Bundling Equilibrium in Combinatorial Auctions[*]

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Abstract:

This paper analyzes individually-rational ex post equilibrium in the VC (Vickrey-Clarke) combinatorial auctions. If Σ is a family of bundles of goods, the organizer may restrict the participants by requiring them to submit their bids only for bundles in Σ. The Σ-VC combinatorial auctions (multi-good auctions) obtained in this way are known to be individually-rational truth-telling mechanisms. In contrast, this paper deals with non-restricted VC auctions, in which the buyers restrict themselves to bids on bundles in Σ, because it is rational for them to do so. That is, it may be that when the buyers report their valuation of the bundles in Σ, they are in an equilibrium. We fully characterize those Σ that induce individually rational equilibrium in every VC auction, and we refer to the associated equilibrium as a bundling equilibrium. The number of bundles in Σ represents the communication complexity of the equilibrium. A special case of bundling equilibrium is partition-based equilibrium, in which Σ is a field, that is, it is generated by a partition. We analyze the tradeoff between communication complexity and economic efficiency of bundling equilibrium, focusing in particular on partition-based equilibrium.

1 Introduction

The Vickrey-Clarke-Groves (VCG) mechanisms [40, 5, 12] are central to the design of protocols with selfish participants (e.g., [28, 37, 39]), and in particular for combinatorial auctions (e.g., [41, 18, 8, 42, 24, 26, 19]), in which the participants submit bids, through which they can express preferences over bundles of goods. The organizer allocates the goods and collects payments based on the participants' bids.[1] These protocols allow to allocate a set of

[1] Motivated by the FCC auctions (see e.g., [6, 21, 22]) there is an extensive recent literature devoted to the design and analysis of multistage combinatorial auctions, in which the bidders express partial preferences over bundles at each stage. See e.g.,[42, 31, 2, 29, 30, 3] .

goods (or services, or tasks) in a socially optimal (surplus maximizing) manner, assuming there are no resource bounds on the agents' computational capabilities.[2] The VCG protocols are designed in a way that truth-revealing of the agents' private information[3] is a dominant strategy to them. Moreover, VCG protocols can be applied in the context of games in informational form, where no probabilistic assumptions about agents' types are required.[4] We shortly define domination and equilibrium in such games. These solutions are called ex post solutions because they have the property that if the players were told about the true state, after they choose their actions, they would not regret their actions.[5]

In this paper we deal with a special type of VCG mechanisms – the VC mechanisms. Amongst the VCG mechanisms the VC mechanisms are characterized by two additional important properties: Truth telling satisfies the participation constraint, that is, it is preferred to non participation,[6] and the seller's revenue is always non negative.

[2]There are at least two sources of computational issues, which arise when dealing with combinatorial auctions; Winner determination –finding the optimal allocation (see e.g.,[33, 26, 38, 34, 10, 1, 35, 14]) , and bid communication – the transfer of information (see e.g., [27]).

[3]This paper deals with the private-values model, in which every buyer knows his own valuations of bundles of goods. In contrast, in a correlated-values model, every buyer receives a signal (possibly about all buyers' valuation functions), and this signal does not completely reveal his own valuation function (see e.g. [23, 15, 20, 7, 32, 31] for discussions of models in which valuations are correlated).

[4]A game in informational form is a pre-Bayesian game. That is, it has all the ingredients of a Bayesian game except for the specification of probabilities. Unlike Bayesian games, games in informational form do not necessarily possess a solution: a recommendation for rational players how to play. However, in many important models such solutions do exist. See Section 2 for a precise definition.

[5]Alternatively, ex post solutions may be called probability-independent solutions because, up to some technicalities concerning the concept of measurable sets, they form Bayesian solutions for every specification of probabilities.

[6]An equilibrium that satisfies the participation constraint is said to be individually-rational.

A famous observation of the theory of mechanism design in economics, termed **the revelation principle** (see e.g. [25]), implies that the discussion of additional individually rational (IR) equilibria of the VC mechanisms may seem unneeded, and indeed it has been ignored by the literature. It can be proved that every mechanism with an ex post equilibrium is economically equivalent to another mechanism – a direct mechanism – in which every agent is required to submit his information. In this direct mechanism, revealing the true type is an ex post dominating strategy for every agent, and it yields the same economics parameters as the original mechanism. However, the two mechanisms differ in the set of inputs that the player submits in equilibrium. This difference may be crucial when we deal with communication complexity. Thus, two mechanisms that are equivalent from the economics point of view, may be considered different mechanisms from the CS point of view.

Thus, tackling the VC mechanisms from a computational perspective introduces a vastly different picture. While the revelation of the agents' types defines one IR equilibrium, there are other (in fact, over-exponentially many) IR equilibria for the VC auctions. Moreover, these equilibria have different communication requirements.

In this paper we analyze ex post equilibria in the VC mechanisms.

Let Σ be a family of bundles of goods. We characterize those Σ, for which the strategy of reporting the true valuation over the bundles in Σ is a player-symmetric IR ex post equilibrium. An equilibrium that is defined by such Σ is called a bundling equilibrium. We prove that Σ induces a bundling equilibrium if and only if it is a quasi field of bundles.[7] The number of bundles in Σ represents the communication complexity of the equilibrium, and the economic efficiency of an equilibrium is measured by the generated social surplus. A special type of bundling equilibria are partition-based equilibria, in which Σ is a field (i.e. it is generated by a partition). The partition-based

[7] A quasi field is a nonempty set of sets that is closed under complements and under disjoint unions.

equilibria are ranked according to the usual partial order on partitions: If one partition is finer than another one, then it yields higher communication complexity as well as higher social surplus.[8]

We analyze the least upper bound (over all possible profiles of valuation functions, one for each buyer) of the ratio between the optimal surplus and the surplus obtained in a partition-based equilibrium. We express this least upper bound in terms of the partition's structure. We provide an upper bound for this ratio, which is proved to be tight in infinitely many cases.

In Section 2 we present the concept of ex post equilibrium in games in informational form. In Section 3 we discuss combinatorial auctions. Together, Sections 2 and 3 provide the reader with a rigorous framework for general analysis of VC protocols for combinatorial auctions. In Section 4 we introduce bundling equilibrium, and provide a full characterization of bundling equilibria for VC protocols. Then we discuss bundling equilibrium that is generated by a partition, titled partition-based equilibrium. In Section 5 we deal with the surplus of VC protocols for combinatorial auctions when following partition-based equilibrium, exploring the spectrum between economic efficiency and communication efficiency.

2 Ex post equilibrium in games in informational form

A game in **informational form** $G = G(N, \Omega, T, (\tilde{t}_i)_{i \in N}, X, (u_i)_{i \in N})$ is defined by the following parameters:

- **Agents**: Let $N = \{1, \ldots, n\}$ be the set of agents.
- **States**: Let Ω be the set of (relevant) states.

[8] It is worth mentioning that the various equilibria cannot be ranked according to the revenue of the seller. That is, under some conditions, a partition-based equilibrium may simultaneously yield more revenue and less communication complexity than the truth revealing equilibrium (an example is provided).

• **Types**: Let T_i be the set of types of agent i, $T = \times_{i \in N} T_i$.

• **Signaling functions**: Let $\tilde{t}_i : \Omega \to T_i$ be the signaling function of agent i. Without loss of generality, it is assumed that every type $t_i \in T_i$ is possible. That is $\tilde{t}_i(\Omega) = T_i$.

