Strategyproof Sharing of Submodular Costs: budget balance versus efficiency*

Hervé Moulin[†] Scott Shenker[‡]

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Abstract

A service is produced for a set of agents. The service is binary, each agent either receives service or not, and the total cost of service is a submodular function of the set receiving service. We investigate strategyproof mechanisms that elicit individual willingness to pay, decide who is served, and then share the cost among them. If such a mechanism is budget balanced (covers cost exactly), it cannot be efficient (serve the surplus maximizing set of users) and vice-versa.

We characterize the rich family of budget balanced and group strategyproof mechanisms; they correspond to the family of cost sharing formulae where an agent's cost share does not decrease when the set of users expand. The mechanism associated with the Shapley value cost sharing formula is characterized by the property that its worst welfare loss is minimal.

When we require efficiency rather than budget balance – the more common route in the literature – we find that there is a single Clarke-Groves mechanism that satisfies certain reasonable conditions. This mechanism, which we call the marginal cost pricing mechanism, is vulnerable to manipulation by coalitions, and also may not fully cover costs (but never runs a budget surplus). We compare the size of the marginal cost pricing mechanism's worst budget surplus with the worst welfare loss of the Shapley value mechanism.

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†Department of Economics, MS 22, Rice University, 6100 Main Street, Houston, TX 77005-1892, (moulin@rice.edu)

‡Corresponding Author: International Computer Science Institute, Suite 600, 1947 Center Street, Berkeley, CA 94704-1198, (shenker@icsi.berkeley.edu)

1 The problem and the results

In this paper we consider a service produced for a given set of agents, and discuss the sharing of these joint costs when the cost structure is submodular. This problem is a simple and fundamental example of production externalities. The standard model has a set N of agents who can either receive service or not (there are no finer gradations of service, it is completely binary), so agent i's preferences are entirely represented by u_i, her willingness to pay for the service. The total cost C of providing service depends on the set of agents served. If the subset (coalition) S of N receives the service, the cost $C(S)$ must be shared between the agents in S; we denote the cost share assigned to agent i by x_i. As the cost function C is not necessarily symmetrical in the different agents, we can think of "service to agent i" as an idiosyncratic good different from "service to agent j".

This simple problem arises in many familiar contexts. Canonical examples include the allocation of overhead costs and transfer pricing (Shubik [1962], Thomas [1980]), sharing the cost of public investments (Straffin and Heaney [1981], Loehman and Whinston [1979]), as well as sharing the cost of access to a network where the network may be distributing water (Okada et al. [1982]), or telecommunication services (Sharkey [1995]), or multicast messages (Herzog et al. [1997]).

Submodularity of the cost function C is a natural – though by no means the only plausible – assumption in this problem: submodularity means that the marginal cost $C(S \cup i) - C(S)$ of adding agent i to a certain set S of other agents does not increase when the set S expands.[1] Given a submodular cost function (also called concave cost function, as in Shapley [1971]) efficiency requires that a single copy of the technology be shared (even if the technology can be duplicated at no fixed cost), a situation that is often called a natural monopoly. Examples of this are discussed in Sharkey [1982], and, in the context of access to a network, by Sharkey [1995].[2]

We study incentive compatible mechanisms to share a submodular cost. We consider mechanisms that elicit from each agent her willingness to pay u_i, then decide who gets service and how the

[1] For instance, if C is a symmetrical function of all the agents (so that "service to an agent" can be viewed as a homogeneous good), submodularity of C means that C is a concave function of $|S|$.

[2] Sharkey [1995] shows that a tree network without congestion yields a submodular cost function. Fix a tree from a source to all potential agents (each agent sitting on a node of the tree) with a cost assigned to each link (or edge) of the tree. The cost of serving any coalition S is simply the sum of the costs of the links (irrespective of the number of agents "downstream" of each link) needed to reach all members of S. An important recent example of such a problem is the allocation of costs in multicast transmissions on the Internet (Herzog et al. [1997]).

cost is shared. We focus exclusively on strategyproof mechanisms, where truthful report of one's willingness to pay is always a dominant strategy. We also discuss a stronger form of incentive compatibility, namely group strategyproofness, where no coalition of agents has an incentive to jointly misreport their true willingness to pay; we require this stronger incentive compatibility property when possible but consider merely strategyproof mechanisms in contexts (to be discussed) where group strategyproofness is impossible to achieve.

