Frugal Path Mechanisms

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Abstract

We consider the problem of selecting a low cost $s-t$ path in a graph, where the edge costs are a secret known only to the various economic agents who own them. To solve this problem, Nisan and Ronen applied the celebrated Vickrey-Clarke-Groves (VCG) mechanism, which pays a premium to induce the edges to reveal their costs truthfully. We observe that this premium can be unacceptably high. There are simple instances where the mechanism pays $\Theta(k)$ times the actual cost of the path, even if there is an alternate path available that costs only $(1+\epsilon)$ times as much. This inspires the *frugal path* problem, which is to design a mechanism that selects a path and induces truthful cost revelation without paying such a high premium.

This paper contributes negative results on the frugal path problem. On two large classes of graphs, including ones having three node-disjoint $s-t$ paths, we prove that *no* reasonable mechanism can always avoid paying a high premium to induce truthtelling. In particular, we introduce a general class of *min function* mechanisms, and show that all min function mechanisms can be forced to overpay just as badly as VCG. On the other hand, we prove that (on two large classes of graphs) *every* truthful mechanism satisfying some reasonable properties is a min function mechanism.

1 Introduction

Many protocols in computer science involve the cooperation of multiple participants. For instance, interactions on the internet typically involve many different components (servers, routers, etc.), controlled by different parties. If we assume that each participant, or *agent*, is selfishly motivated, we cannot expect that the agents will necessarily follow a given protocol, particularly if it is in the agent's self-interest to cheat. The field of *mechanism design*, which grew out of economic social choice theory, studies the design of protocols in which it is never in an agent's self-interest to cheat. These protocols resist manipulation because selfish agents behaving rationally would always choose to follow the rules.

A *mechanism* is a protocol that aggregates the privately known preferences of a group of agents into some *outcome* or "social choice." An agent might misrepresent her preferences in order to manipulate the mechanism. If it is never in a single agent's interest to do so, then the mechanism is called *truthful*, or *strategyproof*. If it is also never in the interest of a coalition of agents to jointly misrepresent their preferences, then the mechanism is called *group strategyproof*. The main tool used for eliciting truthful responses is to pay money to each agent for her participation, where the amounts depend on the responses and the outcome.

In this paper we are interested in mechanisms whose payments are small by some measure. We describe these qualitatively as *frugal*. Typically, mechanism design views the payments only as an inducement to the agents to bid truthfully, while the mechanism cares only about implementing some unrelated objective function (such as a social choice function, or finding the shortest path in a graph, or solving some machine scheduling problem). Of course, the literature on auctions does care about the payments, since the whole point of selling something at auction is to make money.[1] Conversely, when the mechanism is paying to accomplish something, it makes sense to try to keep the total payment low.

We consider the setting where the mechanism must accomplish some task, and it will hire a team of agents to do it. Each agent performs a fixed service and incurs a fixed cost for performing that service. There exist certain teams of agents who can combine to perform the task, and no team is a subset of another. One special case is where the agents are edges in a network, we wish to send something through the network between nodes s and t, and the

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[1]In this case, the payments are negative, because the agents are buying something from the mechanism.

teams are $s - t$ paths. The catch is, an agent's cost is a secret known only to that agent. One can apply the celebrated VCG mechanism [23, 4, 9] to select the team that can perform the task while incurring the minimum cost *to itself*. However, the total amount that the *mechanism must pay* to the members of this team can be very high. In the network example, the cheapest team is just the shortest path with respect to the edge costs. Nisan and Ronen [17] indeed applied the VCG mechanism to solve this shortest path problem.

One drawback to the VCG mechanism when applied to the shortest path problem is that it can be forced to pay a huge amount, compared not only to the actual cost of the path, but also compared to the cost of the cheapest path that is completely disjoint from it. Roughly, we show that all reasonable path selection mechanisms can be forced to overpay just as badly as VCG. Thus, our results are negative. While we phrase everything in terms of paths because of their familiarity, all of our results also hold for the more general problem of hiring a team.

Vickrey auctions and VCG mechanisms. Historically, the first important mechanism is the Vickrey auction [23], or *second-price* auction. The basic version is a sealed-bid auction in which one item is sold to the highest bidder, but the price is equal to the second highest bid. It is easy to see that, in the absence of collusion among agents (bidders), each agent's best strategy is to bid her true value, i.e. the highest price she would be willing to pay for the item, so the mechanism is truthful. Of course, the Vickery auction can be turned around, so that the agents are contractors bidding on a job, and the auctioneer will hire the lowest bidder to do the job, paying her the second lowest bid.

