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Competitive Equilibrium in an Exchange Economy with Indivisibilities*

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We analyze an exchange economy in which (i) all commodities except money are indivisible, (ii) agents' preferences can be described by a reservation value for each bundle of indivisible objects, and (iii) all agents are price-takers. We obtain a necessary and sufficient condition under which market clearing prices exist. Implications for market mechanisms are discussed. *Journal of Economic Literature* Classification Numbers: D51, C78.

1. INTRODUCTION

Consider a seller interested in selling several indivisible objects. If each buyer's valuation of each object is independent of the other object(s) he obtains then a simple auction procedure such as a series of second-price auctions, one for each object, allocates the objects efficiently. If, instead, a buyer's reservation value for an object depends on which other objects he obtains and the buyer has utility for consuming more than one object—we refer to this as interdependent values—then it is not known whether a simple selling mechanism can allocate the objects efficiently.

A recent example of a market with many indivisible objects for sale to buyers with interdependent values is the auction of a few thousand licenses for personal communication services (PCS) spectrum rights conducted by the Federal Communications Commission (FCC). Each bidder's value for a given license depends upon the other licenses it obtains.[1] A second example is the sale of foreclosed real estate by the Resolution Trust Corporation (RTC). Many of the buyers are commercial enterprises who acquire

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[1] See Cramton [6], McMillan [20], and McAfee and McMillan [19] for more on the PCS auction.

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properties for resale. Due to risk aversion, economies of scale and scope, and financial constraints, these buyers can have interdependent values. Consequently, an efficient allocation of resources may not be achieved by a simple auction.

When buyers have interdependent values for several indivisible objects, a Vickrey auction in which bidders submit bids on every conceivable bundle of objects assures an efficient allocation. However, a mechanism in which bids are invited on all bundles is too complex to implement. The combinatorial problem involved in solving for the optimal allocation of bundles to buyers is beyond the capabilities of current optimization techniques.[2] Only simple selling procedures, such as one auction for each object, are practical. Therefore, an immediate question is when do simple auction procedures which determine a price for each object allocate efficiently? And if such auction procedures are usually inefficient, can efficiency be attained in a resale market after the auction?

As a first step towards answering these questions, we analyze an economy in which (i) all commodities except money are indivisible, (ii) agents' preferences can be described by a reservation value for each bundle of indivisible objects, and (iii) all agents are price-takers. We investigate conditions under which perfectly competitive markets are able to allocate resources in this exchange economy. Specifically, we ask whether there exist prices, one for each commodity, such that there is no excess demand for any commodity. After having identified sufficient conditions for existence of market clearing prices, the next step is to investigate whether, under these conditions, simple auction procedures are capable of discovering the competitive equilibrium prices. If market clearing prices do not exist, or if the conditions necessary for existence are too restrictive, then we believe it is unlikely that any simple auction procedure or a post-auction resale market will allocate resources efficiently.

The exchange economy considered here is related to an assignment market (see Koopmans and Beckman [18], and Shapley and Shubik [25]). The difference is that in assignment markets each agent desires only one indivisible objects whereas in our model agents have interdependent values over several indivisible objects. Another related group of papers are Quinzii [22], Gale [11], and Kaneko and Yamamoto [15]. These authors consider an exchange economy which is less general than the one considered in this paper in that each buyer has utility for one object only and more general in that they allow utility functions with income effects.

This paper is also related to the matching literature. Although we do not impose a two-sided restriction in that an agent can be both a buyer

[2] If 50 heterogeneous objects are to be sold to two bidders by means of a Vickrey auction, then 2^{50} possible allocations have to be compared.

and a seller, there is a sense in which this model is equivalent to a two-sided many-to-one matching market. Kelso and Crawford [16] analyze an auction procedure for a two-sided many-to-one matching market. As pointed out later in Section 3.2, Kelso and Crawford's results provide a sufficient condition for existence of competitive equilibrium in our setting.

The problems presented by interdependent values over several indivisible objects have received little attention in the literature on optimal selling mechanisms. In auction theory it is often assumed that either there is one indivisible object for sale or there are several identical units for sale and each buyer's marginal value for more than one unit is zero.[3] Under either assumption, an efficient allocation is achieved by several commonly observed simple auction mechanisms and also by a competitive equilibrium. Other papers, such as Adams and Yellen [1], Harris and Raviv [13], and Palfrey [21], on selling several indivisible objects assume that buyers' values are additive, i.e., independent. The objective in these papers is to devise selling strategies which maximize a monopolist's profits; the existence of efficient mechanisms is not an issue. If buyers' values are additive, then simultaneous sealed bid auctions, one for each object, are efficient under a variety of informational assumptions; moreover, a competitive equilibrium exists and is efficient.

This paper is organized as follows. In Section 3 it is shown that market clearing prices exist if and only if the Pareto-frontier in the exchange economy with indivisible goods coincides with the Pareto-frontier of a transformed economy with only divisible goods. The link between our model and a two-sided matching market is discussed. After stating further assumptions on agents' preferences in Section 4, in Section 5 we apply these necessary and sufficient conditions to the case when agents' reservation value functions are supermodular. Counterexamples to existence of equilibrium are provided in Section 6, and a discussion of implications for simple market mechanisms is given in Section 7. But first, we describe the model in the next section.

2. THE MODEL AND PRELIMINARY RESULTS

Consider an exchange economy with n indivisible commodities and m agents. We assume that there is one unit of each commodity. This is without loss of generality as different units of the same commodity may

[3] Bernheim and Whinston [3] and Kim [17] are exceptions. In these papers, inefficient allocations are likely unless buyers are allowed to bid on bundles.

be treated as different commodities.[4] Each agent $i, i=1, 2, ..., m$, has a reservation value function defined over bundles of objects, $V_i(S)$, for all $S \subseteq N = \{1, 2, ..., n\}$, with $V_i(\phi)=0$. The reservation value functions are weakly increasing, i.e., if $T \subset S$ then $V_i(T) \leqslant V_i(S)$. Free disposal is a sufficient condition for weakly increasing reservation value functions. Agent i is prepared to pay at most $V_i(S)$ for the bundle S. Thus, the utility of an agent with subset S and wealth level w, is

$$U_i(S, w) = V_i(S) + w.$$

This utility function with no income effects is less general than is usually assumed in general equilibrium models; however, this utility specification is common in the auction and bargaining literatures (much of which assumes one indivisible object).

By assumption, wealth is divisible. Agent i's initial endowment of wealth is $\bar{w}_i$. We assume that $\bar{w}_i \geqslant V_i(N)$, $\forall i$: it is feasible for agent i to purchase all commodities in any subset S when the sum of the prices of the objects in S is less than $V_i(S)$. Under this assumption, the initial endowment of objects to the m agents is irrelevant for the existence of market clearing prices. That is, for a given set of reservation value functions for m agents over subsets of n objects, either there exist market clearing prices for all possible initial endowments of n objects to m agents (and the set of market clearing prices and market allocations is the same for all initial endowments) or there do not exist market clearing prices for any initial endowment. Therefore, we leave the initial endowment of objects unspecified.[5] All agents are price takers.[6]

An economy which satisfies the conditions above is denoted by $\mathscr{E}_{\mathrm{I}} = \{N, (V_i, \bar{w}_i), i=1, 2, ..., m\}$.

A *feasible allocation* of objects to agents is an allocation in which no object is assigned more than once. Thus $(S_1, S_2, ..., S_m)$ denotes a feasible allocation in which agent i is allocated the bundle $S_i \subseteq N$, $i=1, 2, ..., m$, and $S_i \cap S_{i'} = \phi$ for all $i \neq i'$. It is possible that in a feasible allocation some agents get nothing ($S_i = \phi$ for some i) and/or some objects are not allocated ($N \backslash (\bigcup_i S_i) \neq \phi$).

[4] See the end of this section for a more precise statement of this claim.

[5] In FCC auctions and in RTC auctions, one agent (the seller) is initially endowed with all the objects.

[6] As there is only one indivisible unit of supply for every object, it may seem unreasonable to assume that sellers are price-takers. However, as explained below, our analysis is easily adapted to the case of many units of each object. Further, an objective of this paper is to investigate when competitive equilibrium prices exist and achieve allocative efficiency so that under these conditions it may be possible to find an auction or some other mechanism capable of discovering these prices.

An *efficient allocation* is a feasible allocation which maximizes the sum of the reservation values of the agents. Thus, $(S_1^*, S_2^*, ..., S_m^*)$ is an efficient allocation if for all other feasible allocations $(S_1, S_2, ..., S_m)$,

$$\sum_{i=1}^{m} V_i(S_i^*) \geqslant \sum_{i=1}^{m} V_i(S_i). \tag{1}$$

As there are a finite number of feasible allocations of commodities to agents, each economy has an efficient allocation. It is possible that at an efficient allocation $(S_1^*, S_2^*, ..., S_m^*)$, some objects are not allocated: $N \backslash (\bigcup_i S_i^*) \neq \phi$.

Clearly, an allocation $(S_1^*, w_1; S_2^*, w_2; ...; S_m^*, w_m)$ of commodities $\{1, 2, ..., n\}$ and wealth $\sum_{i=1}^{m} w_i$ to the agents is Pareto-efficient if and only if $(S_1^*, S_2^*, ..., S_m^*)$ is an efficient allocation. Consequently, the "Pareto-frontier" in the space of reservation values is a hyperplane in $\mathfrak{R}_+^m$ defined by $\sum_{i=1}^{m} v_i = \sum_{i=1}^{m} V_i(S_i^*)$, where $(S_1^*, S_2^*, ..., S_m^*)$ is an efficient allocation. The standard Pareto-frontier in utility space is obtained by translating the Pareto-frontier in reservation value space by an amount equal to the aggregate wealth endowment in the economy.