The above model of cost sharing is formally identical to a cooperative game with transferable utility and has been extensively studied from the points of view of fairness – which allocation of the costs is fair? – and of core stability – which cost allocations are not threatened by the secession of a coalition standing alone to provide its own service? The core stability issue is central to the analysis of contestability, namely the equilibrium under free entry of profit maximizing firms (Baumol, Panzar, and Willig [1982]), but it is not relevant to our problem of cooperative production where the group of potential agents (or the public authority representing them) jointly own the technology and seek to design a mechanism with good efficiency and incentive properties. On the other hand, the issue of fairness in sharing costs is a plausible concern in our design problem, in particular if all agents have equal rights to use the technology. We do not directly address fairness in this paper; however some of our results have interesting consequences in this direction, in the sense that a combination of incentive and efficiency requirements compel us to use a fair mechanism whether we like it or not: see the discussion of Propositions 2 and 3 below.[3]

The literature on strategyproof allocation mechanisms when preferences are quasi-linear emphasizes the tradeoff between efficiency and budget balance: a strategyproof mechanism can be efficient (in our model: serve the joint surplus maximizing set of users) or it can balance the budget (in our model: the total payment collected from users covers the total cost exactly) but it can not be both efficient and budget balanced (Green, Kohlberg, and Laffont [1976], Roberts [1979]). This tradeoff, being particularly clear in our context, is the central concern of our paper.

When the users of the submodular cost technology are in autarchy, balancing the budget is a feasibility constraint; in this case, the mechanism design problem is to characterize the class of reasonable budget balanced and strategyproof mechanisms,[4] and in addition to identify those mechanisms that, in some sense, depart from efficiency as little as possible.

[3] In a similar vein, the same efficiency and incentive requirements force us to use a mechanism whose allocations are in the core, whether we like it or not.

[4] Recent work in this vein includes Moulin and Shenker [1992], Shenker [1992, 1995], Moulin [1994, 1999], Deb and Razzolini [1996, 1999], Ohseto [1995], Serizawa [1996, 1999], Mutuswami [1997].

An alternative, and more often traveled, route is to use a strategyproof mechanism that selects the efficient allocation (and therefore fails to balance the budget): these have been entirely characterized and are called the Clarke-Groves mechanisms (see, for example, Green and Laffont [1979], Clarke [1971], Groves [1973]). Such mechanisms can not be used by a group of agents in autarchy; indeed, they require the presence of an outside banker who can, as the case may be, siphon off the budget surplus or cover the budget deficit.[5]

We now briefly review the contents of the paper. In Section 2, our first result (Proposition 1, borrowed from Moulin [1999]) offers a characterization of a whole class of budget balanced and group strategyproof mechanisms (under some additional requirements that rule out, for instance, coercive taxes – charging an agent who does not get service – and preventing an agent from being served even if she is willing to pay an arbitrarily large amount). To each mechanism in the class, we can associate a particular cost sharing formula that computes a set of cost shares for every possible set S of users (independently of their willingness to pay); the cost share of a given user in S must not increase as the set of users expands from S to $S' \supseteq S$. The revelation mechanism (where each agent reports a willingness to pay and an allocation is implemented by the mechanism) is then equivalent to the "demand game" where each agent simply decides to request the service or not and then, given these demands for service, the cost shares are computed by means of the formula in question.

A key observation is that the class of mechanisms uncovered in Proposition 1 gives the designer tremendous flexibility, including treating equals equally by using an equitable formula (*e.g.*, the Shapley value) or treating equals unequally by means of an asymmetric formula (*e.g.*, ranking the agents in an arbitrary yet fixed fashion and charging stand alone cost $C(i)$ to the first ranked agent i requesting service, incremental cost $C(ij) - C(i)$ to the second ranked agent j requesting service, and so on).[6]

Within the above class we single out the one associated with the Shapley value formula, not because of any fairness properties, but because it generates the smallest potential welfare loss (the smallest deviation from efficiency): Proposition 2 in Section 3. Thus the paramount equitable solution of the cost sharing problem – the Shapley value – ends up being justified solely on incentive grounds,

[5]The standard story is that the "public authority" will play the role of banker, as in Green and Laffont [1979]; a genuinely "micro" example is the Direct Double Auction mechanism of McAfee [1992], where a "trader" absorbs a small budget surplus.