• **Actions**: Let X_i be the set of actions of i, $X = \times_{i \in N} X_i$.

• **Utility functions**: Let $u_i(\omega, x)$ be the utility of i at state ω, when the agents choose the action profile x.

Every $\omega \in \Omega$ defines a game in strategic (normal) form, $G(\omega)$. In this game agent i receives $u_i(\omega, x)$, when it chooses x_i, and all other agents choose x_{-i}. However, the agents do not know which game they play.

For $t_i \in T_i$ let $\Omega_i(t_i)$ be the set of states that generate the signal t_i, that is

$$\Omega_i(t_i) = \{\omega \in \Omega | \tilde{t}_i(\omega) = t_i\}.$$

A **strategy**[9] of i is a function $b_i : T_i \to X_i$; The associated **implied strategy** is the function $\hat{b}_i : \Omega \to X_i$ given by

$$\hat{b}_i(\omega) = b_i(\tilde{t}_i(\omega)).$$

A profile of strategies $b = (b_1, \cdots, b_n)$ is an **ex post equilibrium**, if for every agent i, for every $t_i \in T_i$, for every $\omega \in \Omega_i(t_i)$, and for every $x_i \in X_i$,

$$u_i(\omega, b_i(t_i), \hat{b}_{-i}(\omega)) \geq u_i(\omega, x_i, \hat{b}_{-i}(\omega)).$$

A strategy b_i of i is an **ex post dominant strategy** for i, if for every profile of strategies b_{-i} of the other players, for every $t_i \in T_i$, for every $\omega \in \Omega_i(t_i)$, and for every $x_i \in X_i$,

$$u_i(\omega, b_i(t_i), \hat{b}_{-i}(\omega)) \geq u_i(\omega, x_i, \hat{b}_{-i}(\omega)).$$

Obviously, if b_i is an ex post dominant strategy for every i, b is an ex post equilibrium, but not necessarily vice versa. An ex post equilibrium b, in which every strategy b_i is ex post dominant is called an ex post **domination equilibrium**.

[9] In this paper we do not deal with mixed strategies.

3 Combinatorial auctions

In a combinatorial auction there is a seller, denoted by 0, who wishes to sell a set of m items $A = \{a_1, \ldots, a_m\}$, $m \geq 1$, that are owned by her. There is a set of (potential) buyers $N = \{1, 2, \ldots, n\}$, $n \geq 1$. We take N as the set of agents. Let Γ be the set of all allocations of the goods. That is, every $\gamma \in \Gamma$ is an ordered partition of A, $\gamma = (\gamma_i)_{i \in N \cup \{0\}}$. A **valuation function** of buyer i is a function $v_i : 2^A \rightarrow \Re$, where $\Re$ denotes the set of real numbers, with the normalization $v_i(\emptyset) = 0$. In a more general setup, a buyer may care about the distribution of goods that he does not own. In such a setup the utility of an agent may depend on the whole allocation γ rather than on γ_i. Hence, by dealing with valuation functions we actually assume:

• **No allocative externalities**.[10]

We also assume:

• **Free disposal**: If $B \subseteq C$, $B, C \in 2^A$, then $v_i(B) \leq v_i(C)$.

Let V_i be the set of all possible valuation functions of i (obviously $V_i = V_j$ for all $i, j \in N$), and let $V = \times_{i \in N} V_i$. We refer to V as the set of **states** ($\Omega = V$). In a general model, each buyer receives a signal t_i through a signaling function $\tilde{t}_i$ defined on V. We assume:

• **Private value model**: $\tilde{t}_i(v) = v_i$. That is, $T_i = V_i$ and each buyer knows his valuation function only.[11]

A **mechanism** for allocating the goods is defined by sets of messages X_i, one set for each buyer i, and by a pair (d, c) with $d : X \rightarrow \Gamma$, and $c : X \rightarrow \Re^n$, where $X = \times X_i$. d is called the allocation function and c the transfer function; if the buyers send the profile of messages $x \in X$, buyer i receives the set of goods $d_i(x)$ and pays $c_i(x)$ to the seller. We assume:

• **Quasi linear utilities**: If agent i with the valuation function v_i receives the set of goods γ_i and pays c_i, his utility equals $v_i(\gamma_i) - c_i$.

[10] For auctions in which externalities are assumed see, e.g., [17, 16].

[11] See e.g. [23, 15, 20, 7, 32, 31] for discussions of models in which valuations are correlated and buyers do not know their own valuation.

As the seller cannot force the buyer to participate, a full description of a mechanism should describe the allocation of goods and transfers for cases in which not all agents participate. However, we adopt the way this issue is treated in economics: The mechanism (X, d, c) defines a game in informational form. An ex post equilibrium b in this game satisfies the **participation constraint** if for every buyer i,

$$v_i(d_i(b(v))) - c_i(b(v)) \geq 0 \quad \text{for every } v \in V, \tag{3.1}$$

where $b(v) = (b_1(v_1), \ldots, b_n(v_n))$. If an ex post equilibrium satisfies the participation constraint, we call it **individually rational**. If the buyers use an individually rational ex post equilibrium profile b, then a deviation of a buyer to non participation is not profitable for him.[12] Similarly, a dominant strategy of i, b_i, is individually rational if (3.1) is satisfied for every profile b_{-i} of the other buyers' strategies.

For an allocation γ and a profile of types v we denote by $S(v, \gamma)$ the **total social surplus** of the buyers, that is

$$S(v, \gamma) = \sum_{i \in N} v_i(\gamma_i).$$

We also denote:

$$S_{max}(v) = \max_{\gamma \in \Gamma} S(v, \gamma).$$

Consider a mechanism $M = (X, d, c)$ and an individually rational ex post equilibrium b. For every profile v we denote the surplus generated by b by $S_b^M(v) = S(v, d(b(v)))$, and the revenue collected by the seller by $R_b^M(v) = \sum_{i \in N} c_i(b(v))$.

[12] Thus, b remains an ex post equilibrium profile if every set of messages is extended by a null message, and an agent whose input is null receives no good and pays nothing. Nevertheless, if the issue of uniqueness of equilibrium is important, dealing with ex post individually rational equilibrium instead of dealing with ex post equilibrium in the extended model is not without loss of generality. The extended model may have more equilibrium profiles, that cannot be expressed in the reduced model, that is, equilibrium profiles in which some of the agents do not participate in some of the cases.

Because of the participation constraint,

$$R_b^M(v) \leq S_b^M(v) \leq S_{max}(v) \quad \text{for all } v \in V.$$

A mechanism and an individually rational ex post equilibrium (M, b) are called **socially optimal** if

$$S_b^M(v) = S_{max}(v) \quad \text{for all } v \in V.$$

Note that the seller controls the mechanism, but she does not control the strategies used by the buyers. However, it is assumed that if the mechanism possesses an individually rational ex post equilibrium, the agents use such an equilibrium.[13]

A public seller may wish to generate a socially optimal mechanism, whereas a selfish seller may be interested in the revenue function only. Such a seller would rank mechanisms according to the revenue they generate.