Market clearing prices are prices, one for each commodity, at which there is no excess demand for any commodity. If such prices exist, then the demand for each object is either one unit or zero and the market results in a feasible allocation. Thus, $p_k \geqslant 0$, $k = 1, 2, ..., n$ are *market clearing prices* (or *competitive equilibrium prices*) if there is a feasible allocation $(S_1, S_2, ..., S_m)$ such that

$$V_i(S_i) - \sum_{k \in S_i} p_k \geqslant V_i(S) - \sum_{k \in S} p_k, \qquad \forall S \subseteq N, \quad \forall i = 1, 2, ..., m, \tag{2}$$

$$\sum_{k=1}^{n} p_k = \sum_{i=1}^{m} \sum_{k \in S_i} p_k. \tag{3}$$

When (2) and (3) are satisfied, $(S_1, S_2, ..., S_m)$ is said to be a *market allocation* which is *supported* by prices $p_1, p_2, ..., p_n$, $p_k \geqslant 0$.

We require (2) to hold for all subsets of N rather than only those in an agent's budget set, whatever that set may be.[7] The reason is that any subset S which lies outside an agent's budget set yields less utility than consuming nothing and will never be chosen. This follows from the assumption $\bar{w}_i \geqslant V_i(N) \geqslant V_i(S)$, $\forall i$ and the fact that if a subset S is outside agent i's budget set then $\sum_{k \in S} p_k > \bar{w}_i$.

[7] The budget set of an agent depends on his initial endowment of goods and money which have not been specified.

TABLE 1

Nonexistence of Competitive Equilibrium

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{2, 3\}$	$\{1, 3\}$	$\{1, 2, 3\}$
$V_A(S)$	0	0	0	3	3	3	4
$V_B(S)$	0	0	0	3	3	3	4

Equation (3) is Walras' law. An equivalent statement of (3) is that the price of any object which is unallocated at a market allocation is zero:

$$p_k = 0, \qquad \forall k \in N \Bigg\backslash \left(\bigcup_{i=1}^{m} S_i \right). \tag{4}$$

We refer to $V_i(S_i) - \sum_{k \in S_i} p_k$, the difference between agent i's reservation value for his allocation S_i and the valuation of S_i at market prices, as agent i's consumer surplus.[8]

The following lemma is immediate:

Lemma 1. *Suppose that* $\bar{w}_i \geqslant V_i(N)$ *and* $\bar{w}'_i \geqslant V_i(N), \forall i$. *Then prices* $(p_1, p_2, ..., p_n)$ *support a feasible allocation in the economy* $\mathscr{E}_{\mathrm{I}} = \{N, (V_i, \bar{w}_i), i = 1, 2, ..., m\}$ *if and only if* $(p_1, p_2, ..., p_n)$ *support the same feasible allocation in the economy* $\mathscr{E}'_{\mathrm{I}} = \{N, (V_i, \bar{w}'_i), i = 1, 2, ..., m\}$.

Thus, since we assume that $\bar{w}_i \geqslant V_i(N)$, the initial endowment of wealth has no further bearing on the existence of market clearing prices. The commodities and reservation value functions are enough to specify the economy and we may write $\mathscr{E}_{\mathrm{I}} = \{N, (V_i), i = 1, 2, ..., m\}$.

It is well known that market clearing prices may not exist in the presence of indivisibilities (see Henry [14], Arrow and Hahn [2], and Ellickson [10]). An example is presented below. There are two agents, A and B, and three objects, 1, 2, and 3. The reservation value functions of A and B are given in Table 1.

In this example, there are two efficient allocations: all three objects are assigned to one of the two agents. In light of Proposition 1 below, these two efficient allocations are the only feasible allocations that could be supported by prices. Any prices, p_1, p_2, p_3, that support the efficient allocation $S_A = \{1, 2, 3\}$, $S_B = \phi$ must satisfy $p_1 + p_2 \geqslant 3$, $p_2 + p_3 \geqslant 3$, and $p_3 + p_1 \geqslant 3$, else B could do better than buy nothing. But this implies that A will not buy $S_A = \{1, 2, 3\}$ at these prices as $p_1 + p_2 + p_3 \geqslant 4 \cdot 5 > 4 = V_A(\{1, 2, 3\})$. Hence, no feasible allocation is supported by market clearing prices.

[8] If the initial endowment is with one group of agents (the sellers) and the final endowment after market exchange is with a different group of agents (the buyers), then this definition coincides with the usual notion of consumer surplus.

However, as the next result shows, when the market works at all, it works well.

PROPOSITION 1. *If market clearing prices exist in an economy $\mathscr{E}_I$, then the market allocation must be an efficient allocation.*

Proof. Suppose that $p_1, p_2, ..., p_n$, $p_k \geqslant 0$, are market clearing prices and that $(S_1^*, S_2^*, ..., S_m^*)$ is the market allocation supported by these prices. Let $(S_1, S_2, ..., S_m)$ be any other allocation. Condition (2) implies that

$$V_i(S_i^*) - \sum_{k \in S_i^*} p_k \geqslant V_i(S_i) - \sum_{k \in S_i} p_k, \qquad \forall i = 1, 2, ..., m.$$

Consequently,

$$\sum_{i=1}^{m} V_i(S_i^*) - \sum_{i=1}^{m} \sum_{k \in S_i^*} p_k \geqslant \sum_{i=1}^{m} V_i(S_i) - \sum_{i=1}^{m} \sum_{k \in S_i} p_k.$$

Let $S^* = \bigcup_{i=1}^{m} S_i^*$ and $S = \bigcup_{i=1}^{m} S_i$. From (4) we know that $p_k = 0$ for all $k \in S \backslash S^*$. Thus, the above inequality implies that

$$\begin{aligned}
\sum_{i=1}^{m} V_i(S_i^*) &\geqslant \sum_{i=1}^{m} V_i(S_i) + \sum_{k \in S^*} p_k - \sum_{k \in S} p_k \\
&= \sum_{i=1}^{m} V_i(S_i) + \sum_{k \in S^* \backslash S} p_k - \sum_{k \in S \backslash S^*} \\
&= \sum_{i=1}^{m} V_i(S_i) + \sum_{k \in S^* \backslash S} p_k \\
&\geqslant \sum_{i=1}^{m} V_i(S_i).
\end{aligned}$$

Hence, $(S_1^*, S_2^*, ..., S_m^*)$ is an efficient allocation. ∎

As stated at the beginning of this section, the assumption of one unit supply of each object is without loss of generality. In case there are multiple units of some objects, one can expand the commodity space by treating each unit of an object as a different commodity. It may be verified that market clearing prices exist in the original economy with multiple units per object if and only if market clearing prices exist in the new economy with one unit per object. Moreover, the sets of market allocations supported by equilibrium prices in the two economies (which may be empty sets) are identical, except for relabelling.

3. A NECESSARY AND SUFFICIENT CONDITION

We show that market clearing prices exist in an economy with indivisibilities if and only if the agents' welfare cannot be improved by "making" the commodities divisible in a sense made precise below.

Let $\mathscr{E}_{\mathrm{I}} = \{N, (V_i), i = 1, 2, ..., m\}$ be an economy with indivisible commodities as defined in the previous section. We define a divisible transformation, $\mathscr{E}_{\mathrm{D}}(N, (V_i))$, of $\mathscr{E}_{\mathrm{I}}$ as follows. Let $S^0, S^1, S^2, ..., S^{2^n - 1}$, where $S^0 = \phi$, be an enumeration of all the subsets of N. Each agent i's divisible allocation in $\mathscr{E}_{\mathrm{D}}(N, (V_i))$ consists of a collection of fractions of subsets and is represented as a $2^n - 1$ vector, $X_i = (x_{i1}, x_{i2}, ..., x_{i2^n - 1})$. If agent i gets fraction f of the jth subset, $j = 1, 2, ..., 2^n - 1$, then $x_{ij} = f$. If $x_{ij} = 0$, $\forall j \geqslant 1$, then agent i gets S^0, the empty subset.

Let A be a $n \times (2^n - 1)$ matrix such that if object k is in subset S^j, $j \geqslant 1$, then $a_{kj} = 1$; otherwise $a_{kj} = 0$. Define $\forall y_{ik} \in [0, 1]$, $\forall k = 1, 2, ..., n$, $\forall i = 1, 2, ..., m$

$$W_i(y_{i1}, y_{i2}, ..., y_{in}) \equiv \max_{X_i} \sum_{j=1}^{2^n - 1} V_i(S^j)\, x_{ij} \tag{5}$$

$$\text{s.t.} \quad \sum_{j=1}^{2^n - 1} a_{kj} x_{ij} \leqslant y_{ik}, \qquad \forall k = 1, 2, ..., n \tag{6}$$

$$\sum_{j=1}^{2^n - 1} x_{ij} \leqslant 1 \tag{7}$$

$$x_{ij} \geqslant 0, \qquad \forall j.$$

Constraints (6) in the above linear program ensure that no more than y_{ik} of each commodity k is allocated at any feasible solution, and constraint (7) restricts the sum of fractions of subsets allocated to less than one. We interpret $W_i(Y_i)$, $Y_i = (y_{i1}, y_{i2}, ..., y_{in})$, as a reservation value of agent i in $\mathscr{E}_{\mathrm{D}}(N, (V_i))$ over the divisible commodity bundle Y_i. Thus, the divisible transformation of $\mathscr{E}_{\mathrm{I}} = \{N, (V_i), i = 1, 2, ..., m\}$ is $\mathscr{E}_{\mathrm{D}}(N, (V_i)) \equiv \{N, (W_i), i = 1, 2, ..., m\}$ where W_i, agent i's reservation value function, is defined in (5) on a divisible commodity space—the unit cube in $\mathfrak{R}^n$. The utility function of agents in $\mathscr{E}_{\mathrm{D}}$ is linear in wealth—$U_i(y_{i1}, y_{i2}, ..., y_{in}, w_i) = W_i(y_{i1}, y_{i2}, ..., y_{in}) + w_i$, $\forall i$. The endowments in $\mathscr{E}_{\mathrm{D}}$ are identical to those in $\mathscr{E}_{\mathrm{I}}$.