[6]For functions like C (and later, ξ) whose only variable is a coalition, we use the convenient but abusive notation that $C(i)$ denotes $C(\{i\})$ and $C(ij)$ denotes $C(\{i,j\})$.

without invoking any consideration of equity (not even equal treatments of equals).

Lastly, we explore the alternative route of choosing efficiency over budget balance in Section 4. Within the familiar class of Clarke-Groves mechanisms applied to our problem, Proposition 3 shows that there is exactly one mechanism that does not charge an agent not served and in which all agents participate voluntarily (in the sense that they never end up worse off than by not getting service and paying nothing). This mechanism essentially charges only marginal cost $C(S^*) - C(S^* - i)$ to an agent i in the efficient set S^* of users, and charges nothing to agents outside the efficient set. We call it the Marginal Cost Pricing mechanism. By submodularity of costs, the payments collected from the users fall short of the actual cost of service.

We conclude (in Section 4) that in the problem of sharing submodular costs, the budget balanced (but inefficient) group strategyproof mechanisms may be preferable on two counts to strategyproof mechanisms serving the efficient set of agents (but running a budget deficit): i) the former form a rich class (of linear dimension $n!$) offering much more flexibility to the mechanism designer, whereas the latter contains the single Marginal Cost Pricing mechanism,[7] and ii) the budget balanced mechanisms are group strategyproof, namely immune to joint misreports by a coalitions of agents of any size, whereas the Marginal Cost Pricing mechanism (like every Clarke-Groves mechanism in any context) is extremely vulnerable to such coalitional deviations.

The tradeoff between the budget imbalance of efficient mechanisms and the efficiency loss of budget balanced ones will be influenced by a host of factors that are beyond the scope of our model,[8] yet the relative magnitude of these two quantities is relevant to the community's decision. Thus, in Section 4 we compare the maximal welfare loss of the Shapley value mechanism with the maximal budget deficit of the Marginal Cost Pricing mechanism and find the former to be typically smaller than the latter, although we cannot make rigorous general statements about their relative sizes.

Section 5 collects a handful of concluding comments, and the Appendix contains the proof of Propositions 1, 2, and 3.

Remark 1: We note that our entire analysis can be replicated in the "dual" model of surplus

[7] For instance, the designer has no choice but to treat agents equally, in the sense that if two agents impose equivalent costs ($C(S+i-j) = C(S+j-i)$ for all $S \subseteq N$) then reporting equal willingness to pay ($u_i = u_j$) must result in equal allocations.

[8] In the regulation literature, this tradeoff is often captured by one parameter called the "shadow price of public funds" (see *e.g.*, Laffont and Tirole [1994]). This shadow price is a way of comparing these two kinds of losses via a conversion factor.

sharing, where each agent can contribute an idiosyncratic and binary input (*e.g.*, effort) and the total surplus (output produced) $F(S)$ must be shared between the agents, where S is the subset of these agents contributing their input. If we assume preferences to be quasi-linear in the output, then an individual preference is completely represented by u_i, the (quantity of output equivalent to her) disutility of contributing the output. Assuming *super*modularity of the production function F, all of our present results apply to the surplus sharing case as well. For instance, the analog of the Marginal Cost Pricing mechanism is the Marginal Surplus Sharing mechanism, where an agent in the efficient coalition receives (approximately) her marginal contribution $F(S^*) - F(S^* - i)$ so that actual output is not enough to cover these surplus allocations. In the literature on the exploitation of common property resources (*e.g.*, Weitzman [1974], Sen [1966], Israelsen [1980], Ostrom [1991]) the surplus sharing model is more popular than the cost sharing one though the submodularity of the production function is much more common than its supermodularity. Budget balance is always assumed in this literature.

