beta^2[f]$ is the ratio between the channel input power and the source uncertainty at the encoder before iteration f , thus it increases between iterations as the uncertainty decreases.

The improved performance over [12] is achieved by the use of the same dither by all the agents, combined with the following assumption:

Assumption 1: Let $Q_\Lambda(\cdot)$ be the quantization operation by the lattice Λ , then

$$Q(D + Z) = Q(D) = 0 \quad ,$$

where D is uniform over the basic lattice cell and Z is a “small” Gaussian perturbation .

We will discuss in the sequel the conditions under which this assumption holds with high probability.

We are now ready to state the achieved performance, which equals the bound (22), up to factors which reflect the lattice loss at finite-dimension: The lattice normalized second moment $G(\Lambda)$ and the volume to noise ratio⁴ $\mu(\Lambda)$, see [10].

⁴The volume to noise ratio depends on some overflow probability ϵ , which has to be chosen small enough such that the effect of overflow is not significant. In the notation $\mu(\Lambda)$ we omit this dependence for the sake of conciseness.

Theorem 1: For a suitable choice of α and $\{\beta[f]\}$, the scheme described by (24) approaches:

$$\text{SDR} = 1 + \left[M \cdot \text{SOR} \right] \uparrow \left[M \cdot \text{SNR} \cdot \left(1 + \frac{M \cdot \text{SNR}}{G(\Lambda) \cdot \mu(\Lambda)} \right)^{\rho-1} - 1 \right],$$

provided that Assumption 1 holds

Proof: Denoting $\tilde{D} = [\beta[f]S + D] \bmod \Lambda$, we have that:

$$\begin{aligned} \sqrt{M}X_m[f] &= [\beta[f]S_m + D] \bmod \Lambda \\ &= [\beta[f]S + D + \beta[f]Z_m] \bmod \Lambda \\ &= [\tilde{D} + \beta[f]Z_m] \bmod \Lambda \\ &= \tilde{D} + \beta[f]Z_m, \end{aligned} \quad (25)$$

where for the last transition we used Assumption 1, which is applicable since by the properties of the dithered modulo operation \tilde{D} is uniform over the basic lattice cell. Consequently, defining

$$\tilde{S} = S + \frac{\sum_{m=1}^M Z_m}{M}$$

we have that:

$$\begin{aligned} \tilde{Y}[f] &\triangleq \frac{Y[f]}{\sqrt{M}} = \tilde{D} + \beta[f] \frac{\sum_{m=1}^M Z_m}{M} + \frac{Z_{\text{MAC}}[f]}{\sqrt{M}} \\ &= [\tilde{S} + D] \bmod \Lambda + \frac{Z_{\text{MAC}}[f]}{\sqrt{M}}, \end{aligned}$$

where in the last transition we used again Assumption 1. Now we identify \tilde{Y} as the decoder input in the point-to-point MLM scheme for transmission with side information [10]. Specifically, \tilde{S} is the source, $\hat{S}[f-1]$ is the source SI at the decoder, and the channel signal-to-noise ratio is $M \cdot \text{SNR}$. Noting that after the first (linear) iteration the decoder already knows \tilde{S} with SDR of $1 + M \cdot \text{SNR}$ and incorporating the results of [10], we have that the decoder has SDR of

$$(1 + M \cdot \text{SNR}) \left(1 + \frac{M \cdot \text{SNR}}{G(\Lambda) \cdot \mu(\Lambda)} \right)^{\rho-1}$$

with respect to \tilde{S} , which translates to the desired results with respect to S . ■

The lattice loss factor $G(\Lambda)\mu(\Lambda)$ is always greater than or equal to 1. In order to minimize it, we need high lattice dimension. If we take a sequence of lattices Λ_K which are simultaneously good for source and channel coding, then we have:

$$G(\Lambda_K) \cdot \mu(\Lambda_K) \rightarrow 1, \quad (26)$$

ensuring optimality of our MLM scheme. It remains to verify, though, that Assumption 1 holds; Unfortunately, this demand seems to be in conflict with the high dimension needed for $G(\Lambda)$ and $\mu(\Lambda)$, as we discuss in the sequel. For now, we present one-dimensional results.

Lemma 1: Let P_e be the probability of Assumption 1 not to hold. For one-dimensional lattices, this probability satisfies:

$$P_e < \sqrt{\frac{\pi}{24}} \cdot \sqrt{\frac{\sigma_Z^2}{\sigma^2}},$$

where σ_Z^2 is the power of the perturbation Z and σ^2 is the normalized lattice power.