A mechanism (X, d, c) is called a **direct** mechanism if $X_i = V_i$ for every $i \in N$. That is, in a direct mechanism a buyer's message contains a full description of some valuation function. A direct mechanism is called **truth revealing** if for every buyer i, telling the truth ($b_i(v_i) = v_i$) is an individually rational ex post dominating strategy. (Of course the profile of strategies $b = (b_i)_{i \in N}$ is an individually rational ex post equilibrium.) By the revelation principle,[14] given a mechanism and an individually rational ex post equilibrium (M, b) one can find a direct truth revealing mechanism that yields the same distribution of goods and the same payments (and in particular, the same revenue and surplus functions).[15] It may seem therefore that the concept of ex post equilibrium is not interesting in our setup

[13]For example, the agents may reach the equilibrium by a process of learning (see e.g. [13]).

[14]See e.g. [25].

[15]This strong version of the revelation principle is due to our private values assumption. Otherwise, the revelation principle guarantees the existence of an equivalent direct mechanism in which telling the truth is an individually rational ex post equilibrium (but not necessarily a domination equilibrium).

(private values), and indeed most of the economics literature of mechanism design with private values deals only with direct and truth revealing mechanisms. However, when we deal with computational issues, two mechanism that are equivalent in economics may differ in their complexity. The time, space, and communication required to compute and communicate the message of an agent as well as the chosen allocation may depend on the messages sent in equilibrium. Thus, the concept of ex post equilibrium may be very important even if private values are assumed.

Well-known truth revealing mechanisms are the VC mechanisms. These mechanisms are parameterized by an allocation function d, that is socially optimal. That is, $S_{max}(v) = S(v, d(v))$ for every $v \in V$. The transfer functions are defined as follows:

$$c_i^d(v) = \max_{\gamma \in \Gamma} \sum_{j \neq i} v_j(\gamma_j) - \sum_{j \neq i} v_j(d_j(v)). \tag{3.2}$$

Note that $c_i^d(v) \geq 0$ for every $v \in V$.

The mechanisms differ in the allocation they pick in cases in which there exist more than one socially optimal allocation, and therefore in the second term in (3.2).[16] It is well-known that **all VC mechanisms yield the same utility to a truth telling buyer**: For a VC mechanism d[17] we denote by $u_i^d(v_i, (v_i', v_{-i}))$ the utility of buyer i with the valuation function v_i, when he

[16]By VC mechanisms we refer here to what is also known as Clarke mechanisms or the Pivotal mechanism. More general mechanisms are the VCG mechanisms. Every VCG mechanism is obtained from some VC mechanism by changing the transfer functions: A VCG mechanism is defined by a socially optimal allocation function d and by a family of functions $h = (h_i)_{i \in N}$. The transfer functions are defined by:

$$c_i^d(v) = \max_{\gamma \in \Gamma} \sum_{j \neq i} v_j(\gamma_j) - \sum_{j \neq i} v_j(d_j(v)) + h_i(v_{-i}).$$

Truth telling is an ex post equilibrium in every VCG mechanism, but it is not necessarily an individually rational ex post equilibrium.

[17]Since in all VC mechanisms $M = (X, d, c)$, X is V and c is defined as in (3.2), it is enough to specify d in order to specify the mechanism.

declares v_i' and the other buyers declare v_{-i}. That is,

$$u_i^d(v_i, v') = v_i(d_i(v')) - c_i^d(v'), \tag{3.3}$$

where $v' = (v_i', v_{-i})$. Therefore, by (3.2),

$$u_i^d(v_i, v') = S(v, d(v')) - g_i(v_{-i}), \tag{3.4}$$

where $v = (v_i, v_{-i})$, and

$$g_i(v_{-i}) = \max_{\gamma \in \Gamma} \sum_{j \neq i} v_j(\gamma_j). \tag{3.5}$$

If i declares v_i,

$$u_i^d(v_i, v) = S(v, d(v)) - g_i(v_{-i}) = S_{max}(v) - g_i(v_{-i}). \tag{3.6}$$

As the right-hand side of (3.6) does not depend on d, a truth telling buyer receives the same utility at all VC mechanisms. Note that truth revealing is indeed a dominant strategy in a VC mechanism.

In the next section we discuss other (not truth revealing) individually rational ex post equilibrium profiles in the VC mechanisms. We will focus on **player symmetric** equilibria $b = (b_i)_{i \in N}$, where $b_i = b_j$ for all $i, j \in N$, which are in equilibrium in **every** VC mechanism.

4 Bundling equilibrium

Let $\Sigma \subseteq 2^A$ be a family of bundles of goods. We deal only with such families Σ for which

- $\emptyset \in \Sigma$.

A valuation function v_i is a **Σ-valuation function** if

$$v_i(B) = \max_{C \in \Sigma, C \subseteq B} v_i(C), \quad \text{for every } B \in 2^A.$$

The set of all Σ-valuation functions in V_i is denoted by V_i^Σ. We further denote $V^\Sigma = \times_{i \in N} V_i^\Sigma$. For every valuation function v_i we denote by v_i^Σ its projection on V_i^Σ, that is:

$$v_i^\Sigma(B) = \max_{C \in \Sigma, C \subseteq B} v_i(C), \quad \text{for every } B \in 2^A.$$

Obviously $v_i^\Sigma \in V_i^\Sigma$, and for $v_i \in V_i^\Sigma$, $v_i^\Sigma = v_i$. In particular $(v_i^\Sigma)^\Sigma = v_i^\Sigma$ for every $v_i \in V_i$. Let $f_\Sigma : V_i \to V_i^\Sigma$ be the projection function defined by

$$f^\Sigma(v_i) = v_i^\Sigma.$$

An allocation γ is a Σ-**allocation** if $\gamma_i \in \Sigma$ for every buyer $i \in N$. The set of all Σ-allocations is denoted by Γ^Σ.

We are interested in the following question: For which Σ, do we have that f^Σ is a player-symmetric individually rational ex post equilibrium in every VC mechanism (with any number of buyers)? In such a case we call f^Σ a **bundling equilibrium** for the VC mechanisms and say that Σ induces a bundling equilibrium. The next example shows that not every Σ induces a bundling equilibrium.

Before we present the example we need the following notation: Let $B \in 2^A$, we denote by w_B the following valuation function:

$$\text{If } B \neq \emptyset,\ w_B(C) = 1 \quad \text{if } B \subseteq C, \text{ and} \quad w_B(C) = 0 \quad \text{otherwise.}^{18}$$

$$\text{If } B = \emptyset,\ w_B(C) = 0 \quad \text{for all } C \in 2^A.$$

Example 1

Let A contain four goods a, b, c, d. Let

$$\Sigma = \{a, d, bcd, abc, A, \emptyset\}.^{19}$$

[18] For $B \neq \emptyset$, a valuation function of the form w_B is called a unanimity TU game in cooperative game theory. An agent with such a valuation function is called by Lehmann, O'Callaghan, and Shoham [19] a single-minded agent.

[19] We omit braces and commas when writing subsets of A.

Let $v_2 = w_a, v_3 = w_d$. Consider buyer 1 with $v_1 = w_{bc}$. Note that $v_i \in V_i^{\Sigma}$ for $i = 2, 3$. If buyer 1 uses f^{Σ} he declares $v_1'(bcd) = v_1'(abc) = v_1'(A) = 1$ and $v_1'(C) = 0$ for all other C, and there exists a VC mechanism that allocates a to 2, d to 3 and bc to the seller. In this mechanism the utility of 1 from using f^{Σ} is zero. On the other hand, if agent 1 reports the truth (w_{bc}) he receives (in every VC mechanism) bc and pays nothing. Hence, his utility would be 1. Therefore f^{Σ} is not in equilibrium in this VC mechanism, and hence Σ does not induce a bundling equilibrium.