2 Budget balanced and group strategyproof mechanisms

We first present some basic definitions. We are given a set of agents N ($n = |N|$) and a cost function C which is nondecreasing, nonnegative and submodular. More precisely, for any $S, T \subseteq N$ we assume:

$$C(\emptyset) = 0\ ; \quad S \subseteq T \Rightarrow C(S) \leq C(T);$$

$$C(S \cap T) + C(S \cup T) \leq C(S) + C(T)$$

We denote by u_i, $u_i \geq 0$, the amount agent i is willing to pay for service. A revelation mechanism elicits from each agent her willingness to pay and, based on these reports, chooses who gets service and how the costs are shared.[9] That is, the mechanism takes as input the vector u of the reported willingness to pay, and then produces as output the set of allocations. We use the vectors x and q to denote the resulting allocation decisions; x_i is the cost share imputed to agent i, and q_i describes whether or not agent i is given service ($q_i = 0$ if agent i is not given service, and $q_i = 1$ if agent i is given service). When we want to make the utility profile dependence of these allocations explicit, we will write $q(u)$ and $x(u)$ to denote the allocations resulting at a particular profile. Thus, a *revelation mechanism* is a mapping M associating to all profiles $u \in \Re^n_+$ a coalition $S(u) \subseteq N$ (equivalently expressed by the binary inclusion vector $q(u)$ with $q_i(u) = 1$ for all $i \in S(u)$ and

[9] Following established, but somewhat confusing, notational practice, the vector u will denote the reported values as well as the truthful values. When the two differ, we will make special note of it.

$q_i(u) = 0$ for all $i \notin S(u)$) and a vector $x(u)$ of monetary compensations. We assume quasi-linear utilities, so an agent's utility is simply given by $u_i q_i - x_i$. In Section 5 (comment 1) we explain why in our model this assumption entails no serious loss of generality.

Denote by $w(S, u)$ (or simply $w(S)$ if the underlying utility profile is unambiguous) the stand alone surplus of coalition S given the technology C:

$$w(S, u) = \max_{T \subseteq S} [u_T - C(T)] \tag{1}$$

where we use the notation that $u_T = \sum_{i \in T} u_i$.

To make our incentive properties precise, fix any coalition $T \subseteq N$ and any two profiles u, u' such that $u_j = u'_j$ for all $j \notin T$. Denote by (q, x) and (q', x') the allocations implemented at u and u' respectively. Group strategyproofness requires that if the inequality

$$u_i q'_i - x'_i \geq u_i q_i - x_i$$

holds for all $i \in T$ then it must be an equality for all $i \in T$ as well; that is, if no member in T is made worse off by changing from u to u', then no member of T is made better off. Strategyproofness is merely imposing this requirement for all singleton T's (*i.e.*, $|T| = 1$), so unilateral deviations from the truth never produce better outcomes for the deviating agent.

We are focusing on contexts where agents have the option not to participate and expect to have some control over their destiny. We postulate that a mechanism must satisfy the following three properties:

No Positive Transfers (NPT) The cost shares are nonnegative: $x_i \geq 0$ for all i.

Voluntary Participation (VP) The welfare level corresponding to no service ($q_i = 0$) at no cost ($x_i = 0$) is guaranteed to each agent if they report truthfully.

Consumer Sovereignty (CS) Every agent has a message u_i guaranteeing that, regardless of the other reported values u_{-i}, she gets service ($q_i = 1$).

No Positive Transfers is a standard assumption in cost sharing situations, where positive net transfers to agents are disallowed. It is <u>not</u> a natural requirement for Clarke-Groves mechanisms, which do not balance the budget. However, NPT is not an essential assumption for the characterization of the MC mechanism in Section 4 (see the Remark following Proposition 3).

Voluntary Participation is the standard individual rationality constraint; agent i does not prefer the "status quo" ($q_i = x_i = 0$) allocation to the allocation (q_i, x_i) assigned by the mechanism. In view of No Positive Transfers, this implies that she pays nothing if she gets no service ($q_i = 0 \Rightarrow x_i = 0$).