The following lemma proves that W_i is a concave extension of V_i from the corners of the unit cube in $\mathfrak{R}^n$ to the entire unit cube.

LEMMA 2. (i) *W_i is a well-defined, finite-valued, concave function.*

(ii) *W_i coincides with V_i at all extreme points of the unit cube in $\mathbb{R}^n$, the only points at which V_i is defined.*

Proof. See Appendix. ∎

It may seem that Lemma 2 cannot be true in the case with multiple units of some goods. Consider an economy with two apples and two oranges. Suppose that an agent strictly prefers a 50 : 50 gamble over two apples and two oranges to a bundle with one apple and one orange. That is,

$$0.5\bar{V}(2, 0) + 0.5\bar{V}(0, 2) > \bar{V}(1, 1). \tag{8}$$

How could this reservation value function have a concave extension? Clearly $\bar{V}(\cdot)$ does not have a concave extension in $\mathfrak{R}^2$; Lemma 2 claims that there is a concave extension in $\mathfrak{R}^4$. First, as outlined at the end of Section 2, convert this economy into a unit endowment economy with four commodities. The first two commodities are apples and the next two are oranges. In this new economy, (8) becomes

$$\begin{aligned} &0.5V(1, 1, 0, 0) + 0.5V(0, 0, 1, 1) \\ &\quad > V(1, 0, 1, 0) = V(0, 1, 1, 0) = V(1, 0, 0, 1) = V(0, 1, 0, 1), \end{aligned} \tag{9}$$

where $V(\cdot)$ is the new reservation value function. A concave extension of $V(\cdot)$ on the unit cube in $\mathfrak{R}^4$ is not ruled out by (9); by Lemma 2, we know that this concave extension exists.

The generalizations of feasible and efficient allocations to the divisible economy are straightforward. A *feasible divisible allocation*, $Y_i = (y_{i1}, y_{i2}, ..., y_{in})$, $i = 1, 2, ..., m$, is one which satisfies $\sum_{i=1}^{m} y_{ik} \leqslant 1$, $\forall k = 1, 2, ..., n$. An *efficient divisible allocation* is a feasible divisible allocation, $Y_1^*, Y_2^*, ..., Y_m^*$, such that for any other feasible divisible allocation, $Y_1, Y_2, ..., Y_m$,

$$\sum_{i=1}^{m} W_i(Y_i^*) \geqslant \sum_{i=1}^{m} W_i(Y_i). \tag{10}$$

A feasible allocation $(S_1, S_2, ..., S_m)$ in an economy $\mathscr{E}_{\mathrm{I}}$ induces in its transformed divisible economy a feasible divisible allocation $(Y_1, Y_2, ..., Y_m)$ where for all i, $y_{ik} = 0$ if $k \notin S_i$ and $y_{ik} = 1$ if $k \in S_i$, $\forall k$. An implication of the definitions and of Lemma 2(ii) is that if $S_1^*, S_2^*, ..., S_m^*$, is an efficient allocation in $\mathscr{E}_{\mathrm{I}}$ and $Y_1^*, Y_2^*, ..., Y_m^*$, is an efficient divisible allocation in $\mathscr{E}_{\mathrm{D}}$ then $\sum_{i=1}^{m} W_i(Y_i^*) \geqslant \sum_{i=1}^{m} V_i(S_i^*)$. Thus, the Pareto-frontiers of $\mathscr{E}_{\mathrm{I}}$ and $\mathscr{E}_{\mathrm{D}}$ are parallel hyperplanes in $\mathfrak{R}_+^m$, with the Pareto-frontier of $\mathscr{E}_{\mathrm{D}}$ (weakly) above that of $\mathscr{E}_{\mathrm{I}}$.

As $\mathscr{E}_{\mathrm{D}}$ is an economy in which all goods are divisible and consumers have concave utility functions, any point on the Pareto-frontier of this economy can be allocated through competitive prices. Therefore, it is to be expected that if the Pareto-frontier of $\mathscr{E}_{\mathrm{I}}$ coincides with that of $\mathscr{E}_{\mathrm{D}}$ then market clearing prices exist in $\mathscr{E}_{\mathrm{I}}$. Surprisingly, the converse is also true.

PROPOSITION 2. *Market clearing prices exist in an indivisible economy* $\mathscr{E}_{\mathrm{I}} = \{N, (V_i), i = 1, 2, ..., m\}$ *if and only if an efficient allocation in* $\mathscr{E}_{\mathrm{I}}$ *induces an efficient divisible allocation in* $\mathscr{E}_{\mathrm{D}}(N, (V_i))$.

It follows from the definitions that if one efficient allocation in $\mathscr{E}_{\mathrm{I}}$ induces an efficient divisible allocation in $\mathscr{E}_{\mathrm{D}}$ then all efficient allocations in $\mathscr{E}_{\mathrm{I}}$ induce efficient divisible allocations in $\mathscr{E}_{\mathrm{D}}$. The rest of this section is devoted to proving Proposition 2.

Consider the following:

Integer Program (IP):

$$\max_{X_1, X_2, ..., X_m} \sum_{i=1}^{m} \sum_{j=1}^{2^n - 1} V_i(S^j)\, x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{2^n-1} a_{kj} \sum_{i=1}^{m} x_{ij} \leqslant 1, \qquad \forall k = 1, 2, ..., n \tag{11}$$

$$\sum_{j=1}^{2^n-1} x_{ij} \leqslant 1, \qquad \forall i = 1, 2, ..., m \tag{12}$$

$$x_{ij} = 0 \text{ or } 1, \qquad \forall i, j.$$

The optimal solution set to IP is the set of efficient allocations in $\mathscr{E}_{\mathrm{I}} = \{N, (V_i), i = 1, 2, ..., m\}$. A linear programming relaxation of IP is

Linear Programming Relaxation (LPR):

$$\max_{X_1, X_2, ..., X_m} \sum_{i=1}^{m} \sum_{j=1}^{2^n - 1} V_i(S^j)\, x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{2^n-1} a_{kj} \sum_{i=1}^{m} x_{ij} \leqslant 1, \qquad \forall k = 1, 2, ..., n \tag{13}$$

$$\sum_{j=1}^{2^n-1} x_{ij} \leqslant 1, \qquad \forall i = 1, 2, ..., m \tag{14}$$

$$x_{ij} \geqslant 0, \qquad \forall i, j.$$

The constraints $x_{ij} \leqslant 1$, $\forall i, j$, are implied by (14) and therefore are not included above. LPR has the dual:

Dual of LPR (DLPR):

$$\min_{p_k,\, \pi_i} \sum_{k=1}^{n} p_k + \sum_{i=1}^{m} \pi_i$$

$$\text{s.t.} \quad \sum_{k=1}^{n} a_{kj} p_k + \pi_i \geqslant V_i(S^j), \qquad \forall i, j \tag{15}$$

$$p_k \geqslant 0, \qquad \pi_i \geqslant 0, \qquad \forall i, k.$$

As $\mathscr{E}_{\mathrm{I}}$ always has an efficient allocation, IP has a finite optimal solution. The feasible region of LPR is a nonempty convex polytope. Thus LPR has a finite optimal solution, which in turn implies that DLPR is feasible and has a finite optimal solution. Let M_{IP} denote the value of an optimal solution to IP, and let M_{LPR} and M_{DRLP} be the values of optimal solutions to LPR and DLPR, respectively. Thus,

$$M_{\mathrm{DLPR}} = M_{\mathrm{LPR}} \geqslant M_{\mathrm{IP}}, \tag{16}$$

where the equality follows from the duality theorem of linear programming (see Dantzig [8, p. 125]), and the inequality from the fact that the feasible region of LPR includes the feasible region of IP.

Next, we show that the sum of the reservation values of agents at an efficient divisible allocation in $\mathscr{E}_{\mathrm{D}}(N, (V_i))$ is equal to M_{LPR}.

LEMMA 3. *Let* $(Y_1^*, Y_2^*, ..., Y_m^*)$ *be an efficient divisible allocation in* $\mathscr{E}_{\mathrm{D}}(N, (V_i))$. *Then* $\sum_i W_i(Y_i^*) = M_{\mathrm{LPR}}$.

Proof. See Appendix. ∎

Hence, the necessary and sufficient condition in Proposition 2 may be restated as $M_{\mathrm{IP}} = M_{\mathrm{LPR}}$, and Proposition 2 follows from the following lemma. The proof of the "if" part uses the well known idea that the dual solution may be interpreted as competitive prices supporting an optimal solution of the primal.