4.1 A characterization of bundling equilibria

$\Sigma \subseteq 2^A$ is called a **quasi field** if it satisfies the following properties:[20]

- $B \in \Sigma$ implies that $B^c \in \Sigma$, where $B^c = A \setminus B$.
- $B, C \in \Sigma$ and $B \cap C = \emptyset$ imply that $B \cup C \in \Sigma$.[21]

Theorem 1 Σ *induces a bundling equilibrium if and only if it is a quasi field.*

Proof:

Suppose Σ is a quasi field:

Consider a VC mechanism with an allocation function d. We show that f^{Σ} is an individually rational ex post equilibrium in this VC mechanism.

Assume that every buyer j, $j \neq i$, uses the strategy $b_j = f^{\Sigma}$. Let $v_{-i} \in V_{-i}$. We have to show that for buyer i with valuation v_i, v_i^{Σ} is a best reply to v_{-i}^{Σ}. As truth revealing is a dominating strategy in every VC mechanism, it suffices to show that buyer i's utility when submitting v_i^{Σ} is the same as when submitting v_i.[22] That is, we need to show that

[20] Recall our assumption that we deal only with Σ such that $\emptyset \in \Sigma$.

[21] Equivalently, the union of any number of pairwise disjoint sets in Σ is also in Σ.

[22] Note that this will imply not only that f^{Σ} is in equilibrium but also that it is individually rational.

$$S_{max}(v_i, v^{\Sigma}_{-i}) - \alpha = S((v_i, v^{\Sigma}_{-i}), \gamma) - \alpha,$$

where $\alpha = g_i(v^{\Sigma}_{-i})$, and $\gamma = d(v^{\Sigma}_i, v^{\Sigma}_{-i})$.

Hence, we have to show that

$$S_{max}(v_i, v^{\Sigma}_{-i}) = S((v_i, v^{\Sigma}_{-i}), \gamma). \tag{4.1}$$

Obviously,

$$S_{max}(v_i, v^{\Sigma}_{-i}) \geq S((v_i, v^{\Sigma}_{-i}), \gamma). \tag{4.2}$$

As $v_i(B) \geq v^{\Sigma}_i(B)$ for every $B \in 2^A$,

$$S((v_i, v^{\Sigma}_{-i}), \gamma) \geq S((v^{\Sigma}_i, v^{\Sigma}_{-i}), \gamma) = S_{max}(v^{\Sigma}_i, v^{\Sigma}_{-i}). \tag{4.3}$$

Let $\xi = d(v_i, v^{\Sigma}_{-i})$. For $j \neq i$ and $j \neq 0$, let $\xi^{\Sigma}_j \in \Sigma$ be such that $\xi^{\Sigma}_j \subseteq \xi_j$ and $v^{\Sigma}_j(\xi^{\Sigma}_j) = v^{\Sigma}_j(\xi_j)$. Let $\xi^{\Sigma}_i = (\cup_{j \neq 0, i} \xi^{\Sigma}_j)^c$, and let $\xi^{\Sigma}_0 = \emptyset$.

Because Σ is a quasi field, $\xi^{\Sigma}_i \in \Sigma$, and hence $\xi^{\Sigma} \in \Gamma^{\Sigma}$. As $\xi_i \subseteq \xi^{\Sigma}_i$, ξ^{Σ} is also optimal for (v_i, v^{Σ}_{-i}). However

$$S((v_i, v^{\Sigma}_{-i}), \xi^{\Sigma}) = S((v^{\Sigma}_i, v^{\Sigma}_{-i}), \xi^{\Sigma}) \leq S_{max}(v^{\Sigma}_i, v^{\Sigma}_{-i}). \tag{4.4}$$

Combining (4.2), (4.3), and (4.4) yields

$$S_{max}(v^{\Sigma}_i, v^{\Sigma}_{-i}) \geq S_{max}(v_i, v^{\Sigma}_{-i}) \geq S((v_i, v^{\Sigma}_{-i}), \gamma) \geq S_{max}(v^{\Sigma}_i, v^{\Sigma}_{-i}).$$

Therefore (4.1) holds.

Suppose Σ induces a bundling equilibrium:

We first show that if $B \in \Sigma$, then $B^c \in \Sigma$. If $B = A$ then by definition $B^c = \emptyset \in \Sigma$. Let $B \subset A$. Assume, for the sake of contradiction, that $B^c \notin \Sigma$. Let $v_2 = w_B$ and $v_1 = w_{B^c}$. Note that $v^{\Sigma}_2 = v_2$. Thus, if buyer 2 uses f^{Σ}, he declares v_2. If buyer 1 uses f^{Σ}, he declares v^{Σ}_1, where $v^{\Sigma}_1(B^c) = 0$. Hence, there exists a VC mechanism d, that allocates B to agent 2 and B^c to the seller. However, if buyer 1 deviates and declares his true valuation, then this

VC mechanism allocates to him B^c, and he pays nothing. Hence, there is a profitable deviation from f^{Σ}, a contradiction.

Next, we show that if $B, C \in \Sigma$ are disjoint then $B \cup C \in \Sigma$. By the first part of the proof, it suffices to show that $(B \cup C)^c \in \Sigma$. Clearly, we may assume that the sets B, C, and $(B \cup C)^c$ are all non empty. Assume, for the sake of contradiction, that $(B \cup C)^c \notin \Sigma$. Consider three buyers with valuations $v_1 = w_{(B \cup C)^c}$, $v_2 = w_B$, $v_3 = w_C$. Proceeding as in the first part of the current part of the proof yields a similar contradiction. ▮

It may be useful to note that if f^{Σ} is a buyer-symmetric equilibrium for a fixed set of buyers, then Σ is not necessarily a quasi field. For example, if there is only one buyer, every Σ such that $A \in \Sigma$ induces an equilibrium. In the case of two buyers, being closed under complements is necessary and sufficient for Σ to induce an equilibrium. However, it can be deduced from the proof of the only if part of Theorem 1, that for a fixed set of buyers N, if $n = |N| \geq 3$, then Σ must be a quasi field if it induces an equilibrium for the set of buyers N.

4.2 Partition-based equilibrium

Let $\pi = \{A_1, ..., A_k\}$ be a partition of A into non empty parts. That is, $A_i \neq \emptyset$ for every $A_i \in \pi$, $\cup_{i=1}^{k} A_i = A$, and $A_i \cap A_j = \emptyset$ for every $i \neq j$. Let Σ_π be the field generated by π. That is, Σ_π contains all the sets of goods of the form $\cup_{i \in I} A_i$, where $I \subseteq \{1, ..., k\}$. To avoid confusion: $\emptyset \in \Sigma_\pi$. For convenience, we will use f^π to denote f^{Σ_π}. A corollary of Theorem 1 is:

Corollary 1 *f^π is a bundling equilibrium.*

Proof: As Σ_π is a field it is in particular a quasi field. Hence, the proof follows from Theorem 1. ▮

A bundling equilibrium of the form f^π, where π is a partition, will be called a **partition-based equilibrium**. Thus, a partition-based equilibrium is a bundling equilibrium f^{Σ} that is based on a field $\Sigma = \Sigma_\pi$. It is important to note that there exist quasi fields, which are not fields. For example, let $A = \{a, b, c, d\}$. $\Sigma = \{ab, cd, ac, bd, A, \emptyset\}$ is a quasi field, which is not a field. We note, however, that when $m = |A| \leq 3$, the notions of quasi field and field coincide.