Proof: For a scalar lattice, the basic cell is $[-r, r]$, where $r = \sqrt{3\sigma^2}$. Then we have:

$$\begin{aligned} P_e &\triangleq \Pr\{Q_\Lambda(D + Z) \neq 0\} \\ &= \frac{1}{2 \cdot r} \int_{-r}^r [\Pr\{Z > r - v\} + \Pr\{Z < -r - v\}] dv \\ &= \frac{1}{r} \int_0^{2 \cdot r} \Pr\{Z > v\} dv \\ &< \frac{1}{r} \int_0^\infty \Pr\{Z > v\} dv \\ &< \frac{1}{r} \int_0^\infty \exp\left\{-\frac{v^2}{2\sigma_Z^2}\right\} dv \\ &= \frac{\sqrt{2\pi\sigma_Z^2}}{2 \cdot r} = \sqrt{\frac{\pi}{24}} \cdot \sqrt{\frac{\sigma_Z^2}{\sigma^2}}. \end{aligned}$$

The first equality follows, since by the properties of dithered quantization, $v = [D + Z] \bmod \Lambda$ is uniform over the basic lattice cell. The rest follows by algebraic manipulation and the known bound on the integral over a Gaussian distribution ■

This result can be extended to find the probability of error in at least one agent of M , where D is equal for all but the perturbations are independent. Since the error events in the different agents are highly dependent, the union bound is far from being tight. This error probability can be bounded by P_e of the Lemma, multiplied by a factor

$$A(M) \triangleq \sum_{m=1}^M \frac{(-1)^{m+1}}{\sqrt{m}} \cdot \binom{M}{m}$$

which grows very slowly with M : Numerical examination shows that it is slower than $\log(\log M)$.

Applying this to our setting, we seek the condition for the first transition in (25) to hold with high probability. In this case, the lattice normalized power is P while the perturbation power at iteration f is $\beta^2[f] \text{Var}\{Z_m\}$. Going back to the proof of the MLM scheme in [10], we find that the zooming factor β satisfies:

$$\begin{aligned} \beta^2[f] &\leq \frac{P}{\text{Var}\{\hat{S}[f-1] - \tilde{S}\}} \\ &= (1 + M \cdot \text{SNR}) \left(1 + \frac{M \cdot \text{SNR}}{G(\Lambda)\mu(\Lambda)} \right)^{f-2} \cdot \frac{P}{\text{Var}\{\tilde{S}\}}. \end{aligned} \quad (27)$$

Since this is growing with f , we find that the worst probability of error is at the last iteration, where:

$$P_e < \sqrt{\frac{\pi}{24}} \cdot \frac{(1 + M \cdot \text{SNR}) \left(1 + \frac{M \cdot \text{SNR}}{G(\Lambda)\mu(\Lambda)} \right)^{\rho-2}}{\frac{1}{M} + \text{SOR}}. \quad (28)$$

Consequently, Assumption 1 will hold with high probability throughout the iterations as long as:

$$\text{SOR} \gg A^2(M) \cdot (1 + M \cdot \text{SNR}) \left(1 + \frac{M \cdot \text{SNR}}{G(\Lambda)\mu(\Lambda)} \right)^{\rho-2}. \quad (29)$$

This implies continuity of the achievable distortion as $\text{SOR} \rightarrow \infty$, but a more interesting case from the practical point of view occurs when

$$M \cdot \text{SOR} \cong (1 + M \cdot \text{SNR})^\rho ,$$

i.e. the contributions of the observation noise and of the channel noise to the cooperation-bound distortion (22) are similar. In that case, ignoring the finite dimension effects, the condition becomes:

$$\text{SNR} \gg A^2(M) . \quad (30)$$

Recalling the slow growth of $A(M)$, the required SNR is reasonable.

Trying to close the gap to the cooperation bound by increasing the lattice dimension turns out to be a difficult task⁵. This occurs, since “good” lattices in the sense of (26) must have ball-like cells, and consequently most of their mass is concentrated close to their surface in the limit of high dimension, where a small perturbation may take the signal outside the cell. Furthermore, this shape of the cells means that an error would cause a shift in all coordinates. There may exist lattices, different than the conventional “good” ones, which strike a good balance between the desired properties of low normalized second moment, immunity to channel noise and immunity to source perturbation; This subject is left for further research.

Remarks about the MLM scheme:

1. **Optimality for the SI setting:** Clearly, this also presents a scheme for the joint CEO/MAC problem with source SI in the decoder.

2. **Adding a channel interference:** Suppose that a dirty-paper interference, known at all the encoders, is added to the channel output. This would not affect performance, since part of this interference can be subtracted before the modulo operation at each encoder, resulting in an operation equivalent to interference cancellation in the point-to-point setting [10]. On the other hand, if a sum of interferences is added, where each encoder only knows one of them - then the different subtraction which has to be made at each encoder cannot coexist with the local linearity demand reflected in Assumption 1. This problem bears similarity to the loss of the dirty MAC problem [16].

3. **Working with non-integer ρ :** If ρ is non-integer, then the iterative approach cannot utilize the full channel BW. An alternative approach which will work for any $\rho \geq 1$, is performing source prediction at the decoder in the time domain, and using the predictor output as SI - as done by the Analog Matching scheme [9] in the point-to-point setting. The same approach can be used to exploit the memory of a colored source.

4. **Bandwidth compression:** For $\rho < 1$, channel prediction (“precoding”) has to be performed at the encoder, and the predictor output can be seen as a dirty-paper SI, see [9]. When

⁵Actually, we know that it cannot be completely closed, since for any finite SOR the bound of [5] is tighter.