LEMMA 4. *Market clearing prices exist in* $\mathscr{E}_{\mathrm{I}}$ *if and only if any optimal solution to* IP *is an optimal solution to* LPR, *i.e.*, $M_{\mathrm{IP}} = M_{\mathrm{LPR}}$.

Proof. Let $X^* = (x_{ij}^*)$, $i = 1, 2, ..., m$, $j = 1, 2, ..., 2^n - 1$ be an optimal solution to LPR and let $P^* = (p_1^*, p_2^*, ..., p_n^*)$ and $\Pi^* = (\pi_1^*, \pi_2^*, ..., \pi_n^*)$ be an optimal solution to DLPR. That is, X^* is LPR feasible, (P^*, Π^*) is DLPR feasible and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^n} V_i(S^j)\, x_{ij}^* = \sum_{k=1}^{n} p_k^* + \sum_{i=1}^{m} \pi_i^*.$$

The complementary slackness conditions (see Dantzig [8, pp. 135–136]) are

$$\left[1-\sum_{j=1}^{2^n-1} a_{kj} \sum_{i=1}^{m} x_{ij}^*\right] p_k^* = 0, \qquad \forall k, \tag{17}$$

$$\left[1-\sum_{j=1}^{2^n-1} x_{ij}^*\right] \pi_i^* = 0, \qquad \forall i, \tag{18}$$

$$\left[\sum_{k=1}^{n} a_{kj} p_k^* + \pi_i^* - V_i(S^j)\right] x_{ij}^* = 0, \qquad \forall i, j. \tag{19}$$

We show below that $p_k^*, k=1, 2, ..., n$, are prices of the n objects which "support" the efficient divisible allocation X^*. That is, if $x_{ij}^* > 0$ then these prices satisfy (2) with $S_i = S^j$. And if an object is not fully used up at the allocation X^*, then its price is zero. In addition, we show that π_i^* is agent i's consumer surplus, $i=1, 2, ..., m$.[9]

First, observe that (17) implies that if $\sum_{j=1}^{2^n-1} a_{kj} \sum_{i=1}^{m} x_{ij}^* < 1$ then $p_k^* = 0$. Next, if $x_{ij}^* > 0$ then

$$\begin{aligned} V_i(S^j) - \sum_{k=1}^{n} a_{kj} p_k^* &= \pi_i^* \\ &\geqslant V_i(S^{j'}) - \sum_{k=1}^{n} a_{kj'} p_k^*, \qquad \forall j', \end{aligned} \tag{20}$$

where the equality follows from (19) and the inequality from the feasibility of (P^*, Π^*) in DLPR. This, together with $\pi_i^* \geqslant 0$, implies that the prices $p_1^*, p_2^*, ..., p_n^*$ support the allocation x_{ij}^*.

From (20) it is clear that each agent i's per unit subset consumer surplus is identical for all subsets of which he receives positive amounts; this surplus is equal to π_i^*. From (18) it follows that if $\sum_{j=1}^{2^n-1} x_{ij}^* < 1$ then $\pi_i^* = 0$. Thus, π_i^* is agent i's consumer surplus.

To prove sufficiency, suppose that $M_{\mathrm{IP}} = M_{\mathrm{LPR}}$. Consequently, there exists a solution $X^* = x_{ij}^*$, $i=1, 2, ..., m$, $j=1, 2, ..., 2^n-1$ which is feasible and optimal for both IP and LPR. Moreover, X^* is a feasible and efficient allocation in $\mathscr{E}_{\mathrm{I}}$. The preceding argument implies that the DLPR optimal variables $p_1^*, p_2^*, ..., p_n^*$ are prices which support X^* in $\mathscr{E}_{\mathrm{I}}$.

To prove necessity, suppose that $p_1^*, p_2^*, ..., p_n^*$, $p_k^* \geqslant 0$, are market clearing prices which support $(S^{j_1}, S^{j_2}, ..., S^{j_m})$,[10] a feasible allocation in $\mathscr{E}_{\mathrm{I}}$.

[9] The definition of consumer surplus in Section 2 is modified to $\sum_j x_{ij}[V_i(S^j) - \sum_{k \in S^j} p_k]$.

[10] For each i, S^{j_i}, the subset allocated to agent i, is the j_ith element, $0 \leqslant j_i \leqslant 2^n - 1$ in the enumeration of subsets of N.

From Proposition 1 we know that $(S^{j_1}, S^{j_2}, ..., S^{j_m})$ is an efficient allocation. Define

$$\pi_i^* \equiv V_i(S^{j_i}) - \sum_{k \in S^{j_i}} p_k^*, \qquad \forall i.$$

As the prices $p_1^*, p_2^*, ..., p_n^*$ support $(S^{j_1}, S^{j_2}, ..., S^{j_m})$, we have $\pi_i^* \geqslant 0$ and

$$\pi_i^* = V_i(S^{j_i}) - \sum_{k \in S^{j_i}} p_k^* \geqslant V_i(S^j) - \sum_{k \in S^j} p_k^*, \qquad \forall i, j.$$

Thus, p_k^* and π_i^* are dual feasible. This, together with (16), implies that

$$\begin{aligned} M_{\mathrm{LPR}} &= M_{\mathrm{DLPR}} \\ &\leqslant \sum_{i=1}^{m} \pi_i^* + \sum_{k=1}^{n} p_k^* \\ &= \sum_{i=1}^{m} \left[V_i(S^{j_i}) - \sum_{k \in S^{j_i}} p_k^* \right] + \sum_{k=1}^{n} p_k^* \\ &= \sum_{i=1}^{m} V_i(S^{j_i}) \\ &= M_{\mathrm{IP}}, \end{aligned}$$

where the second to last equality follows from the fact that $(S^{j_1}, S^{j_2}, ..., S^{j_m})$ is feasible and that $p_k^* = 0$ for all $k \notin \bigcup_{i=1}^{m} S^{j_i}$, and the last equality from the efficiency of $(S^{j_1}, S^{j_2}, ..., S^{j_m})$. Thus (16) implies that $M_{\mathrm{LPR}} = M_{\mathrm{IP}}$. ∎

The optimal solutions to the integer programming formulation of an assignment market and to the associated linear programming relaxation are the same. Therefore, the dual linear program yields prices that decentralize efficient allocations in assignment markets (see Koopmans and Beckman [18]). In our setting, the equivalence between the integer programming formulation and its linear programming relaxation depends on the existence of market clearing prices.

In the proof of Lemma 4, X^* and (P^*, Π^*) were arbitrary optimal solutions to LPR and DLPR respectively. Therefore, the following corollary is immediate:

Corollary 1. *If one efficient allocation in* $\mathscr{E}_{\mathrm{I}}$ *is supported by a price vector* $(p_1^*, p_2^*, ..., p_n^*)$, *then all efficient allocations in* $\mathscr{E}_{\mathrm{I}}$ *are supported by* $(p_1^*, p_2^*, ..., p_n^*)$.

It is clear from the definition of competitive equilibrium that market clearing prices are nonnegative and bounded from above by $\max_i V_i(N)$.

In addition, the proof of Lemma 4 implies that if market clearing prices exist in $\mathscr{E}_{\mathrm{I}}$, then these prices constitute part of an optimal solution to DLPR. As the optimal solution set to a linear program is closed and convex, we have[11]

COROLLARY 2. *The set of market clearing prices in $\mathscr{E}_{\mathrm{I}}$ is a closed, bounded, convex (and possibly empty) set.*

The balancedness condition in cooperative game theory (see Bondareva [5] and Shapley [24]) states that a function $V(\cdot)$ defined on subsets S^j of a finite set N is balanced if for any vector $(\lambda_1, \lambda_2, ..., \lambda_{2^n-1})$, $\lambda_j \geqslant 0$, such that $\forall k \in N, \sum_{j:\, k \in S^j} \lambda_j = 1$ we have $\sum_{j=1}^{2^n-1} \lambda_j V(S^j) \leqslant V(N)$. The vector $(\lambda_1, \lambda_2, ..., \lambda_{2^n-1})$ is called a collection of balancing weights. When agents have identical reservation values, this condition is sufficient for existence of market clearing prices.[12]

COROLLARY 3. *If all agents have the same reservation value function $V(\cdot)$, and if $V(\cdot)$ is balanced then market clearing prices exist.*

Proof. Suppose that x_{ij}, $i = 1, 2, ..., m$, $j = 1, 2, ..., 2^n - 1$ is in the feasible region of LPR. Thus, $\sum_{j:\, k \in S^j} \sum_{i=1}^{m} x_{ij} \leqslant 1$, $\forall k \in N$.

For $k = 1, 2, ..., n$, let $j(k)$ be the index number of the set $\{k\}$. That is, $S^{j(k)} = \{k\}$. Define $\lambda_{j(k)} \equiv [\sum_{i=1}^{m} x_{ij(k)} + (1 - \sum_{j:\, k \in S^j} \sum_{i=1}^{m} x_{ij})]$. For all j such that $|S^j| \geqslant 2$, define $\lambda_j = \sum_{i=1}^{m} x_{ij}$. Clearly, (λ_j) is a vector of balancing weights and $\lambda_j \geqslant \sum_{i=1}^{m} x_{ij}$, $j = 1, 2, ..., 2^n - 1$. Thus,

$$\sum_{j=1}^{2^n-1} \sum_{i=1}^{m} x_{ij} V(S^j) \leqslant \sum_{j=1}^{2^n-1} \lambda_j V(S^j) \leqslant V(N),$$

where the first inequality follows from $\lambda_j \geqslant \sum_{i=1}^{m} x_{ij}$, $\forall j$ and the second from balancedness. Thus, LPR has an integer optimal solution and $M_{\mathrm{IP}} = M_{\mathrm{LPR}}$. ∎

The necessary and sufficient condition obtained in this section can be used to check for existence in specific examples.[13] This condition could also be used to obtain conditions on agents' preferences which ensure existence. Before considering several assumptions on the agents' reservation value functions in Section 4, we present three extensions of our results in

[11] Corollary 2 can also be proved directly from (2) and (4).