5 Surplus and communication complexity

Let $\Sigma \subseteq 2^A$. If every buyer uses f^Σ, then in every VC mechanism d, the total surplus generated when the types of the buyers are given by $v \in V$ is $S(v^\Sigma, d(v^\Sigma)) = S_{max}(v^\Sigma)$.

We denote

$$S_{\Sigma-max}(v) = \max_{\gamma \in \Gamma^\Sigma} S(v, \gamma).$$

Obviously,

$$S_{\Sigma-max}(v) = S_{\Sigma-max}(v^\Sigma) = S_{max}(v^\Sigma), \quad \text{for every } v \in V.$$

For convenience we denote $S_{\Sigma-max}$ by S_Σ, and we call S_Σ the **Σ-optimal surplus function** (note that $S_{2^A} = S_{max}$). When Σ is a field generated by a partition π we write S_π for S_{Σ_π}.

If Σ is a quasi field we say that the **communication complexity** of the equilibrium f^Σ is the number of bundles in Σ, that is $|\Sigma|$. Notice that this is a natural definition because a buyer who is using f^Σ has to submit a vector of $|\Sigma|$ numbers to the seller.[23] Thus, if π is a partition, the communication complexity is $2^{|\pi|}$. If $\Sigma_1 \subseteq \Sigma_2$, then $S_{\Sigma_1}(v) \leq S_{\Sigma_2}(v)$ for every $v \in V$. So, Σ_2 induces more surplus (a proxy for economic efficiency) than Σ_1, but Σ_2 also induces higher communication complexity. Hence, there is a tradeoff between economic efficiency and computational complexity. The next example shows

[23] A discussion of the way this can be extended to deal with the introduction of concise bidding languages [26, 4] is beyond the scope of this paper.

that as far as the **revenue** of the seller is concerned, there is no clear comparison between the revenues obtained by quasi fields ranked by inclusion.[24] Before we present the example, note that for two partitions π_1, π_2, $\Sigma_{\pi_1} \subseteq \Sigma_{\pi_2}$ if and only if π_2 refines π_1.

Example 2

Assume there are two buyers, $N = \{1, 2\}$, and two goods, $A = \{a, b\}$. Assume $v_1 = w_a$ and $v_2 = w_b$. In any VC mechanism, in the truth revealing equilibrium buyer 1 gets a, buyer 2 gets b, and they pay nothing. Hence the level of social surplus is 2 and the revenue of the seller at $v = (v_1, v_2)$ is zero. Let π be the trivial partition $\{A\}$ ($\Sigma_\pi = \{\emptyset, A\}$). If each buyer uses the equilibrium strategy f^π, they both report w_A. Hence, one of the buyers gets ab and pays 1. The seller collects a revenue of 1, and the social surplus equals 1. Hence, $S_{max}(v) > S_\pi(v)$ and $R(v) < R_\pi(v)$. On the other hand, if $N = \{1, 2, 3, 4\}$ where v_1 and v_2 are defined as before and $v_3 = v_1$ and $v_4 = v_2$, $S_{max}(v) = 2$ and $R(v) = 2$ while $S_\pi(v) = 1$ and $R_\pi(v) = 1$.

For every family of bundles Σ with $A \in \Sigma$ we define

$$r^n_\Sigma = \sup_{v \in V, v \neq 0} \frac{S_{max}(v)}{S_\Sigma(v)}, \tag{5.1}$$

where $V = V_1 \times \cdots \times V_n$. Thus, r^n_Σ is a worst-case measure of the economic inefficiency that may result from using the strategy f^Σ when there are n buyers. Obviously $r^n_\Sigma \geq 1$, and equality holds for $\Sigma = 2^A$. A standard argument using homogeneity and continuity of S_{max}/S_Σ shows that the supremum in (5.1) is attained, i.e., it is a maximum.

The following remark gives a simple upper bound on the inefficiency associated with Σ.

Remark 1 For every $\Sigma \subseteq 2^A$ with $A \in \Sigma$, and for every $v \in V$,

$$S_{max}(v) \leq n S_\Sigma(v),$$

[24]In spite of our example, it is commonly believed that social optimality is a good proxy for revenue. This was proved to be asymptotically correct when the number of buyers is large, and the organizer has a Bayesian belief over the distribution of valuation functions, which assumes independence across buyers (see [24]).

where n is the number of potential buyers. Consequently,

$$r_\Sigma^n \leq n.$$

Proof: Let $\gamma = d(v)$, where d is any VC mechanism.

$$S_{max}(v) = S(v, \gamma) = \sum_{i \in N} v_i(\gamma_i) \leq \sum_{i \in N} v_i(A) = \sum_{i \in N} v_i^\Sigma(A) \leq n S_\Sigma(v).$$

∎

However, we are interested mainly in upper bounds on the economic inefficiency that are independent of the number of buyers. For every family of bundles Σ with $A \in \Sigma$ we define

$$r_\Sigma = \sup_{n \geq 1} r_\Sigma^n. \tag{5.2}$$

It is easy to see that, since any allocation assigns non empty bundles to at most $m = |A|$ buyers, the supremum in (5.2) is attained for some $n \leq m$. When $\Sigma = \Sigma_\pi$ for a partition π, we write r_π instead of r_{Σ_π}.

In the following subsection we characterize and estimate r_π, thereby obtaining a quantitative form of the tradeoff between communication and economic efficiency in partition-based equilibria.

5.1 Communication efficiency vs. economic efficiency in partition-based equilibria

We first express r_π in terms of the partition $\pi = \{A_1, ..., A_k\}$ only. A **feasible** family for π is a family $\Delta = (H_i)_{i=1}^s$ of (not necessarily distinct) subsets of $\{1, ..., k\}$ satisfying the following two conditions:

- $H_i \cap H_j \neq \emptyset$ for every $1 \leq i, j \leq s$.
- $|\{i : l \in H_i\}| \leq |A_l|$ for every $1 \leq l \leq k$.

We write $s = s(\Delta)$ for the number of sets in the family Δ (counted with repetitions).

Theorem 2 *For every partition π,*

$$r_\pi = \max s(\Delta),$$

where the maximum is taken over all families Δ that are feasible for π.

Proof:

We first prove that $r_\pi \leq \max s(\Delta)$. It suffices to show that for every $v \in V$ there exists a feasible family Δ for π such that

$$S_{max}(v) \leq s(\Delta) \cdot S_\pi(v).$$

Let $v \in V$. Let γ be a socially optimal allocation. That is,

$$S_{max}(v) = \sum_{i \in N} v_i(\gamma_i).$$

For every γ_i let γ_i^π be the minimal set in Σ_π that contains γ_i. That is $\gamma_i^\pi = \cup_{l \in J_i} A_l$ where $J_i = \{l \in \{1, \ldots, k\} : A_l \cap \gamma_i \neq \emptyset\}$.