The VCG mechanism addresses the more general problem of selecting from an arbitrary set of outcomes. In a VCG mechanism, each agent declares a real *valuation* function, which represents her preferences over the possible outcomes. The mechanism then selects the outcome that maximizes the total valuation of all agents, and computes a payment to each agent (which could be negative, if the agent is buying something). Assuming that each agent is trying to maximize the sum of her valuation and payment, the VCG mechanism is truthful (but not group strategyproof). Essentially, the VCG mechanism induces truthtelling by paying such that each agent's profit is equal to the value it adds to the system.

As discussed above, Nisan and Ronen [17] applied the VCG mechanism to network routing, in particular, finding shortest paths in a graph. If the mechanism chooses a path through edge e, this edge incurs the (privately known) cost c_e (i.e. its valuation is $-c_e$). The VCG mechanism asks the edges to report their costs and chooses the shortest path according to the reported costs. Viewing this as a type of auction, we will refer to the reported costs as *bids*, and use b_e to denote edge e's bid. If there exists a cut edge e separating s from t, then the chosen path would have to use e no matter what, so no mechanism can elicit truthtelling by that edge. Thus, we assume there exist two edge-disjoint paths from s to t. The mechanism's payments are such that edge e's profit is exactly the value e adds to the system, i.e. the amount by which e's presence decreases the length of the shortest path. The mechanism pays zero to all edges not on the chosen path. To an edge e on the chosen path, it pays $\mathrm{cost}|_{b_e=\infty} - \mathrm{cost}|_{b_e=0}$. Here $\mathrm{cost}|_{b_e=0}$ denotes the cost of the shortest path if e had bid zero, and $\mathrm{cost}|_{b_e=\infty}$ is the cost of the shortest path in the graph with e deleted. If e bids truthfully, then its profit is $\mathrm{cost}|_{b_e=\infty} - \mathrm{cost}|_{b_e=c_e}$, its added value. In the case that the graph consists of just some parallel arcs from s to t, the VCG mechanism for shortest paths reduces to the Vickery auction.

Frugality. The Vickrey auction has a nice property that we would hope to carry over to the shortest path problem. If we are buying a good or service from the lowest bidder, we would like to know that we are not losing too much by paying the second lowest bid. Of course, if the lowest bid is 1 and the second lowest is 100, then we are paying a big premium to enforce truthfulness. Since there are other truthful auction mechanisms, we might hope to find one that never pays more than some constant times the lowest bid. Unfortunately, elementary arguments show this is impossible. Since Vickrey pays the second lowest bid, if there is tight competition in the market then the premium for truthfulness is probably not high, because the two lowest bids are likely to be close together.

In contrast, the VCG shortest path mechanism can pay many times the cost of the chosen path, even if there is a disjoint alternate path of similar cost. Consider a graph consisting of two disjoint $s-t$ paths, P and Q. Suppose the length of path P (according to the bids) is L, the length of Q is $L(1+\epsilon)$, and P has k edges. Then the VCG mechanism chooses the path P. For each edge $e \in P$, the threshold bid is $b_e + \epsilon L$, since this is the highest it can bid before Q becomes the shorter path. Thus, the total payment made to P is $L(1+k\epsilon)$. If $k\epsilon$ is large, then we could pay a large factor times the actual cost of the chosen path. Moreover, for fixed ϵ, we pay $\Theta(k)$ times the cost of the *longer* path. The problem here is that the removal of any edge of P forces us to take path

Q. So, given the presence of the other edges, the marginal "added value" of an edge $e \in P$ is ϵL, even though it actually requires all of these edges to reap this benefit. If this entire path were one entity, then the VCG mechanism would pay it a profit of only ϵL, but since it is composed of k entities each of whom must be induced to bid truthfully, the mechanism must pay a total profit of $k\epsilon L$.

Our results. We show that no "reasonable" truthful mechanism avoids this pitfall. First we note that, roughly, the payment scheme is entirely determined by the path selection rule – the payment on an edge e in the chosen path should equal the *threshold* bid beyond which e would no longer be on the chosen path. Thus, if we restrict ourselves to selecting the shortest path, then VCG is the *only* truthful mechanism. However, it is conceivable that choosing an almost-shortest path might yield a better payment. Since each edge on the chosen path is paid at least its bid, we will still want to choose a fairly short path (with respect to the bids). The VCG mechanism paid a lot because it had to pay profits to many different edges, which suggests that we should give preference to paths with fewer edges.

A natural class of selection rules that allows us to easily encode such preferences is what we call *min function* mechanisms. For each path, we evaluate some function of the bids on its edges, and select the path with the minimum function value. The VCG mechanism is a special case, where the function is just the sum of the bids. Unfortunately, an example similar to the bad example for VCG shows that min function mechanisms can be made to perform similarly poorly.