[12] In cooperative games with transferable utility, the balancedness condition is applied to the characteristic function which represents values of coalitions of players. Here, we apply it to the reservation value function which represents values of coalitions of goods. Thus, there is a difference in interpretation.

[13] In Table 1, for instance, it can be verified that $4 = M_{\mathrm{IP}} < M_{\mathrm{LPR}} = 4.5$ implying non-existence of equilibrium prices.

Section 3.1 and relate the indivisible economy $\mathscr{E}_{\mathrm{I}}$ to a matching market in Section 3.2.

3.1. *Extensions*

1. If the endowment of at least one object is greater than one unit, then by converting this economy into an equivalent economy with one unit of each object (as explained at the end of Section 2) the results of this section provide necessary and sufficient conditions for existence of equilibrium prices in the original economy (with multiple units of some objects). Alternatively, one can obtain necessary and sufficient conditions for existence by directly transforming the economy with multiple units of some objects into an equivalent divisible economy using a straightforward extension of (5). With the exception of the analog of Lemma 2(ii), all the results proved so far remain true.

2. We have assumed throughout that agents are free to buy any subset they want. One could, instead, assume that agents' choices are limited to $\mathscr{T}$, a subset of the set of all subsets of N. This would entail the following changes:

(i) A feasible allocation $(S_1, S_2, ..., S_m)$ must satisfy $S_i \cap S_{i'} = \phi$, $\forall i \neq i'$, and $\forall i$, $S_i \in \mathscr{T}$ (rather than $S_i \subseteq N$).

(ii) The qualifier $\forall S \subseteq N$ in (2) is replaced by $\forall S \in \mathscr{T}$.

(iii) The definition of efficient allocation remains the same; however, an efficient allocation under the earlier definition of a feasible allocation may no longer be feasible.

(iv) In the proofs and definitions, all summations $\sum_{j=1}^{2^n-1}$ are replaced by $\sum_{S \in \mathscr{T}}$.

With these changes, all the results proved so far, except the analog of Lemma 2(ii), remain true.

3. Another divisible transformation of the indivisible economy is obtained by excluding the constraint $\sum_j x_{ij} \leqslant 1$ from the linear program (5) (and therefore, excluding constraints (14) from LPR and the variables π_i from DLPR). The analog of Lemma 2(ii) does not hold for this divisible transformation, and Proposition 2 is modified to: Market clearing prices which give each agent zero consumer surplus exist in an indivisible economy if and only if an efficient allocation in the indivisible economy induces an efficient divisible allocation in the divisible economy. This necessary and sufficient condition is satisfied when agents' reservation values are additive.

The proofs of each of these extensions mimic arguments used in this section and are omitted.

3.2. *An Exchange Economy as a Two-Sided Matching Market*

A transfer of an indivisible object from one agent to another in the economy considered here may be thought of as a match between the two agents. Thus, the exchange economy is a matching market without the two-sidedness restriction as each agent in this economy may be a buyer of certain objects and a seller of others.[14] One can transform an exchange economy $\mathscr{E}_{\mathrm{I}}$ into a two-sided exchange economy $\mathscr{E}_{\mathrm{TS}}$ such that market clearing prices exist in $\mathscr{E}_{\mathrm{I}}$ if and only if market clearing prices exist in $\mathscr{E}_{\mathrm{TS}}$. This is shown below.

Given an economy $\mathscr{E}_{\mathrm{I}} = \{N, (V_i), i = 1, 2, ..., m\}$ and any initial endowment of the set N to the agents $i = 1, 2, ..., m$, define two economies with the same set N of indivisible commodities: $\mathscr{E}'_{\mathrm{I}} = \{N, (\hat{V}_i), i = 1, 2, ..., m, m+1, ..., m+n\}$ and $\mathscr{E}_{\mathrm{TS}} = \{N, (\hat{V}_i), i = 1, 2, ..., m, m+1, ..., m+n\}$. The reservation value functions $\hat{V}_i$ are defined as

$$\hat{V}_i(S) = \begin{cases} V_i(S), & \forall S \subseteq N, \quad \forall i = 1, 2, ..., m, \\ 0, & \forall S \subseteq N, \quad \forall i = m+1, m+2, ..., m+n. \end{cases}$$

In $\mathscr{E}'_{\mathrm{I}}$, the initial endowment of the indivisible objects is the same as in $\mathscr{E}_{\mathrm{I}}$. As agents $m+1, m+2, ..., m+n$ have no initial endowments and have no utility for the objects, they play no role in $\mathscr{E}'_{\mathrm{I}}$. Thus, market clearing prices exist in $\mathscr{E}_{\mathrm{I}}$ if and only if market clearing prices exist in $\mathscr{E}'_{\mathrm{I}}$.

In $\mathscr{E}_{\mathrm{TS}}$, agents $i = 1, 2, ..., m$ are (potential) buyers and have no initial endowment of any indivisible object. Agents $i = m+1, m+2, ..., m+n$ are sellers with agent $m+k$ being endowed with one unit of object k, $k = 1, 2, ..., n$. As noted in Section 2, since there are no budget constraints (i.e., $\bar{w}_i \geqslant V_i(N)$) the existence of market clearing prices depends only on the agents' reservation values and on the aggregate endowments (i.e., the set N) but not on how the aggregate endowments are distributed among the agents. Hence, market clearing prices exist in $\mathscr{E}_{\mathrm{TS}}$ if and only if market clearing prices exist in $\mathscr{E}'_{\mathrm{I}}$, and by the argument in the preceding paragraph, if and only if market clearing prices exist in $\mathscr{E}_{\mathrm{I}}$. The efficient allocations and (as an examination of (2) and (4) reveals) the set of market clearing prices in $\mathscr{E}_{\mathrm{I}}$, $\mathscr{E}'_{\mathrm{I}}$, and $\mathscr{E}_{\mathrm{TS}}$ are identical.

Kelso and Crawford [16] analyze two-sided matching markets with money. The two sides they consider—firms and workers—correspond to buyers and sellers in $\mathscr{E}_{\mathrm{TS}}$ in that many sellers (workers) may be matched to one buyer (firm) but not *vice versa*. As workers have intrinsic preferences over which firm they are matched with, it is natural in the Kelso and Crawford model to allow different salaries (prices) for a worker depending

[14] In FCC auctions and in RTC auctions, buyers and sellers are distinct entities. Hence, these and other auctions are two-sided matching markets.

where he works. In our setting, as sellers will sell only to a buyer who offers the highest price, the set of competitive equilibrium outcomes would be unchanged even if one allowed prices to be buyer dependent. Hence, the results in Kelso and Crawford apply directly here.

Consider the following condition from Kelso and Crawford's paper. For any price vector $P=(p_1, p_2, ..., p_k, ..., p_n)$, let

$$M^i(P) \equiv \arg\max_S \left[V_i(S) - \sum_{k \in S} p_k \right], \qquad i=1, 2, ..., m.$$

For any two price vectors P and $\tilde{P}$ and subset S define

$$T(S, P, \tilde{P}) \equiv \{k \mid k \in S \text{ and } p_k = \tilde{p}_k\}.$$

An exchange economy satisfies the *gross substitutes* assumption if for every agent i, if $S^i \in M^i(P)$ and $\tilde{P} \geqslant P$, then there exists $\tilde{S}^i \in M^i(\tilde{P})$ such that $T(S^i, P, \tilde{P}) \subseteq \tilde{S}^i$. Thus, different objects are substitutes in the sense that the demand for an object does not decrease if prices of some other objects increase. Kelso and Crawford provide an ascending-price auction procedure in which buyers propose prices for objects—a variant of the deferred acceptance algorithm of Gale and Shapley [12]. This auction procedure leads to a competitive equilibrium when the gross substitutes assumption is satisfied. Hence, the gross substitutes condition is sufficient for existence of market clearing prices. Further, they show that the set of competitive equilibrium payoffs in $\mathscr{E}_{\mathrm{TS}}$ coincides with its core (whether or not the gross substitutes assumption is satisfied). Roth [23] generalizes the Kelso–Crawford framework to a two-sided many-to-many matching market and provides two auction procedures (one in which buyers propose and another in which sellers propose) which lead to a competitive equilibrium under the gross substitutes assumption.

By viewing an exchange economy as a two-sided matching market one obtains market mechanisms which, under certain conditions, converge to competitive equilibrium outcomes. In a setting where many heterogeneous objects are sold to buyers who buy at most one object, Demange, Gale, and Sotomayor [9] show that an ascending price mechanism (a special case of an algorithm proposed by Crawford and Knoer [7]) converges to the minimum competitive price and that this mechanism is not individually manipulable by the buyers.