Consumer Sovereignty means that by reporting a high enough willingness to pay, any agent is guaranteed to get the service. For instance, in the mechanisms discussed below, reporting any $u_i \geq C(i)$ (their stand alone cost) ensures that $q_i = 1$ no matter what the other reported u_j are. Note that we do not require that reporting $u_i = 0$ ensures that $q_i = 0$; as long as $x_i = 0$ when $u_i = 0$, we can have $q_i = 1$. If $C(S)$ is strictly increasing in S, however, the later configuration ($u_i = 0$ but $q_i = 1$) will not happen in any of the mechanisms discussed below.[10]

The strategyproof and budget balanced mechanisms we consider are derived from an underlying *cost sharing method*. A cost sharing method is a formula ξ associating to each cost function C and to each coalition S of agents an allocation of the total cost $C(S)$ among these agents in the form of nonnegative cost shares $\xi_i(S)$.[11] These cost shares must satisfy the budget balancedness constraint $\xi_S(S) = C(S)$.

We now explain why these cost sharing methods underlie the relevant strategyproof and budget balanced mechanisms. Given a cost sharing method and a profile u of willingness to pay, consider the (normal form) *demand game* where each agent decides whether or not to request service (agent i's strategy choice is $q_i = 0$ or $q_i = 1$) and costs are shared among those who request service according to the cost sharing method ξ. If it so happens that for all profiles u the demand game has a unique α-equilibrium (where α-equilibrium can be either a Nash equilibrium or a strong equilibrium) then this α-equilibrium outcome defines a strategyproof revelation mechanism. That is, this mechanism elicits u_i, $i \in N$, and chooses the allocations of the unique α-equilibrium outcome of the reported demand game. The fact that such a mechanism is strategyproof is a classic result of the implementation literature: see Dasgupta, Hammond, and Maskin [1979] and Comment 1 in Section 5. The key to the above property (a unique α-equilibrium for all profiles) is that the cost sharing method must be *cross-monotonic*, so agent i's cost share cannot increase when the set of agents receiving service expands:[12]

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[10] If $u_i = 0$ but $C(S+i) > C(S-i)$ for all $S \subseteq N$, then i is not part of the efficient coalition. In the mechanisms discussed below, the coalition served is always a subset of the efficient coalition, so $u_i = 0 \Rightarrow q_i = 0$ whenever agent i imposes strictly positive incremental costs.

[11] In the language of cooperative game theory, a cost sharing method is a "value" defined for all reduced games (S, C).

[12] In some of the literature, the cross monotonicity property is known as *population monotonicity*, such as in

$$S \subseteq T, \quad i \in S \Rightarrow \xi_i(T) \leq \xi_i(S) \tag{2}$$

(recall that $\xi_i(S)$ is agent i's cost share when S is the set of agents who get service). Under cross monotonicity, the demand game has nondecreasing best reply functions: if $q_i = 1$ is a best reply when a coalition S gets service, then $q_i = 1$ remains a best reply when any superset of S gets service. Therefore, the demand game possesses a Pareto superior Nash equilibrium which is, in addition, a strong equilibrium (Aumann [1959]). This equilibrium can be computed by the simultaneous Cournot tatonnement starting at N, that is to say the equilibrium outcome $\tilde{S}(\xi, u)$ is the limit of the following inclusion monotonic sequence:

$$S^0 = N \; ; \quad S^{t+1} = \{i | u_i \geq \xi_i(S^t)\} \tag{3}$$

In the first step of the algorithm all agents are offered service at the personalized price $\xi_i(N)$, and only those in S^1 accept; in the second step, the agents in S^1 are offered service at the (higher) price $\xi_i(S^1)$ and those in S^2, $S^2 \subseteq S^1$ accept; and so on. This algorithm defines an inclusion monotonic sequence in N, and its outcome[13] $\tilde{S}(\xi, u)$ is the unique strong equilibrium (there may be other strong equilibrium q's but they are all welfare equivalent, *i.e.*, produce the same welfare for every agent) of the demand game (N, ξ, u). We associate to the cost sharing method ξ the following revelation mechanism denoted by $M(\xi)$:

$$S(u) = \tilde{S}(\xi, u); x_i(u) = \xi_i(\tilde{S}(\xi, u)) \; if \;\; i \in \tilde{S}(\xi, u); \;\; x_i(u) = 0 \; otherwise \tag{4}$$

If ξ is cross-monotonic then $M(\xi)$ is strategyproof. In fact, $M(\xi)$ is group strategyproof as well. Moreover, the converse statement holds.