Since min function mechanisms exhibit essentially the same bad behavior as VCG, it is logical to look for other mechanisms. We outline some basic natural properties that we argue any reasonable path selection rule should have, and verify that min function mechanisms satisfy them. We then prove a characterization result that on two reasonably large classes of graphs, every mechanism satisfying our reasonable properties is indeed a min function mechanism. One corollary of our result is that on any graph containing three node-disjoint $s-t$ paths, every mechanism satisfying our properties can be forced to exhibit the same bad behavior as the VCG mechanism. The two classes of graphs for which we prove our result are: (a) graphs with an $s-t$ arc, and (b) connected graphs containing s and t, with the addition of two node-disjoint $s-t$ paths. We also show that the result fails on any graph consisting of only two node-disjoint $s-t$ paths. We conjecture that our characterization result holds for all graphs with three node-disjoint $s-t$ paths.

2 Related Work

Recently there has been a spate of research at the border of economics, game theory and computer science. See [20] for a recent overview. The economics and game theory literature contains an enormous body of work on mechanism design, also called *implementation theory* or the *theory of incentives*. See [15, chapter 23] or [19, chapter 10] for an introduction to the field, or the surveys [13, 10]. The celebrated Vickery-Clarke-Groves (VCG) mechanism [23, 4, 9] is the main general positive result here. Nisan and Ronen [17] applied the mechanism design framework to some standard optimization problems in computer science, such as shortest paths and scheduling on unrelated machines, and raised some associated computational issues. The papers [11] and [18] address computational difficulties with the VCG mechanism. The topic of revenue maximizing auctions has received a huge amount of attention. See the recent papers [1, 3, 6, 7, 8, 14, 16, 21, 22], or [5] for a recent survey on combinatorial auctions.

The issue of frugality is raised in [2], which presents a truthful mechanism for minimizing makespan on machines with speeds. Their mechanism gives a 3-approximation and pays the machines only a logarithmic factor more than the actual costs incurred by the machines, provided no one machine dominates the total processing power. This contrasts starkly with the main result of this paper that, under suitable assumptions, truthful path selection mechanisms can be forced to pay $\Theta(n)$ times the actual costs incurred, even in a tight market.

3 Preliminaries

The costs that the edges of the graph announce for themselves we call *bids*. We let b denote the vector of bids, b_e the bid on edge e, b_{-e} the vector of all bids besides b_e, b_P the vector of bids along a path (or any other set of edges) P, and b_{-P} the vector of all bids besides b_P. We often refer to the path selected by the mechanism as the *winning* path, and say that edges on this path *win* while edges not on the path *lose*. It is easy to see that in a truthful mechanism, the payment to edge e should depend only on b_{-e} and whether e wins or loses. Thus, for fixed b_{-e}, there are only two possible payments that e can receive. It follows that there must be some threshold bid T such that if e bids below T it will win, and if e bids above T it will lose. We require that the mechanism satisfy *voluntary participation* , which means that every edge incurs non-negative profit. Since we want to minimize the payments, we should pay an edge zero if it loses. Truthfulness then requires that if e wins, we pay it T, its threshold bid. Therefore, our only flexibility is in choosing the path selection rule, since this uniquely determines the payments required to induce truthful bidding. A path selection rule is *monotone* if a losing edge can never cause itself to win by raising its bid. Thus, the truthful mechanisms that pay zero to losing edges correspond exactly to the monotone path selection rules in which any edge can always bid high enough to lose.

The edges are allowed to bid any positive real number, but not zero or ∞. However, we will come across cases where a bid is so large that it is "effectively" infinite, and we will use "bidding ∞" as a convenient shorthand for this. We will make this concept precise when we need it.

4 Desired properties for truthful mechanisms

We discuss some properties that we argue any reasonable path selection rule ought to possess. First, each edge or path should be able to control its own fate. In an ordinary auction (with the agents selling), it is a standard requirement that any agent bidding high enough will lose and any agent bidding low enough will win, provided the other agents' bids are fixed. The first requirement is necessary for a truthful mechanism, since the payment to an edge on the winning path equals its threshold bid, so in particular there must be a finite threshold bid. It would be unreasonable to enforce the second property, since it could be the case that every path containing edge e contains other expensive edges. However, if all edges along a path bid sufficiently low, that path should win. Second, since we are comparing the merits of various paths, it makes sense that raising the bids of edges not on path P should only make P more attractive. In particular, if P is the winner, it should still win even if other edges bid higher. We call this the *independence* property, because it implies that our comparison between paths P and Q is independent of the bids on all other edges. We summarize these three properties below.

- (edge autonomy) Given any edge e and any bids b_{-e} for the other edges, there exists some bid T such that if e bids T or higher, no path using e will win.
- (path autonomy) Given any bids b_{-P} for the edges off the path P, if all the edges on P bid sufficiently low then P will win.
- (independence) If path P wins, and an edge $e \notin P$ raises its bid, then P will still win.