4. FURTHER ASSUMPTIONS ON AGENTS' PREFERENCES

A reservation value function, V_i, is *superadditive* if for all $S, T \subseteq N$ such that $S \cap T = \phi$

$$V_i(S) + V_i(T) \leqslant V_i(S \cup T). \qquad (21)$$

The example in Table 1 of Section 2 implies that superadditivity is not sufficient for the existence of market clearing prices. The following assumption is stronger.

A reservation value function, V_i, is *supermodular* if for all $S, T \subseteq N$

$$V_i(S) + V_i(T) \leqslant V_i(S \cup T) + V_i(S \cap T). \tag{22}$$

Supermodularity implies that the products are complements. To see this, note that an alternative (and equivalent) definition of supermodularity is the following. For all $T_1, T_2, T_3 \subseteq N$ such that $T_1 \cap T_2 = T_2 \cap T_3 = T_3 \cap T_1 = \phi$,

$$V_i(T_1 \cup T_3) - V_i(T_1) \leqslant V_i(T_1 \cup T_2 \cup T_3) - V_i(T_1 \cup T_2). \tag{23}$$

Under supermodular preferences, the marginal value for T_3 increases with an agent's current level of possessions.

Preferences are strictly supermodular if the inequality in (22) is strict whenever $S \not\subseteq T \not\subseteq S$ (or equivalently if the inequality in (23) is strict whenever $T_2 \neq \phi$ and $T_3 \neq \phi$). From (22) it is clear that supermodularity implies superadditivity. If N contains at most two objects then superadditivity implies supermodularity.

Agents' preferences are (i) *subadditive* if the inequality in (21) is reversed, and (ii) *submodular* if the inequality in (22) is reversed. Submodular preferences are subadditive. Preferences are *additive* if the inequality in (22) is replaced by an equality, or equivalently, if

$$V_i(S) = \mu_i(S), \qquad \forall S \subseteq N, \quad \forall i, \tag{24}$$

where μ_i is an additive measure. Prices $p_k = \max_i[\mu_i(\{k\})]$ support an efficient allocation in an economy where agents' reservation value functions satisfy (24).

5. SUPERMODULAR PREFERENCES

A supermodular reservation value function is balanced. Therefore, Corollary 3 of Section 3 implies that if all agents have the same supermodular reservation value function then market clearing prices exist. The following proposition extends this to two types of agents with supermodular reservation value functions which are strictly increasing (i.e., $V_i(S) > V_i(T)$ for all $T \subset S$, $S \not\subseteq T$).

PROPOSITION 3. *Suppose there are two types of agents in an indivisible economy $\mathscr{E}_{\mathrm{I}}$—type* 1 *agents with reservation value function V^1 and type* 2

TABLE 2

Nonexistence with Supermodular Preferences

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{2, 3\}$	$\{1, 3\}$	$\{1, 2, 3\}$
$V_A(S)$	1	1	1	30	3	3	40
$V_B(S)$	1	1	1	3	30	3	40
$V_C(S)$	1	1	1	3	3	30	40

agents with reservation value function V^2. Further, suppose that both V^1 and V^2 are strictly supermodular and strictly increasing. Then market clearing prices exist.

Proof. See Appendix. ∎

The agents' consumer surpluses at the market clearing prices constructed in the proof of Proposition 3 is zero. This is not accidental. Using Extension 3 of Section 3, it can shown that if agents' reservation values are supermodular and market clearing prices exist, then there also exist market clearing prices which give each agent zero consumer surplus.

The following example (for which we are grateful to Bhaskar Dutta) shows that supermodularity is not enough to ensure the existence of market clearing prices. There are three objects, 1, 2, and 3, and three agents, A, B, and C in this example (Table 2). An efficient indivisible allocation is $S_A = \{1, 2, 3\}$, $S_B = S_C = \phi$ which yields $M_{\mathrm{IP}} = 40$, whereas the efficient divisible allocation is $S_A = \frac{1}{2}\{1, 2\}$, $S_B = \frac{1}{2}\{2, 3\}$, $S_C = \frac{1}{2}\{1, 3\}$ with $M_{\mathrm{LP}} = 45$. As $M_{\mathrm{IP}} < M_{\mathrm{LP}}$, equilibrium prices do not exist.

This example also shows that Proposition 3 cannot be generalized to three types of agents with supermodular reservation values.

6. COUNTEREXAMPLES AND OPEN QUESTIONS

Although we obtain a necessary and sufficient condition for existence of market clearing prices (Proposition 2), we have been less successful at identifying sufficient conditions on agents' preferences. Market clearing prices exist when (i) all agents have additive reservation values, and when (ii) all agents have supermodular reservation value functions and are either identical or of two types. The example in Table 2 of Section 5, shows that supermodular preferences are not sufficient for existence of competitive equilibrium. We provide a few counterexamples to existence under other assumptions on preferences.

First, consider the following example from Kelso and Crawford [16]. As shown in Table 3 below, two buyers, A and B, have submodular

TABLE 3

Nonexistence with Submodular Preferences

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{2, 3\}$	$\{1, 3\}$	$\{1, 2, 3\}$
$V_A(S)$	4	4	4.25	7.5	7	7	9
$V_B(S)$	4	4.25	4	7	7	7.5	9

reservation values over sets containing up to three objects, 1, 2, and 3. An efficient indivisible allocation is $\{1, 2\}$ to agent A and $\{3\}$ to agent B, with $M_{\text{IP}} = 11.5$. An efficient divisible allocation is $\frac{1}{2}\{1, 2\} + \frac{1}{2}\{3\}$ to agent A and $\frac{1}{2}\{1, 3\} + \frac{1}{2}\{2\}$ to agent B, with $M_{\text{LPR}} = 11.75$. Nonexistence of equilibrium prices is implied by Lemma 4 as $11.5 = M_{\text{IP}} < M_{\text{LPR}} = 11.75$.

There are two (types of) agents in the above counterexample. Therefore, a version of Proposition 3 for submodular preferences is ruled out. Whether there exist equilibrium prices when all agents have identical submodular reservation value functions remains an open question.

Consider the following property, which is stronger than submodularity. A reservation value function, V_i, is a *concave measure function* if it can be written as

$$V_i(S) = f_i(\mu_i(S)), \qquad \forall S \subseteq N,$$

where $f_i\colon \Re \to \Re$ is a concave function and μ_i is an additive measure defined on all the subsets of N. When reservation values are concave measure functions, price increases of objects other than object 1, say, will not make an agent want to stop buying object 1. Hence, concave measure functions satisfy the gross substitutes assumption of Kelso and Crawford. Therefore, market clearing prices exist when agent's reservation values are concave measure functions.

It may be argued that in the FCC spectrum auctions, buyers' reservation value functions exhibit complementarity initially and then substitutability. That is, each buyer's marginal value for additional licenses increases initially until the buyer has enough licenses to offer a basic portfolio of services to his customers; after this point, the marginal value for licenses decreases. Assuming for simplicity that all licenses (objects) are identical, let $V(\cdot)$ be a reservation value function over 5 objects. Suppose that there is one seller whose reservation value for the objects is zero, and each of the

TABLE 4

Nonexistence with Complementary and Substitute Preferences

S	$\#S = 1$	$\#S = 2$	$\#S = 3$	$\#S = 4$	$\#S = 5$
$V(S)$	1	3	6	8	9

TABLE 5

Nonexistence with Supermodular and Submodular Preferences

S	$\{1\}$	$\{2\}$	$\{1, 2\}$
$V_A(S)$	1	1	3
$V_B(S)$	2	2	3

m buyers have the reservation value function defined in Table 4.[15] An efficient allocation is to give subset $\{1, 2, 3\}$ to buyer 1, and subset $\{4, 5\}$ to buyer 2, whereas an efficient divisible allocation is to give $\frac{1}{2}\{1, 2, 3\} + \frac{1}{2}\{1, 2, 4, 5\}$ to buyer 1 and $\frac{1}{2}\{3, 4, 5\}$ to buyer 2. As $9 = M_{\mathrm{IP}} < M_{\mathrm{LPR}} = 10$, nonexistence of equilibrium prices is implied.

Finally, consider an example in which one agent, A, has supermodular preferences and the other agent, B, has submodular preferences (Table 5). An efficient allocation in this example is $S_A = \{1, 2\}$, $S_B = \phi$. An efficient divisible allocation is $\frac{1}{2}\{1, 2\}$ to A and $\frac{1}{2}\{1\} + \frac{1}{2}\{2\}$ to B. Thus, $3 = M_{\mathrm{IP}} < M_{\mathrm{LPR}} = 3.5$ and Lemma 4 implies nonexistence of equilibrium prices.

7. IMPLICATIONS FOR MARKET MECHANISMS

It is well known that when one indivisible object is for sale, several common auction forms implement the competitive equilibrium outcome at the minimum competitive price. Under complete information about buyer reservation values, the minimum market clearing price and allocation is implemented in the sealed bid first-price auction and in the oral descending-price auction. The sealed-bid second-price auction and the oral ascending-price auction attain this competitive equilibrium outcome under both complete and incomplete information.

Do there exist simple market mechanisms (i.e., mechanisms that assign a price to each object) which efficiently allocate multiple indivisible objects when market clearing prices exist?[16] As mentioned in Section 3.2, one setting in which such a mechanism exists is when buyers have zero marginal utility

[15] This reservation value function is, roughly speaking, supermodular over smaller sets and submodular over larger sets. That is, $V(\cdot)$ satisfies (22) when S and T each have two or less elements, and satisfies (22) with the inequality reversed when S and T each have three or more elements.