Let ξ be a partition of N to r subsets, such that for every $i, j \in I \in \xi$, $i \neq j$, $\gamma_i^\pi \cap \gamma_j^\pi = \emptyset$. Assume r is the **minimal** cardinality of such a partition. For every $I \in \xi$ let $H_I = \cup_{i \in I} J_i$. That is, each H_I is a set of indices of parts A_l in π that should be allocated to the buyers in I in order for each of them to get the goods they received in the optimal allocation γ. Note that if $I \neq J$, $H_I \cap H_J \neq \emptyset$, otherwise we can join I and J together in contradiction to the minimality of the cardinality of ξ. Hence, $\Delta = (H_I)_{I \in \xi}$ is a family of subsets of $\{1, ..., k\}$ that satisfies that any two subsets in Δ intersect. Furthermore, the second condition for feasibility is also satisfied, because for any given $l \in \{1, ..., k\}$ there are at most $|A_l|$ buyers i with $\gamma_i \cap A_l \neq \emptyset$, and hence at most $|A_l|$ parts $I \in \xi$ such that $l \in H_I$. Thus, Δ is a feasible family for π with $s(\Delta) = r$.

Every H_I, $I \in \xi$ defines a Σ_π-allocation. In this allocation every $i \in I$ receives γ_i^π, and the seller receives all other goods. Therefore $\sum_{i \in I} v_i(\gamma_i^\pi) \leq S_\pi(v)$ for every $I \in \xi$. Hence,

$$S_{max}(v) \leq \sum_{i \in N} v_i(\gamma_i^\pi) = \sum_{I \in \xi} \sum_{i \in I} v_i(\gamma_i^\pi) \leq \sum_{I \in \xi} S_\pi(v) = r S_\pi(v).$$

Next, we prove that $r_\pi \geq \max s(\Delta)$. It suffices to show that for every feasible family Δ for π there exists a profile of valuations $v = (v_1, ..., v_n) \neq 0$ for some number n of buyers satisfying

$$S_{max}(v) \geq s(\Delta) \cdot S_\pi(v).$$

Let $\Delta = (H_i)_{i=1}^s$ be a feasible family for π. By the second condition of feasibility, we can associate with each H_i a set of goods B_i containing one good from each A_l such that $l \in H_i$, in such a way that the sets B_i are pairwise disjoint. By the first condition of feasibility, for every $1 \leq i, j \leq s$ there can be no two disjoint sets $C_i, C_j \in \Sigma_\pi$ such that $B_i \subseteq C_i$, $B_j \subseteq C_j$.

Now, we take $n = s$ buyers, and let buyer i have the valuation $v_i = w_{B_i}$. Then $S_{max}(v) = s$ whereas $S_\pi(v) = 1$. ▮

Theorem 2 reduces the determination of the economic inefficiency measure r_π to a purely combinatorial problem. However, this combinatorial problem does not admit an easy solution.[25] Nevertheless, we will use Theorem 2 to calculate r_π in some special cases, and to obtain a general upper bound for it which is tight in infinitely many cases.

The following proposition determines r_π for partitions π with a small number of parts. We use the notations $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ for the lower and upper integer rounding functions, respectively.

Proposition 1 *Let $|A| = m$, and let $\pi = \{A_1, ..., A_k\}$ be a partition of A into k non empty sets.*

- *If $k = 1$ then $r_\pi = m$.*
- *If $k = 2$ then $r_\pi = \max\{|A_1|, |A_2|\}$. Consequently, the minimum of r_π over all partitions of A into 2 parts is $\lceil \frac{m}{2} \rceil$.*

[25] The special case of this problem, in which $|A_i| = |A_j|$ for all $A_i, A_j \in \pi$, has been treated in the combinatorial literature using a different but equivalent terminology (see e.g. [11]). But even in this case, a precise formula for $\max s(\Delta)$ seems out of reach.

- *If $k = 3$ then $r_\pi = \max\{|A_1|, |A_2|, |A_3|, \lfloor \frac{m}{2} \rfloor\}$. Consequently, the minimum of r_π over all partitions of A into 3 parts is $\lfloor \frac{m}{2} \rfloor$.*

Proof:

In each case, we determine the maximum of $s(\Delta)$ over all families Δ that are feasible for π.

For $k = 1$, a feasible family consists of at most $|A_1| = m$ copies of $\{1\}$, and therefore $\max s(\Delta) = m$.

A feasible family for $k = 2$ cannot contain two sets, H_i and H_j, such that $1 \notin H_i$ and $2 \notin H_j$, because such sets would be disjoint. Hence, for any feasible family Δ, either all sets contain 1 or all of them contain 2. Therefore, $s(\Delta) \leq \max\{|A_1|, |A_2|\}$. On the other hand, feasible families of size $|A_1|$, $|A_2|$ trivially exist.

Suppose $k = 3$, and denote

$$\beta_l = |A_l| \quad \text{for } l = 1, 2, 3.$$

We first show that $s(\Delta) \leq \max\{\beta_1, \beta_2, \beta_3, \lfloor \frac{m}{2} \rfloor\}$ for every feasible family Δ. If Δ contains some singleton $\{l\}$, then all sets in Δ must contain l, and hence $s(\Delta) \leq \beta_l$. Otherwise, Δ consists of s_{12} copies of $\{1, 2\}$, s_{13} copies of $\{1, 3\}$, s_{23} copies of $\{2, 3\}$, and s_{123} copies of $\{1, 2, 3\}$, for some non negative integers $s_{12}, s_{13}, s_{23}, s_{123}$. We have the following inequalities:

$$s_{12} + s_{13} + s_{123} \leq \beta_1,$$

$$s_{12} + s_{23} + s_{123} \leq \beta_2,$$

$$s_{13} + s_{23} + s_{123} \leq \beta_3.$$

Upon adding these inequalities we obtain

$$2(s_{12} + s_{13} + s_{23}) + 3s_{123} \leq m,$$

which implies

$$s(\Delta) = s_{12} + s_{13} + s_{23} + s_{123} \leq \lfloor \frac{m}{2} \rfloor.$$

We show next that there exists a feasible family Δ with $s(\Delta) = \max\{\beta_1, \beta_2, \beta_3, \lfloor \frac{m}{2} \rfloor\}$. If this maximum is one of the β_l's, this is trivial. So assume that $\beta_l < \lfloor \frac{m}{2} \rfloor$ for $l = 1, 2, 3$. If m is even then the family Δ that consists of

$$s_{12} = \frac{\beta_1 + \beta_2 - \beta_3}{2} \quad \text{copies of } \{1, 2\},$$

$$s_{13} = \frac{\beta_1 + \beta_3 - \beta_2}{2} \quad \text{copies of } \{1, 3\},$$

$$s_{23} = \frac{\beta_2 + \beta_3 - \beta_1}{2} \quad \text{copies of } \{2, 3\},$$

is feasible (note that the prescribed numbers are non negative because $\beta_l < \lfloor \frac{m}{2} \rfloor$ for $l = 1, 2, 3$, and they are integers because $\beta_1 + \beta_2 + \beta_3 = m$ is even). The size of this family is $s(\Delta) = s_{12} + s_{13} + s_{23} = \frac{m}{2}$. If m is odd, we make slight changes in the values of s_{12}, s_{13}, s_{23}: we add $\frac{1}{2}$ to one of them and subtract $\frac{1}{2}$ from the other two. In this way we get a family Δ with $s(\Delta) = \lfloor \frac{m}{2} \rfloor$. ∎

We see from Proposition 1 that if we use partitions into two parts (entailing a communication complexity of 4), the best we can do in terms of economic efficiency is $r_\pi = \lceil \frac{m}{2} \rceil$, and this is achieved by partitioning A into equal or nearly equal parts. Allowing for three parts (and therefore a communication complexity of 8) permits only a small gain in r_π (in fact, no gain at all when m is even).