Our last property, which we call *sensitivity*, is the most subtle. It essentially says that if two paths are "tied," then perturbing the bids appropriately should break the tie.

Definition 4.1. *Suppose that for some set of bids, path P wins. Suppose that there exists an edge e such that arbitrarily small perturbations of e's bid cause another path Q to win. Then we say paths P and Q are* tied for the lead.[2]

Sensitivity just says that the mechanism should pay attention to all of the bids. If P wins but is tied with Q for the lead, then any increase in the bid of any edge in $P - Q$ or any decrease in the bid of any edge in $Q - P$ should make Q more attractive than P. Thus, any of these perturbations should make P lose. However, they may not make P win, since there may be other paths also tied for the lead. The rigorous definition appears below.

- (sensitivity) Suppose path P wins, and Q is tied with P for the lead. Then increasing b_e for any $e \in P - Q$ or decreasing b_e for any $e \in Q - P$ causes P to lose.

We now record a simple consequence of the independence property to which we have already alluded.

Proposition 4.2. *If lowering the bid on some edge causes the winning path to change, then the new winning path must contain that edge.*

Proof: Suppose P wins, but lowering the bid on edge e causes path Q to win, where $e \notin Q$. Raising e's bid back to its original value changes the winning p ath back to P, contradicting independence. ■

5 Min function mechanisms

We define a broad class of truthful mechanisms called min function mechanisms, that allow us to easily introduce a bias toward certain paths, say ones with fewer edges. For each $s-t$ path P in the graph, define a positive real valued function f_P of the vector of bids b_P. The mechanism evaluates each function f_P and selects the path with the lowest function value, breaking ties arbitrarily. (Since there are possibly exponentially many paths, we cannot necessarily compute these mechanisms in polynomial time, even if the functions are simple. But this is of no concern to us, since we are after a characterization result.) In

[2] Note that, by independence, e belongs to either P or Q.

order for the mechanism to be truthful and satisfy the desired properties, we require the functions to satisfy the following:

- $f_P(b_P)$ is continuous and strictly increasing in b_e, for each $e \in P$
- (infinite limit) $\lim_{b_e \to \infty} f_P(b_P) = \infty$ for each $e \in P$
- (zero limit) $\lim_{b_P \to 0} f_P(b_P) = 0$.

Notice that the VCG shortest path mechanism [17] is a special case, where the function f_P simply sums the bids along the path P. We will show that the path selection rule just described is monotone and hence gives a truthful mechanism.

THEOREM 5.1. *The min function path selection rule yields a truthful mechanism.*

Proof: We must show that this path selection rule is monotone, i.e. a losing edge can never cause itself to win by raising its bid. If P is currently the winning path and $e \notin P$, then $f_P(b_P)$ is the minimum function value. Raising b_e only increases the function values for all paths including e, but does not affect f_P, so path P still wins. Thus, the mechanism is monotone.

We now show that every edge in the winning path has a threshold bid. Let P be the path not including e whose function value is minimum. Let T be the largest bid for e such that some path Q containing e has $f_Q(b_{Q-e}, T) \leq f_P(b_P)$. As e raises its bid towards T, the winning path could change, but always contains e. Once b_e is raised beyond T, path P wins. ■

Notice that our proof relies only on the functions f_P increasing with infinite limit. Continuity and the zero limit are not important for truthfulness, and we can dispense with the strictness if we employ an appropriate tie breaking rule (such as numbering the paths and breaking ties by the lowest numbered path). We included these extra requirements on the f_P's so that min function mechanisms will satisfy our desired properties.

THEOREM 5.2. *Min function mechanisms satisfy the autonomy, independence and sensitivity properties.*

Proof: Edge autonomy follows from the infinite limit property, since the graph is 2-edge connected. Path autonomy follows from the zero limit property since the functions are positive valued. Independence follows because the f_P's are strictly increasing and unaffected by the edges not on P. Sensitivity follows because the f_P's are continuous and *strictly* increasing. ■

Theorems 5.1 and 5.2 show that on any graph, a min function mechanism is truthful and satisfies the desired properties. We now show a partial converse, namely that on two large classes of graphs, every truthful mechanism satisfying the desired properties is a min function mechanism. The first class is all graphs that contain an $s-t$ arc. The second class is all graphs consisting of a connected graph containing s and t, plus two parallel $s-t$ paths whose nodes are disjoint from each other and from the rest of the graph.

First we introduce a convenient bit of notation. Since we require the bids to be positive real numbers, a bid of ∞ is not allowed. But if the bids along any one path P are bounded, then for each edge $e \notin P$ there is some T such that if e bids at least T, it will never be on the chosen path (by edge autonomy). As shorthand for "e bids above T", we write that e bids ∞.