[16] A Vickrey auction is not a simple market mechanism according to this definition.

for consuming more than one object (from a set of heterogeneous objects). Under this scenario, a competitive equilibrium exists. Demange, Gale, and Sotomayor [9] provide a simple (buyer) incentive-compatible ascending-bid mechanism which converges to the minimum competitive price vector under incomplete information about buyer preferences. Whether there are simple incentive compatible market mechanisms which converge to a competitive equilibrium (whenever one exists) under the more general condition that buyers may want to consume more than one object is an open question.

Simultaneous sealed bid auctions—one for each of the n objects—are analyzed in Bikhchandani [4]. Under complete information, Nash equilibria in simultaneous first-price auctions and in simultaneous second-price auctions implement the set of competitive equilibrium outcomes (when this set is nonempty). The set of pure strategy Nash equilibrium allocations in simultaneous first-price auctions is identical to the set of competitive equilibrium allocations.

Given a probability distribution on buyers' and sellers' valuations, it can be verified using Lemma 4 whether there is a positive probability that the agents will draw values which lead to nonexistence of competitive equilibrium. If with probability one market clearing prices exist, then does there exist a simple market mechanism capable of discovering these prices? Observe that when reservation values are additive (i.e., they satisfy (24)) then competitive equilibrium prices exist; a simple market mechanism which arrives at competitive equilibrium prices under a variety of informational assumptions is a set of simultaneous sealed bid auctions, one for each object. Other assumptions under which simple market mechanisms may be efficient are: (i) buyers have a common unknown balanced reservation value function (Corollary 3), and (ii) buyers' preferences satisfy the hypothesis of Proposition 3, with each buyer's type being private information.

We close with two implications for market mechanisms when market clearing prices do not exist. First, when a single object is sold (or equivalently when several units of an object are sold but each buyer needs at most one unit), then, under a wide variety of informational assumptions, the oral ascending price auction has the *no regret* property. That is, along the equilibrium path bidders do not wish to change their bids after learning the history of dropout behavior of others. Nonexistence of market clearing prices implies that when bidders value more than one object and have interdependent values, then simultaneous oral ascending price auctions will not have the no regret property, at least when each bidder knows his reservation value function. No matter how the objects are allocated by the auction, some bidder would want to change bids *ex post* if he could.

Second, it would be surprising to find a simple market mechanism that achieves allocative efficiency in a wide variety of circumstances in which the competitive equilibrium fails.[17] If market clearing prices do not exist then bundling a few of the objects together may lead to existence, with some loss of efficiency. An alternative approach is to set prices for some bundles, say those containing two or three objects, as well as for individual objects within these bundles.

APPENDIX: PROOFS OF LEMMAS 2, 3, AND PROPOSITION 3

Proof of Lemma 2. (i) The feasible region in the above linear program (5) is a nonempty convex polyhedron. Thus, $W_i(\cdot)$ is well-defined and finite at each $y_{i1}, y_{i2}, ..., y_{in}$. Moreover, from the duality theorem of linear programming, the dual linear program of (5) has a finite optimal solution for each $y_{i1}, y_{i2}, ..., y_{in}$ and the value of this optimal solution is $W_i(y_{i1}, y_{i2}, ..., y_{in})$. The dual of (5) is

$$\begin{aligned} W_i(y_{i1}, y_{i2}, ..., y_{in}) \equiv \min_{\pi_i, p_k} \ & \pi_i + \sum_{k=1}^{n} y_{ik} p_k \\ \text{s.t.} \quad & \pi_i + \sum_{k=1}^{n} a_{kj} p_k \geqslant V_i(S^j), \qquad \forall j \\ & \pi_i \geqslant 0, \qquad p_k \geqslant 0, \qquad \forall k, \end{aligned} \tag{25}$$

where we use the fact that the value of the dual objective function at the optimal solution is $W_i(y_{i1}, y_{i2}, ..., y_{in})$. Let $Y^1 = (y^1_{i2}, y^1_{i2}, ..., y^1_{in})$, $Y^2 = (y^2_{i1}, y^2_{i2}, ..., y^2_{in})$ be two points in the unit cube. For any $\lambda \in (0, 1)$, let $Y^\lambda = \lambda Y^1 + (1-\lambda) Y^2$. Let $(p^l_1, p^l_2, ..., p^l_n, \pi^l_i)$ be an optimal solution to the dual linear program defining $W_i(Y^l)$, $l = 1, 2$. That is,

$$W_i(Y^l) = \pi^l_i + \sum_{k=1}^{n} y_{ik} p^l_k \leqslant \pi_i + \sum_{k=1}^{n} y_{ik} p_k, \qquad l = 1, 2, \tag{26}$$

for all feasible $(p_1, p_2, ..., p_n, \pi_i)$. Similarly, let $(p^\lambda_1, p^\lambda_2, ..., p^\lambda_n, \pi^\lambda_i)$ be an optimal solution to the dual program defining $W_i(Y^\lambda)$. Observe that the feasible region of the dual linear program (25) does not depend

[17] In specific cases a simple market mechanism may work even though competitive equilibrium fails. A mechanism which sets the price of each object to zero and allocates everything to a randomly chosen buyer would allocate efficiently in the example in Section 2, Table 1, but not in the example in Section 6, Table 3.

on $y_{i1}, y_{i2}, ..., y_{in}$. Therefore, $(p_1^\lambda, p_2^\lambda, ..., p_n^\lambda, \pi_i^\lambda)$ is feasible in the dual programs defining $W_i(Y^1)$ and $W_i(Y^2)$. Consequently,

$$\begin{aligned} W_i(Y^\lambda) &= \pi_i^\lambda + \sum_{k=1}^{n} y_{ik}^\lambda p_k^\lambda \\ &= \lambda \left[\pi_i^\lambda + \sum_{k=1}^{n} y_{ik}^1 p_k^\lambda \right] + (1-\lambda) \left[\pi_i^\lambda + \sum_{k=1}^{n} y_{ik}^2 p_k^\lambda \right] \\ &\geqslant \lambda \left[\pi_i^1 + \sum_{k=1}^{n} y_{ik}^1 p_k^1 \right] + (1-\lambda) \left[\pi_i^2 + \sum_{k=1}^{n} y_{ik}^2 p_k^2 \right] \\ &= \lambda W_i(Y^1) +)1-\lambda)\ W_i(Y^2), \end{aligned}$$

where the inequality follows from (26). This establishes the concavity of W_i.

(ii) There is a one-to-one correspondence between the extreme points of the unit cube in $\Re^n$ and the subsets of N. For any $S \subseteq N$, let $Y^S = (y_1^s, y_2^s, ..., y_n^s)$ be such that $y_k^s = 1$ if $k \in S$, and $y_k^s = 0$ if $k \notin S$, $\forall k$. We show that $W_i(Y^S) = V_i(S)$.

Observe that

$$x_{ij} = \begin{cases} 1, & \text{if} \quad S^j = S \\ 0, & \text{otherwise,} \end{cases}$$

is feasible in the linear program (5) which defines W_i at Y^S. The objective function value at this feasible solution is $V_i(S)$ and, therefore, $W_i(Y^S) \geqslant V_i(S)$.

Let $X_i^* = (x_{i1}^*, x_{i2}^*, ..., x_{i2^n-1}^*)$ be an optimal (feasible) solution to (5). Suppose that $S^l \not\subseteq S$. Take any $k \in S^l \backslash S$. Thus, $y_k^s = 0$. The feasibility of X_i^* implies that

$$0 \leqslant x_{il}^* = a_{kl} x_{il}^* \leqslant \sum_{j=1}^{2^n-1} a_{kj} x_{ij}^* \leqslant y_k^s = 0.$$

Therefore, $x_{il}^* = 0$ for all $S^l \not\subseteq S$, or equivalently, if $x_{il}^* > 0$ then $S^l \subseteq S$. As the reservation price function is increasing, $V_i(S^l) \leqslant V_i(S)$, for all $S^l \subseteq S$. Thus

$$\begin{aligned} W_i(Y^S) = \sum_{j=1}^{2^n-1} V_i(S^j)\, x_{ij}^* &= \sum_{l:\, x_{il}^* > 0} V_i(S^l)\, x_{il}^* \\ &\leqslant V_i(S) \sum_{l:\, x_{il}^* > 0} x_{il}^* \leqslant V_i(S). \end{aligned}$$

This completes the proof. ▮

Proof of Lemma 3. The optimal solutions to the following linear program constitute the set of efficient divisible allocations in $\mathscr{E}_{\mathrm{D}}(N, (V_i))$:

$$\max_{y_{ik}} \sum_{i=1}^{m} W_i(y_{i1}, y_{i2}, ..., y_{in}) \tag{27}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_{ik} \leqslant 1, \qquad \forall k = 1, 2, ..., n$$

$$y_{ik} \geqslant 0, \qquad \forall i, k.$$

Let $(Y_1^*, Y_2^*, ..., Y_m^*)$ be an optimal (feasible) solution to the linear program (27). For each i, let (x_{ij}^*), $j = 1, 2, ..., 2^n - 1$ be an optimal solution to the linear program (5) which defines $W_i(Y_i^*)$. As $(Y_1^*, Y_2^*, ..., Y_m^*)$ is feasible in (27), (x_{ij}^*), $\forall i, \forall j$, is feasible in LPR. Thus,

$$\sum_{i=1}^{m} W_i(Y_i^*) = \sum_{i=1}^{m} \sum_{j=1}^{2^n - 1} V_i(S^j)\, x_{ij}^* \leqslant M_{\mathrm{LPR}}.$$