Proposition 1 *For any cross monotonic cost sharing method ξ, the mechanism $M(\xi)$ (defined by (4)) is budget balanced, meets NPT, VP, CS, and is group strategyproof. Conversely, for any revelation mechanism M satisfying budget balance, NPT, VP, CS, and group strategyproofness there exists a cross monotonic cost sharing method ξ such that $M(\xi)$ is welfare equivalent to M.*

The Proposition is identical to Theorem 2 in Moulin [1999]. In the special case of an excludable public good ($C(S) = 1$ for all nonempty S), Deb and Razzolini [1996, 1999] and Ohseto [1995] provide similar yet different characterization results.

Sprumont [1990], Thomson [1983].

[13] We will use the notation $\tilde{S}(u)$ if, from the context, the underlying cost sharing method is unambiguous.

We do not provide a proof of the "converse" statement in Proposition 1; however, the "direct" statement is easy to prove and we repeat its proof in Appendix A.

Remark 2: We note that all three assumptions NPT, VP and CS are needed in the characterization result. This is shown by the following three examples. Denote by e the "egalitarian" allocation $q_i = 1$ and $x_i = C(N)/n$ for all i. Consider two voting mechanisms between e and the "null" allocation ($q_i = x_i = 0$ for all i). If unanimity is required to implement e (*i.e.*, every agent can force the null outcome), we have a mechanism satisfying all properties listed in Proposition 1 but Consumer Sovereignty. If unanimity is required to implement the null outcome (every agent can force e), the mechanism meets all properties but Voluntary Participation. Our third mechanism involves two agents (for simplicity) and meets all properties but No Positive Transfers:

$$S(u) = \{12\},\ x(u) = (C(12), 0)\ if\ u_1 \geq C(12)\ and\ u_2 \geq C(12) - C(1)$$

$$S(u) = \{1\},\ x(u) = (C(12), C(1) - C(12))\ if\ u_1 \geq C(12)\ and\ u_2 < C(12) - C(1)$$

$$S(u) = \{2\},\ x(u) = (0, C(2))\ if\ u_1 < C(12)\ and\ u_2 \geq C(2)$$

$$S(u) = \emptyset,\ x(u) = (0, 0)\ if\ u_1 < C(12)\ and\ u_2 < C(2)$$

The family of revelation mechanisms uncovered by Proposition 1 leaves a fair degree of flexibility to the mechanism designer. Suppose first the agents have equal right to the use of the technology C, so that the property "equal treatment of equals" is deemed desirable: this requires that if the function C is symmetrical in i and j (*i.e.*, $C(S+i) = C(S+j)$ for all $S \subseteq N - \{i,j\}$), the cost shares of agents i and j should be symmetrical as well.

Then the designer can choose from a variety of formulae, including the Shapley value (Shapley [1953])[14], the egalitarian solution (Dutta and Ray [1989])[15], convex combinations of these, and more.[16] For instance, in the case of two agents $N = \{1, 2\}$, these two values are

[14]For a proof that this value is cross monotonic when C is submodular, see, for example, Sprumont [1990].

[15]The egalitarian solution of the cooperative game (N, c) picks the most egalitarian profile of cost shares x^* within the stand alone core of this game ($x^*_S \leq C(S)$ for all S). Dutta [1990] shows that this satisfies cross monotonicity.

[16]See Sprumont [1990] for a systematic discussion of the cross monotonicity property. Note that the nucleolus (Schmeidler [1969]), another familiar value for cooperative games, is not cross monotonic (see Sonmez [1993]).