We will now state the two parts of our main result.

Theorem 3 *Let $\pi = \{A_1, ..., A_k\}$ be a partition of A into k non empty sets of maximum size $\beta(\pi)$. (That is, $\beta(\pi) = \max\{|A_1|, ..., |A_k|\}$.) Then*

$$r_\pi \leq \beta(\pi) \cdot \varphi(k),$$

where

$$\varphi(k) = \max_{j=1,\dots,k} \min\{j, \frac{k}{j}\}.$$

The proof of Theorem 3 is given in the following subsection.

Note that

$$\varphi(k) \leq \sqrt{k}.$$

In particular, if all sets in π have equal size $\frac{m}{k}$, we obtain the upper bound

$$r_\pi \leq \frac{m}{\sqrt{k}}.$$

Now, consider the case when, for some non negative integer q, we have

$$k = q^2 + q + 1, \quad \text{and} \tag{5.3}$$

$$|A_i| = q + 1 \quad \text{for } i = 1, ..., k. \tag{5.4}$$

In this case

$$\varphi(k) = \frac{q^2 + q + 1}{q + 1},$$

and hence the upper bound of Theorem 3 takes the form

$$r_\pi \leq k.$$

The second part of our main result implies that in infinitely many of these cases this upper bound is tight.

Theorem 4 *Let $\pi = \{A_1, ..., A_k\}$ be a partition that satisfies (5.3) and (5.4) for some q which is either 0 or 1 or of the form p^l where p is a prime number and l is a positive integer. Then*

$$r_\pi = k.$$

We prove Theorems 3 and 4 in the following subsection.

5.2 Proofs of Theorems 3 and 4

We begin with some preparations. Let $\Delta = (H_i)_{i=1}^s$ be a family of (not necessarily distinct) subsets of $\{1, ..., k\}$. A vector of non negative numbers $\delta = (\delta_i)_{i=1}^s$ is called a **semi balanced**[26] vector for Δ if for every $l \in \{1, ..., k\}$,

$$\sum_{i:l \in H_i} \delta_i \leq 1.$$

Proposition 2 *Let* $\Delta = (H_i)_{i=1}^s$ *be a family of (not necessarily distinct) subsets of* $\{1, ..., k\}$ *such that* $H_i \cap H_j \neq \emptyset$ *for every* $1 \leq i, j \leq s$. *Let* $\delta = (\delta_i)_{i=1}^s$ *be a semi balanced vector for* Δ. *Then*

$$\sum_{i=1}^s \delta_i \leq \varphi(k),$$

where

$$\varphi(k) = \max_{j=1,...,k} \min\{j, \frac{k}{j}\}.$$

Proof:

Assume without loss of generality that $h = |H_1|$ is the minimal number of elements in a member of Δ. The proposition will be proved if we prove the following two claims:

Claim 1: $\sum_{i=1}^s \delta_i \leq h$.

Claim 2: $\sum_{i=1}^s \delta_i \leq \frac{k}{h}$.

Proof of Claim 1:

Let $z = \sum_{l \in H_1} \sum_{i:l \in H_i} \delta_i$. As every H_i intersects H_1, every δ_i appears in z at least once. Therefore, $z \geq \sum_{i=1}^s \delta_i$. Because δ is semi balanced, $\sum_{i:l \in H_i} \delta_i \leq 1$ for every l, and in particular for $l \in H_1$. Hence, $z \leq \sum_{l \in H_1} 1 = h$. ∎

Proof of Claim 2:

[26] This concept is equivalent to what is called a fractional matching in combinatorics. We chose the term semi balanced, because balanced vectors, defined by requiring equality instead of weak inequality, are a familiar concept in game theory (see e.g. [36]).

Let $w = \sum_{l=1}^{k} \sum_{i: l \in H_i} \delta_i$. Every δ_i appears in w exactly $|H_i|$ times. Since $|H_i| \geq h$ for every i, we have $w \geq h \sum_{i=1}^{s} \delta_i$. On the other hand, as in the proof of Claim 1, we obtain $w \leq \sum_{l=1}^{k} 1 = k$. Combining the two inequalities, we get $\sum_{i=1}^{s} \delta_i \leq \frac{k}{h}$. ∎

Therefore,

$$\sum_{i=1}^{s} \delta_i \leq \min\{h, \frac{k}{h}\} \leq \varphi(k).$$

∎

We are now ready for the proof of Theorem 3:

Proof of Theorem 3:

Let $\pi = \{A_1, ..., A_k\}$ be a partition of A into k non empty sets of maximum size $\beta(\pi)$. We have to prove that $r_\pi \leq \beta(\pi) \cdot \varphi(k)$. By Theorem 2, it suffices to show that for every feasible family Δ for π, we have

$$s(\Delta) \leq \beta(\pi) \cdot \varphi(k).$$

Let $\Delta = (H_i)_{i=1}^{s}$ be such a family. Consider the vector $\delta = (\delta_i)_{i=1}^{s}$ with

$$\delta_i = \frac{1}{\beta(\pi)}, \quad i = 1, ..., s.$$

By the second condition of feasibility, this vector is semi balanced. Hence we may apply Proposition 2 and conclude that

$$\sum_{i=1}^{s} \delta_i \leq \varphi(k),$$

or equivalently,

$$\frac{s}{\beta(\pi)} \leq \varphi(k),$$

as required. ∎

In order to prove Theorem 4 we invoke a result about finite geometries (see e.g. [9]). A **finite projective plane** of order q is a system consisting of a set Π of points and a set Λ of lines (in this abstract setting, a line is just a set of points, i.e., $L \subseteq \Pi$ for every $L \in \Lambda$), satisfying the following conditions:

- $|\Pi| = |\Lambda| = q^2 + q + 1$.
- Every point is incident to $q+1$ lines and every line contains $q+1$ points.
- There is exactly one line containing any two points, and there is exactly one point common to any two lines.

Such a system does not exist for every q. However, it trivially exists for $q = 0$ (a single point) and for $q = 1$ (a triangle) and it is known to exist for every q of the form $q = p^l$, where p is a prime number and l is a positive integer. The first non trivial example, corresponding to $q = 2$, is called the **Fano plane**:

$$\Pi = \{1, 2, 3, 4, 5, 6, 7\},$$

$$\Lambda = \{124, 235, 346, 457, 561, 672, 713\}.$$

Proof of Theorem 4:

Let $\pi = \{A_1, ..., A_k\}$ be a partition that satisfies (5.3) and (5.4) for some q which is either 0 or 1 or of the form p^l where p is a prime number and l is a positive integer. As $r_\pi \leq k$ follows from Theorem 3 (see the discussion preceding the statement of Theorem 4), we need to prove only that $r_\pi \geq k$. By Theorem 2, it suffices to show that there exists a family Δ with $s(\Delta) = k$ which is feasible for π. Such a family is given by the system of lines of a projective plane of order q, when the points are identified with $1, ..., k$. ∎

5.3 More on the ranking of equilibria

The tradeoff between communication complexity and economic efficiency, as delineated above, may be made concrete by the following scenario. Suppose that a set A of m goods is given, and we are in a position to recommend to the potential buyers an equilibrium strategy. Assume further that a certain level M of communication complexity is considered the maximum acceptable level. If we are going to recommend a partition-based equilibrium f^π, then the number of parts in π should be at most $k = \lfloor \log_2 M \rfloor$. From the viewpoint of economic efficiency, we would like to choose such a partition π with r_π as low as possible. Which partition should it be?