We would like to have some concept of the bids on one path being "less than" the bids on another path. As we have noted, the independence property implies that, when deciding the winner between paths P and Q, the bids on the other edges are irrelevant. Thus, we might as well assume that the other edges all bid ∞. Suppose P bids b_P, Q bids b_Q, and all other edges bid ∞. If P wins, then we will write $b_P \prec b_Q$. For this to even make sense, it is necessary that the bids b_P and b_Q can be *realized simultaneously*, i.e. they agree on the common edges $P \cap Q$. Note that if P and Q share a node (other than s and t) then we can put together pieces of P and Q to create another finite path L. If L is the winner, then we cannot necessarily compare b_P and b_Q. In this case, we say that b_P and b_Q are *incomparable*. If P and Q are node-disjoint then they are the only two finite paths, so one of them must win and the bids are comparable. As a matter of notation, if v is an explicit vector that we intend to interpret as a vector of bids for path P, we will write it as $(v)_P$.

LEMMA 5.1. *Given a mechanism satisfying the independence property, suppose P, Q, and L are three $s-t$ paths, the bids b_P and b_L can be realized simultaneously and are comparable, and Q is node-disjoint from P and L. If $b_P \prec b_Q$ and $b_Q \prec b_L$, then $b_P \prec b_L$.*

Proof: Suppose on the contrary that when P bids b_P, L bids b_L, and the other edges all bid ∞, path L wins. Now lower the bids on Q to b_Q. By Proposition 4.2, either L still wins, or some path containing an edge in Q wins. But since Q is node-disjoint from P and L, every such path besides Q

contains an edge that bid ∞. Thus, either L or Q wins. If L wins, then independence implies it still wins after we raise the bids on all edges in $P - L$ to ∞, but this contradicts the assumption $b_Q \prec b_L$. If Q wins, then it still wins after we raise all the bids on $L - P$ to ∞, which contradicts $b_P \prec b_L$. ■

THEOREM 5.3. *If graph G contains the edge $s - t$, then any truthful mechanism for the $s - t$ frugal path problem on G that satisfies the independence, sensitivity and edge and path autonomy properties is a min function mechanism.*

Proof: For each path P with bids b_P, we must first define a value for $f_P(b_P)$. Let R denote the arc from source to sink. We call this the *reference* arc. Since R is node-disjoint from every other $s-t$ path, $(c)_R$ is comparable to every bid for every other path. Define $f_R(c) = c$ and for each other path P and set of bids b_P, define

$$f_P(b_P) = \sup\{c : (c)_R \prec b_P\} \tag{5.1}$$

In other words, $f_P(b_P)$ is the threshold bid for the reference arc. Since the mechanism is truthful, R will win if it bids below this threshold, and lose if it bids above the threshold.

Now we must show that the functions f_P select the correct path. Suppose on the contrary that $f_P(b_P) < f_Q(b_Q)$, but the mechanism selects path Q. By independence, if we raise all other bids to ∞, then Q still wins, so $b_Q \prec b_P$. By definition of f_P and f_Q, neither P nor Q can be R. Now let $c \in (f_P(b_P), f_Q(b_Q))$, so $b_P \prec (c)_R$ and $(c)_R \prec b_Q$. Applying Lemma 5.1 yields $b_P \prec b_Q$, which is a contradiction.

Finally, we must prove that the functions f_P satisfy the three properties we require of the functions defining a min function mechanism. Edge autonomy implies that $f_P(b_P) \to \infty$ as any bid along P goes to ∞, and path autonomy implies that $f_P(b_P) \to 0$ as $b_P \to 0$, since no matter what the reference arc bids, P must be able to lose to or beat it. Truthfulness implies the functions f_P are monotone increasing, since raising b_e can never cause a path containing e to beat the reference arc. Strict monotonicity and continuity are obvious for f_R, but are a bit more involved for the other paths.

To show the f_P's are strictly increasing, we fix some path P, some $e \in P$, and fix b_P to some bid $\bar{b}_P$. Set R's bid to $f_P(\bar{b}_P)$, and all other bids to ∞. Suppose that R wins (a similar argument holds when P wins). Then any increase to b_R causes P to win. Thus any decrease to b_e causes P to win (by sensitivity). Let us decrease b_e by ϵ, so $b_P = (\bar{b}_{P-e}, \bar{b}_e - \epsilon)$, so P wins. We want to show that the threshold bid for R is lower, which shows that $f_P(b_P)$ is lower. If not, than any decrease to R's bid causes R to win. So by sensitivity, any increase in b_e causes R to win. In particular, raising b_e by $\frac{\epsilon}{2}$ so that $b_P = (\bar{b}_{P-e}, \bar{b}_e - \frac{\epsilon}{2})$ causes R to win. But we already argued that P wins in this situation, so we've reached a contradiction.