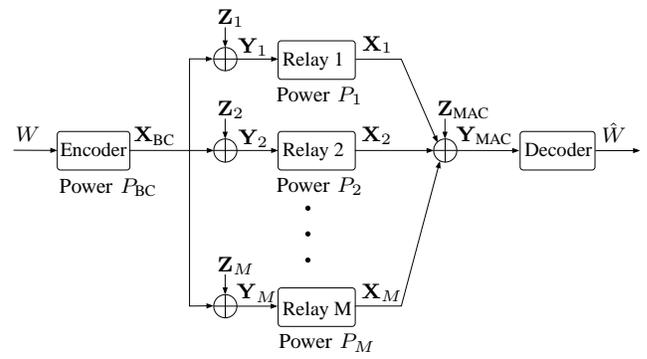


Figure 6: The parallel relay network.

the SOR is high enough, this SI is almost identical at all the agents, allowing optimum operation as in point 2 above.

IV. APPLICATION TO THE PARALLEL RELAY NETWORK

The parallel relay network [18] consists of a single encoder which needs to convey a message W to a single decoder, with the help of M parallel relays, see Fig. 6. It is convenient to look at this network as the concatenation of two parts, a broadcast (BC) section from the encoder to the relays, and a MAC section from the relays to the decoder. We assume a Gaussian symmetric setting, i.e. all noises are Gaussian and mutually independent, and the noise variances in different branches are equal. We define the signal to noise ratios of the two sections as:

$$\text{SNR}_{\text{BC}} \triangleq \frac{P_{\text{BC}}}{\text{Var}\{Z_m\}} \quad (31a)$$

$$\text{SNR}_{\text{MAC}} \triangleq \frac{\sum_{m=1}^M P_m}{\text{Var}\{Z_{\text{MAC}}\}} . \quad (31b)$$

When the bandwidths of the BC and MAC sections are equal, the decode and forward (D&F) strategy is optimal for $\text{SNR}_{\text{BC}} \geq M \cdot \text{SNR}_{\text{MAC}}$, while the amplify and forward (A&F) strategy is optimal in the limit $\text{SNR}_{\text{MAC}} \gg \text{SNR}_{\text{BC}}$. As mentioned in the introduction, A&F can be seen as a joint source/channel approach. Namely, we identify the codeword transmitted to the channel, X_{BC} , as the source, the BC section noises as observation noises and the relays and agents. For the resulting joint CEO/MAC with equal BW, A&F simply means that the agents apply the optimal analog strategy.

A recent work [8] presented the *rematch and forward* (R&F) strategy, which extends the advantages of A&F to the bandwidth mismatch case, where we are allowed ρ MAC-section uses per each BC-section use. In this strategy, the codebook always has the MAC section BW. Again the chosen codeword is seen as a Gaussian signal, but now it is not identical to the channel input since there is a BW mismatch between this codeword and the BC. The R&F strategy uses joint source/channel coding to overcome this mismatch. The “source” reconstruction at the relays is now seen as the agents observation, resulting in an equivalent CEO problem with

$$\text{SOR} = (1 + \text{SNR}_{\text{BC}})^{\frac{1}{\rho}} - 1 . \quad (32)$$

For any $\rho < 1$ we can directly apply (12) to find that the codeword can be reconstructed at the final decoder with

$$\text{SDR} = 1 + M \cdot \left([(1 + \text{SNR}_{\text{BC}})^{\frac{1}{\rho}} - 1] \uparrow [\text{SNR}_{\text{MAC}}] \right), \quad (33)$$

and transmission with rate (normalized to BC section uses)

$$R_{\text{R\&F}} = \frac{\rho}{2} \log(\text{SDR}) \quad (34)$$

is feasible.

For $\rho > 1$, the original R&F strategy encounters the problem of correlated observation noises: When the source BW is larger than the channel BW, the channel does not supply enough degrees of freedom for the reconstruction errors to be independent. Using joint CEO/MAC coding with BW expansion, we can now overcome this problem. We change the R&F strategy, then, by choosing the codebook BW to be the minimum between the BW of the two sections. For $\rho > 1$ it equals the BC section BW, thus the encoder directly transmits the chosen codeword and the relays perform the bandwidth expansion. Under the conditions discussed in the previous section and ignoring the finite lattice dimension losses, we know that the encoder can estimate the codeword with SDR approaching the cooperation bound (22), with SNR_{BC} playing the part of the SOR. The resulting rate is given by $\frac{1}{2} \log(\text{SDR})$, again normalized to BC section uses. Seeing that (33) also approaches the cooperation bound for high enough SNR_{BC} , we conclude that for any ρ we can have:

$$R_{\text{R\&F}} \cong \frac{1}{2} \log \left([M \cdot \text{SNR}_{\text{BC}}] \uparrow [(M \cdot \text{SNR}_{\text{MAC}})^{\rho}] \right). \quad (35)$$

Finally, we turn back to verify that Assumption 1 actually holds in cases of interest. Recalling (30), we find that when the BC and MAC sections have similar capacities, i.e. neither D&F nor C&F are optimal, then the assumption does hold with high probability, for reasonably large SNR_{MAC} .

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