Conversely, let (x_{ij}^o), $\forall i, \forall j$, be an optimal solution to LPR. Define, $y_{ik}^o \equiv \sum_j a_{kj} x_{ij}^o$, $\forall i, k$, and $Y_i^o = (y_{i1}^o, y_{i2}^o, ..., y_{in}^o)$. For each i, (x_{ij}^o), $j = 1, 2, ..., 2^n - 1$ is feasible in the linear program (5) which defines $W_i(Y_i^o)$. Therefore, $W_i(Y_i^o) \geqslant \sum_j V_i(S^j)\, x_{ij}^o$, $\forall i$. Moreover, as (x_{ij}^o), $\forall i, \forall j$, is feasible in LPR, $(Y_1^o, Y_2^o, ..., Y_n^o)$ is feasible in (27). Thus,

$$\sum_{i=1}^{m} W_i(Y_i^*) \geqslant \sum_{i=1}^{m} W_i(Y_i^o) \geqslant \sum_{i=1}^{m} \sum_{j=1}^{2^n - 1} V_i(S^j)\, x_{ij}^o = M_{\mathrm{LPR}}.$$

This completes the proof. ▮

Proof of Proposition 3. Suppose that agents $i \in T^1 \equiv \{1, 2, ..., m_1\}$ are of type 1 and agents $i \in T^2 \equiv \{m_1 + 1, m_1 + 2, ..., m\}$ are of type 2. Let $X_i^* = (x_{i1}^*, x_{i2}^*, ..., x_{i2^n - 1}^*)$, $i = 1, 2, ..., m$ be an optimal solution to LPR and let $P^* = (p_1^*, p_2^*, ..., p_n^*)$ and $\Pi^* = (\pi_1^*, \pi_2^*, ..., \pi_n^*)$ be an optimal solution to DLPR.

It is easy to show that at any optimal solution to DLPR, the consumer surplus for all agents of the same type is the same. That is $\pi_i^* = \pi_{i'}^*$. for all $i, i' \in T^l$, $l = 1, 2$, and we may define $\pi^l \equiv \pi_i^*$ where $i \in T^l$.

Suppose that at this optimal solution to LPR, we have $x_{ij_1}^* > 0$ and $x_{i'j_2}^* > 0$, where $i, i' \in T^1$. Then (19) implies that

$$\sum_{k=1}^{n} a_{kj_1} p_k^* + \pi^1 = V^1(S^{j_1}), \tag{28}$$

$$\sum_{k=1}^{n} a_{kj_2} p_k^* + \pi^1 = V^1(S^{j_2}). \tag{29}$$

Suppose further that $S_{j_1} \nsubseteq S_{j_2} \nsubseteq S_{j1}$. Then (28), (29), and strict supermodularity imply that

$$\sum_{k \in S^{j_1} \cup S^{j_2}} p_k^* + \sum_{k \in S^{j_1} \cup S^{j_2}} p_k^* + 2\pi^1 = V^1(S^{j_1}) + V^1(S^{j_2})$$
$$< V^1(S^{j_1} \cup S^{j_2}) + V^1(S^{j_1} \cap S^{j_2}).$$

Since, by (15),

$$\sum_{k \in S^{j_1} \cup S^{j_2}} p_k^* + \pi^1 \geqslant V^1(S^{j_1} \cup S^{j_2})$$

and

$$\sum_{k \in S^{j_1} \cup S^{j_2}} p_k^* + \pi^1 \geqslant V^1(S^{j_1} \cap S^{j_2}),$$

we have a contradiction. Hence, either $S^{j_1} \subseteq S^{j_2}$ or $S^{j_2} \subseteq S^{j_1}$.

Consequently, if sets $S^{j_1}, S^{j_2}, ..., S^{j_r}$ are the only ones that are allocated to type 1 buyers, i.e., if $\sum_{i=1}^{m_1} x_{ij_l}^* > 0$, $\forall l = 1, 2, ..., r$, and $\sum_{i=1}^{m_1} x_{ij}^* = 0$, $\forall j \notin \{j_1, j_2, ..., j_r\}$, then we may assume without loss of generality that

$$S^{j_1} \subset S^{j_2} \subset \cdots \subset S^{j_r}. \tag{30}$$

(A similar result holds for the sets that are allocated to type 2 buyers.) Further, relabeling the objects if necessary, we may write

$$S^{j_l} = \{1, 2, ..., k_l\}, \qquad \forall l = 1, 2, ..., r, \tag{31}$$

where $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n$.

Let $\mathcal{N}^1 \equiv \{S^{j_1}, S^{j_2}, ..., S^{j_r}\}$ be the collection of subsets of N which are allocated to type 1 buyers at this optimal solution to LPR. That is, if $S^j \in \mathcal{N}^1$ then $x_{ij}^* > 0$ for some $i \in T^1$, and if $S^{j'} \notin \mathcal{N}^1$ then $x_{ij'}^* = 0$ for all $i \in T^1$. Similarly, $\mathcal{N}^2$ is the set of subsets of N which are allocated to type 2 buyers. Then

$$\sum_{k \in S} p_k^* + \pi^l = V^l(S), \qquad \forall S \in \mathcal{N}^l, \quad l = 1, 2, \tag{32}$$

$$\sum_{k \in S} p_k^* + \pi^l \geqslant V^l(S), \qquad \forall S \in \mathcal{N}^l, \quad l = 1, 2, \tag{33}$$

where (32) follows from (19) and (33) from (15). The remainder of the proof divides into two cases.

Case I: *At least two agents of each type.* If $\pi^1 > 0$ then (18) implies that $\sum_j x_{ij}^* = 1$ for each $i \in T^1$. But as there are at least two type 1 agents, (30) implies that at the feasible (and optimal) solution $(X_1^*, X_2^*, ..., X_m^*)$ contraint (13) of LPR is violated for all $k \in S^{j_1}$. Consequently, we must have $\pi^1 = 0$. A symmetric argument establishes that $\pi^2 = 0$. Thus, (32) and (33) may be written as

$$\sum_{k \in S} p_k^* = V^l(S), \qquad \forall S \in \mathcal{N}^l, \quad l = 1, 2, \tag{34}$$

$$\sum_{k \in S} p_k^* \geqslant V^l(S), \qquad \forall S \in \mathcal{N}^l, \quad l = 1, 2, \tag{35}$$

We construct an efficient allocation in $\mathscr{E}_{\mathrm{I}}$ which is supported by these prices.

Define

$$A_k \equiv 1 - \sum_{j=1}^{2^n - 1} a_{kj} \sum_{i=1}^{m_1} x_{ij}^*, \qquad \forall k,$$

the amount of commodity k available to type 2 buyers. As reservation value functions are strictly increasing, it is easy to show that constraints (13) are binding at the optimal solution. Thus, we must have

$$\sum_{j=1}^{2^n - 1} a_{kj} \sum_{i=m_1+1}^{m} x_{ij}^* = A_k, \qquad \forall k. \tag{36}$$

Let $k_0 = 0$. It follows from (31) that

$$\begin{aligned} &A_k = A_{k'}, \qquad \text{if} \quad k_l < k, \quad k' \leqslant k_{l+1}, \qquad \text{for some} \quad l = 0, 1, 2, ..., r, \\ &A_k > A_{k'}, \qquad \text{if} \quad k' \leqslant k_l < k, \qquad\qquad \text{for some} \quad l = 1, 2, ..., r. \end{aligned} \tag{37}$$

If $A_n = 0$ then $A_k = 0, \forall k$, and therefore (36) implies that $X_i^* \equiv 0$ for all $i = m_1 + 1, m_1 + 2, ..., m$. Hence $\mathcal{N}^2 = \phi$. Further, (37) implies that $r = 1$ and $\mathcal{N}^1 = \{N\}$. That is, a type 1 buyer either gets a fraction of N or nothing. Consequently, (34) and (35) imply that prices $p_1^*, p_2^*, ..., p_n^*$ support the feasible allocation $S_1 = N$, $S_i = \phi$, $i = 2, 3, ..., m$ in $\mathscr{E}_{\mathrm{I}}$.

Suppose, instead, that $1 > A_n > 0$. Then $S^{j_r} = N$, and once again the allocation $S_1 = N$, $S_i = \phi$, $i = 2, 3, ..., m$ in $\mathscr{E}_{\mathrm{I}}$ is supported by prices $p_1^*, p_2^*, ..., p_n^*$.

Finally, suppose that $A_n = 1$. Then $k_r < n$, where $S^{j_r} = \{1, 2, ..., k_r\}$. It can be verified that as X_i^*, $i = m_1 + 1, m_1 + 2, ..., m$, must satisfy (36) and (37), and because for any $S, S' \in \mathcal{N}^2$ either $S \subset S'$ or $S' \subset S$, we must have

$\{k_{r+1}, k_{r+1}+1, ..., n\} \in \mathcal{N}^2$. Therefore, (34) and (35) imply that prices $p_1^*, p_2^*, ..., p_n^*$ support the feasible allocation $S_1 = \{1, 2, ..., k_r\}$, $S_m = \{k_{r+1}, k_{r+1}+1, ..., n\}$, $S_i = \phi$, $i = 2, 3, ..., m-1$, in $\mathscr{E}_{\mathrm{I}}$.

Case II: *Only one agent of type* 1 *or only one agent of type* 2. This case includes the case of two agents, each of a different type. The proof uses arguments similar to those in Case I, and is omitted. ∎

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