To show continuity, we again argue by contradiction. Suppose we have some path P such that f_P is not continuous. Then f_P is discontinuous in some coordinate b_e, where e is an edge of P. Since f_P is monotone increasing in b_e, there must be some jump discontinuity at some point $b_P = \bar{b}_P$, where $f_P(\bar{b}_{P-e}, \bar{b}_e^-) = x$, $f_P(\bar{b}_{P-e}, \bar{b}_e^+) = y$, $x < y$, and the $^+$ and $^-$ denote limits as $b_e \to \bar{b}_e$ from above and below. Suppose $f_P(\bar{b}_P) = x$ (a similar argument holds when $f_P(\bar{b}_P) = y$). Suppose R bids $\frac{x+y}{2}$ and P bids $\bar{b}_P$, so P wins. Any increase in e's bid would raise $f_P(b_P)$ above y, so R would win. Thus, by sensitivity, any decrease in R's bid causes R to win, in particular for $b_R \in (x, \frac{x+y}{2})$, which is a contradiction. ■

THEOREM 5.4. *If the graph G consists of some connected graph including nodes s and t, plus two extra $s - t$ paths that are disjoint from the rest of graph, then any truthful mechanism for the $s-t$ frugal path problem on G that satisfies the independence, sensitivity and edge and path autonomy properties is a min function mechanism.*

Proof: Let $\mathcal{M}$ denote the mechanism. Let us call one of the disjoint paths R (for *reference* path) and the other disjoint path S (for *secondary* reference path). Define $f_R(c \cdot \mathbf{1}) = c$, where $\mathbf{1}$ denotes the vector of all ones, i.e. every edge on R bids c. For any $P \neq R$, define

$$f_P(b_P) = \sup\{c : (c \cdot \mathbf{1})_R \prec b_P\}.$$

We also need to define $f_R(b_R)$ when the bids are not all the same, but let us postpone this for now. The proof that the functions f_P for $P \neq R$ are strictly increasing, continuous, and have the two limit properties, is exactly the same as in the proof of Theorem 5.3, except that every time we set the reference arc bid to c in that argument, we set it to $c \cdot \mathbf{1}$ here. Therefore, the range of each function f_P (for $P \neq R$) is $(0, \infty)$.

We want to show that the mechanism $\mathcal{M}$ always selects a path of minimum function value. In other words, if $f_P(b_P) < f_Q(b_Q)$, we need to rule out the possibility that Q wins. If Q were to win, then, by the independence property, it would still win after we raise the bids of all edges not in P or Q to ∞, so we would have $b_Q \prec b_P$. Thus, it suffices to rule out this possibility. In other words, we want to show that if $f_P(b_P) < f_Q(b_Q)$ and b_P and b_Q are comparable, then $b_P \prec b_Q$. There are several cases, depending on whether P or Q is one of the two reference paths.

Case 1: neither P nor Q is R. In this case, we note that if $b_R = c \cdot \mathbf{1}$ where $c \in (f_P(b_P), f_Q(b_Q))$, then $b_P \prec b_R \prec b_Q$ so we can apply Lemma 5.1 to obtain $b_P \prec b_Q$.

Before we can consider the other cases, we must define $f_R(b_R)$ when the bids along R are not all the same. Let

$$f_R(b_R) = \sup\{c : \exists b_S \text{ s.t. } f_S(b_S) = c \text{ and } b_S \prec b_R\}.$$

Now that we have shown the other functions f_P to have the properties we want, we can use them to prove the properties for f_R.

Case 2: $P \neq S$ and $f_P(b_P) < f_R(b_R)$. In this case, by definition of f_R, there exist bids $\bar{b}_S$ for path S such that $\bar{b}_S \prec b_R$ and $f_S(\bar{b}_S) \in (f_P(b_P), f_R(b_R))$. By Case 1 we have $b_P \prec \bar{b}_S$. Applying Lemma 5.1 gives $b_P \prec b_R$. The argument is similar when $f_P(b_P) > f_R(b_R)$.

Case 3: $f_S(b_S) < f_R(b_R)$. Select any other $s-t$ path P in the graph. Since the range of f_P is $(0, \infty)$, there exists a bid $\bar{b}_P$ such that $f_P(\bar{b}_P) \in (f_S(b_S), f_R(b_R))$. By Case 1 we have $b_S \prec \bar{b}_P$, and by Case 2a we have $\bar{b}_P \prec b_R$, so by Lemma 5.1 we have $b_S \prec b_R$. The argument is similar when $f_S(b_S) > f_R(b_R)$.