According to Theorem 3, we obtain the lowest guarantee on r_π by making the maximum size of a part in π as small as possible, which means splitting A into k equal (or nearly equal, depending on divisibility) parts. This leads to the question whether, for given m and k, the lowest value of r_π itself (not of our upper bound) over all partitions π of A into k parts is achieved at an equi-partition, i.e., a partition $\pi = \{A_1, ..., A_k\}$ such that $\lfloor \frac{m}{k} \rfloor \leq |A_i| \leq \lceil \frac{m}{k} \rceil$, $i = 1, ..., k$.

While Proposition 1 gives an affirmative answer for $k = 1, 2, 3$, it turns out, somewhat surprisingly, that this is not always the case. This is shown in the following example.

Example 3

Let $m = 21$ and $k = 7$. If π is an equi-partition of the 21 goods into 7 triples then, by Theorem 4, $r_\pi = 7$. Consider now a partition $\pi' = \{A_1, ..., A_7\}$ in which

$$|A_1| = 2, |A_2| = 4, |A_3| = \cdots = |A_7| = 3.$$

We claim that $r_{\pi'} \leq 6$.

In order to prove this, it suffices to show that there exists no feasible family of 7 sets for π'. Suppose, for the sake of contradiction, that $\Delta = (H_i)_{i=1}^7$ is such a family. Let H_i be an arbitrary set in Δ. It follows from the second condition of feasibility that if H_i contains the element 1 then it shares it with at most one other set in Δ. Similarly, if H_i contains the element 2 then it shares it with at most three other sets in Δ. For $l = 3, ..., 7$, if H_i contains the element l then it shares it with at most two other sets in Δ. This implies that H_i must contain at least three elements (because it must share an element with every other set, and $3 + 2 < 6$). Moreover, if H_i contains exactly three elements and one of them is 1, then it also contains 2 (since

$1+2+2<6$). On the other hand, we have

$$\sum_{i=1}^{7}|H_i| = \sum_{i=1}^{7}\sum_{l\in H_i}1 = \sum_{l=1}^{7}\sum_{i:l\in H_i}1 = \sum_{l=1}^{7}|\{i : l\in H_i\}| \le \sum_{l=1}^{7}|A_l| = 21.$$

Since every H_i has at least three elements, it follows that every H_i has exactly three elements, and all the weak inequalities $|\{i : l \in H_i\}| \le |A_l|$ must in fact hold as equalities. In particular, there exist two sets in Δ, say H_i and H_j, that contain the element 1. By the above, they both contain 2 as well. Let l be the third element of H_i. Then among the remaining five sets in Δ, the set H_i shares the element 1 with none of them, it shares the element 2 with two of them, and the element l with at most two of them. This contradicts the fact that H_i intersects every other set in Δ.

It can be checked that in fact $r_{\pi'} = 6$ and this is the lowest achievable value among all partitions of 21 goods into 7 sets. We omit the detailed verification of this.

The tradeoff between communication complexity and economic efficiency was quantitatively analyzed above only for partition-based equilibria. It is natural to ask whether it is possible to beat this tradeoff using the more general bundling equilibria. The answer is, in a sense made precise below: sometimes yes, but not by much.

Example 4

Assume that the number of goods m is even, and let the set of goods A be partitioned into two equal parts B and C. Consider $\Sigma \subseteq 2^A$ defined by

$$\Sigma = \{D \subseteq A : |D\cap B| = |D \cap C|\}.$$

It is easy to check that Σ is a quasi field, and hence it induces a bundling equilibrium. The communication complexity is

$$|\Sigma| = \sum_{j=0}^{m/2}\binom{m/2}{j}^2 = \sum_{j=0}^{m/2}\binom{m/2}{j}\binom{m/2}{m/2-j} = \binom{m}{m/2}.$$

We claim that $r_\Sigma = 2$. That $r_\Sigma \ge 2$ can be seen by taking two buyers with valuations w_B and w_C, respectively. To see that $r_\Sigma \le 2$, suppose that

v is a profile of valuations for a set of buyers N, and let γ be an optimal allocation. Split the set N into the following two sets:

$$N_B = \{i \in N : |\gamma_i \cap B| \geq |\gamma_i \cap C|\},$$

$$N_C = \{i \in N : |\gamma_i \cap B| < |\gamma_i \cap C|\}.$$

Note that the sets of goods $\gamma_i, i \in N_B$, can be expanded to pairwise disjoint sets of goods that belong to Σ. In other words, there exists a Σ-allocation ξ such that $\gamma_i \subseteq \xi_i$ for every $i \in N_B$. Similarly, there exists a Σ-allocation η such that $\gamma_i \subseteq \eta_i$ for every $i \in N_C$. Hence

$$S_{max}(v) = \sum_{i \in N} v_i(\gamma_i) = \sum_{i \in N_B} v_i(\gamma_i) + \sum_{i \in N_C} v_i(\gamma_i) \leq \sum_{i \in N} v_i(\xi_i) + \sum_{i \in N} v_i(\eta_i) \leq 2S_\Sigma(v).$$

Thus, $r_\Sigma \leq 2$.

We claim further that if a partition π of A satisfies $r_\pi \leq 2$ then $|\Sigma_\pi| \geq 2^{m-2}$. Indeed, suppose $\pi = \{A_1, ..., A_k\}$. It is easy to find a feasible family of 3 sets for π if one of the A_l's has three or more elements, or if three of the A_l's have two elements each. Therefore, $r_\pi \leq 2$ implies that at most two of the sets $A_1, ..., A_k$ have two elements and the rest are singletons. Thus $k \geq m - 2$ and $|\Sigma_\pi| \geq 2^{m-2}$.

Since $\binom{m}{m/2} < 2^{m-2}$ for all even $m \geq 10$, we have the following conclusion: If $m \geq 10$ then every partition-based equilibrium that matches the economic efficiency of f^Σ has a higher communication complexity than f^Σ. In other words, the quasi field Σ offers an efficiency/complexity combination that cannot be achieved or improved upon (in the Pareto sense) by any field.

The above example notwithstanding, the efficiency/complexity combinations which arise from arbitrary quasi fields are still subject to a tradeoff that is not much better than for fields. This is the content of our final remark.

Remark 2 Let $m = |A|$ and let k be a positive integer. Any quasi field $\Sigma \subseteq 2^A$ with $r_\Sigma \leq \frac{m}{k}$ must contain a partition of A into k non empty parts, and therefore must satisfy $|\Sigma| \geq 2^k$.

Proof:

Let there be m buyers, each with valuation w_a for a distinct $a \in A$. For this v we have $S_{max}(v) = m$. If $r_\Sigma \leq \frac{m}{k}$ then we must have $S_\Sigma(v) \geq k$. Hence an optimal Σ-allocation has to assign non empty bundles of goods to at least k buyers. Thus Σ contains k pairwise disjoint non empty sets of goods, and therefore, being a quasi field, also a partition of A into k non empty parts. ∎

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