For $P \neq R$, we already know that f_P is strictly increasing, continuous, and has the two limit properties. Now that we know the mechanism selects a path of smallest function value as the winner, we can prove that f_R also has these properties, using arguments similar to the proof of Theorem 5.3. Thus, our functions satisfy the required properties to define a min function mechanism, and $\mathcal{M}$ always selects a path of minimum function value. Since a min function mechanism may break ties arbitrarily, we choose ours to break ties the same way that $\mathcal{M}$ does. Thus, we have successfully represented $\mathcal{M}$ as a min function mechanism. ■

We conjecture that Theorem 5.4 extends to all graphs containing 3 node-disjoint $s-t$ paths P, Q, and R. In such a graph, when all other edges incident to an internal node of P or Q bid ∞, the mechanism reduces to a case covered by Theorem 5.4. It seems one ought to be able to extend the min function representation to the whole graph.

6 Costly example for min function mechanisms

We show here that any min function mechanism can be forced to pay $L(1+k\epsilon)$, where L is the cost of and k is the number of edges on the winning path, and $L(1+\epsilon)$ is the cost of some node-disjoint path. If $k\epsilon$ is large, then we would pay a large factor times the actual cost of the chosen path, even though we are in a "tight market." Moreover, for fixed ϵ, we pay $\Theta(k)$ times the cost of the *longer* path. This contrasts with the Vickrey auction, where we only pay as much as the 2nd lowest bid.

Consider a min function mechanism on a graph consisting of two node-disjoint $s-t$ paths, P and Q. Fix some $L, \epsilon \in \mathbb{R}^+$, and let $|P|$ and $|Q|$ denote the number of edges in P and Q. Let b_P^i denote the vector of bids along P where all edges bid $\frac{L}{|P|}$, except for the ith edge, which bids $\frac{L}{|P|} + \epsilon L$. Similarly, let b_Q^i denote the bids along Q where all edges bid $\frac{L}{|Q|}$, except for the ith edge, which bids $\frac{L}{|Q|} + \epsilon L$. Suppose without loss of generality that $f_Q(b_Q^1)$ is the maximum of the function values $f_P(b_P^1), \ldots, f_P(b_P^{|P|}), f_Q(b_Q^1), \ldots, f_Q(b_Q^{|Q|})$. Let b_P^0 denote the vector of bids when every edge along P bids $\frac{L}{|P|}$. Now consider how the mechanism acts when P bids b_P^0 and Q bids b_Q^1. Since $f_P(b_P^0) < f_P(b_P^1) \leq f_Q(b_Q^1)$, path P wins. To determine the payments, we must figure out the threshold bid for each edge in P. Since, for $i = 1, \ldots, |P|$, $f_P(b_P^i) \leq f_Q(b_Q^1)$, P still wins (or ties) even if one of the edges along P raises its bid by ϵL. Thus, the threshold bid for each edge in P is at least $\frac{L}{|P|} + \epsilon L$, so the total amount paid by the mechanism is at least $L(1 + |P|\epsilon)$, even though the length of P is L and Q is only $L(1+\epsilon)$. If the graph consists of more than just P and Q, we can achieve the same result by setting the bids of all other edges to be ∞.

Theorem 6.1. *Any truthful mechanism $\mathcal{M}$ on a graph that contains either an $s-t$ arc or three edge-disjoint $s-t$ paths and satisfies the desired properties can be forced to pay $L(1+k\epsilon)$, where the winning path has k edges and costs L, and $L(1+\epsilon)$ is the cost of some other node-disjoint path.*

Proof: If G belongs to one of the two classes covered by Theorems 5.3 and 5.4, then the mechanism is a min function mechanism. Otherwise, select three node-disjoint paths, P, Q, and R. When all of the other edges of G bid ∞, we can essentially ignore them, by the independence property. We can then apply Theorem 5.4 to the remaining graph, which consists of P, Q, and R. Thus, $\mathcal{M}$ reduces in this

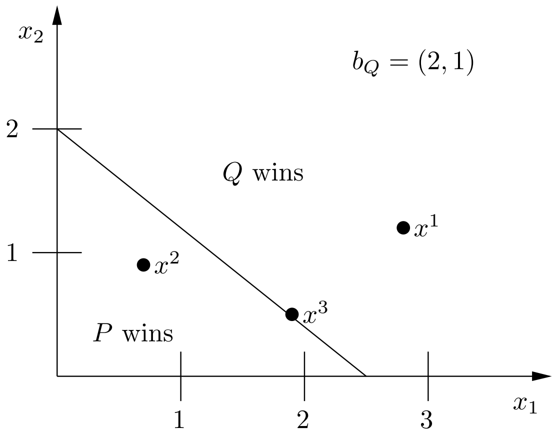

Figure 1: The winning path is determined by this "phase diagram."

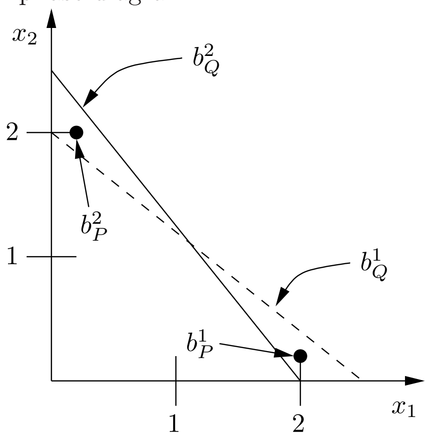

Figure 2: This figure shows that the $\prec$ relation on bids is non-transitive.

case to a min function mechanism. The preceding discussion shows how any min function mechanism can be forced to pay $\Theta(k)$ times the cost of the second-shortest path. ∎

7 An exceptional mechanism

In this section, we exhibit a truthful mechanism that satisfies the desired properties but is not a min function mechanism. Let G be a 4-node graph consisting of 2 $s-t$ paths P and Q, each having two edges. Let x_1 and x_2 be the bids along P, and y_1 and y_2 be the bids along Q. Let us describe the winning path via a "phase diagram." For a given (y_1, y_2), draw a dividing line in the $x_1 - x_2$ plane with x_1-intercept $(y_1 + \frac{y_2}{2}, 0)$ and x_2-intercept $(0, y_2 + \frac{y_1}{2})$. If (x_1, x_2) lies on or below the line, then P wins. Otherwise Q wins. In Figure 1, P loses if it bids x^1, wins if it bids x^2, and ties but wins the tiebreaker if it bids x^3.

It is easy to check that this mechanism is truthful and satisfies the desired properties. For P, the threshold bids are where the horizontal and vertical lines through (x_1, x_2) cross the dividing line in the phase diagram, provided this crossing occurs in the first quadrant, and zero otherwise. Increasing either bid along Q pushes the dividing line both up and to the right in the phase diagram, which cannot cause Q to win unless it was already the winner. Since increasing either bid can push the dividing line out arbitrarily far, the edges of Q have threshold bids as well. Thus, the mechanism is truthful and satisfies edge autonomy. Since, for fixed (y_1, y_2), the region of the phase diagram where P wins contains a neighborhood of the origin, P can win by bidding low enough. Path Q can also win by bidding sufficiently low, since this moves the dividing line arbitrarily close to the origin. Hence path autonomy holds. Independence holds trivially because there are only two paths from which to choose. If (x_1, x_2) is on the dividing line, then P wins, increasing either bid on P causes the point to move above the line so Q wins, and decreasing either bid on Q causes the line to move down and left, so Q wins. Thus, sensitivity holds.

To show that this mechanism cannot be represented by a min function mechanism, consider the bids $b_P^1 = (2, \frac{1}{5})$, $b_P^2 = (\frac{1}{5}, 2)$, $b_Q^1 = (2, 1)$ and $b_Q^2 = (1, 2)$. Figure 2 overlays the phase diagrams. The dashed dividing line corresponds to Q's bid b_Q^1 and the solid line to b_Q^2. From the phase diagram, we see that $b_P^1 \prec b_Q^1 \prec b_P^2 \prec b_Q^2 \prec b_P^1$, and perturbing any of these bids a small amount would not change the outcome. If the mechanism were representable by a min function mechanism, then we would have $f_P(b_P^1) < f_Q(b_Q^1) < f_P(b_P^2) < f_Q(b_Q^2) < f_P(b_P^1)$, which is a contradiction.

This example is easily extended to two node-disjoint $s-t$ paths, each with at least 2 edges. If the bids on P are $x_1, \ldots, x_k$ and the bids on Q are $y_1, \ldots, y_l$, then in the phase diagram we represent b_Q by the dividing line with intercepts $\sum_i x_i - \frac{x_2}{2}$ and $\sum_i x_i - \frac{x_1}{2}$, and we represent b_P by the point $(\sum_i x_i - x_2, x_2)$. Thus, the extra bids on Q just move the dividing line outward and the extra bids on P just move the point to the right.

8 Conclusion

The only way in which we relied on the graph structure of our problem is that no path is a subset of another path. If we consider the more general problem mentioned in the introduction of hiring a team to accomplish a task, all of our results hold and the arguments go through without modification. In the path selection problem, the teams are paths.

Thus, our two classes of graphs correspond to (a) instances including some team of one agent and (b) instances in which there are at least three teams, two of whose members belong to only one team.

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