Shapley $\xi_1(12) = \frac{1}{2}(C(12) + C(1) - C(2))$; $\xi_2(12) = \frac{1}{2}(C(12) + C(2) - C(1))$

Egalitarian $\xi_1(12) = med\{\frac{C(12)}{2}, C(1), C(12) - C(2)\}$; $\xi_2(12) = med\{\frac{C(12)}{2}, C(2), C(12) - C(1)\}$
(where $med\{\cdot\}$ is the median operator)

Another possibility is that the agents have unequal claims on the technology C (perhaps because of seniority, earlier contributions to its development, and so on) and that the designer wishes to choose a cost sharing formula that fairly reflects this fact. For simplicity, suppose that the agents are ordered as $\{1, 2, \ldots, n\}$ and that their claims are lexicographic: agent 1's claim outweighs any other claim; agent 2's claim outweighs those of any agent $3, 4, \ldots, n$, and so on. A consequence of Proposition 1 is that this hierarchy of claims determines entirely the choice of our designer (within the class defined by NPT, VP, CS, and group strategyproofness). Indeed, in any cross monotonic cost sharing method, agent 1 must pay out at least $C(S) - C(S-1)$ where S is the coalition of agents who get the service; this follows from $\xi_{S-1}(S) \leq \xi_{S-1}(S-1) = C(S-1)$ and $\xi_1(S) = C(S) - \xi_{S-1}(S)$. Thus, agent 1's superior claim entitles him to this minimal cost share $C(S) - C(S-1)$. Similarly, agent 2's cost share must be at least $C(S-1) - C(S - \{1,2\})$ (by repeating the above argument), and so on. The resulting cost sharing method is the familiar *hierarchical stand alone* method (also called the marginal contribution method) where the agent i with the largest index in S (*i.e.*, the agent with the least claim to the resources) pays her stand alone cost $C(i)$, the agent with the next largest index j pays $C(ij) - C(j)$, and so on. This method is cross monotonic given a submodular cost function. Recall that the Shapley value is merely the arithmetic average of such methods over all possible orderings of N.

3 The Shapley Value Mechanism

We now turn to the outstanding position of the Shapley value within the class of cross monotonic cost sharing methods. Given such a method ξ, we denote by $\gamma(\xi, u)$ the welfare loss at a particular profile u and let $\gamma(\xi)$ be the maximal welfare loss, over all profiles, of the mechanism $M(\xi)$ defined by (4);

$$\gamma(\xi) = \sup_{u \in \Re^n_+} \gamma(\xi, u) = \sup_{u \in \Re^n_+} [w(N, u) - (u_{\tilde{S}(\xi,u)} - C(\tilde{S}(\xi, u)))]$$

where, recall, $w(S, u) = \max_{T \subseteq S}[u_T - C(T)]$.

We now seek to minimize this maximal loss through the choice of ξ. This leads us to the Shapley value cost sharing method, which we write as

$$\xi_i^*(S) = \sum_{T \subseteq S-i} \frac{|T|!(|S|-|T|-1)!}{|S|!}[C(T \cup i) - C(T)] \;\; for\ all\ S\ and\ i \in S$$

We refer to the corresponding revelation mechanism $M(\xi^*)$ as SH.

Proposition 2 *Among all mechanisms $M(\xi)$ derived from cross monotonic cost sharing methods, SH has the uniquely smallest maximal efficiency loss*

$$\gamma(\xi^*) < \gamma(\xi) \;\; for\ all\ \xi \neq \xi^*$$

which is given by the formula below.

$$\gamma(\xi^*) = \left(\sum_{S \subseteq N} \frac{(|S|-1)!(n-|S|)!}{n!} C(S) \right) - C(N) \tag{5}$$

The proof of Proposition 2 is presented in Appendix B. The function P defined in formula (5) is the *potential* function introduced by Hart and Mas-Colell [1989]. Our proof relies on the representation of the Shapley value by the potential function, due to these authors. [17]

Remark 3: In contrast to Proposition 2 that singles out the Shapley value among all cross monotonic cost sharing methods, Mutuswami [1997] demonstrates that the egalitarian cost sharing method of Dutta and Ray [1989] is characterized by the property that the expected size of the coalition receiving service is maximal. This result assumes IID willingness to pay with monotone hazard rate.

4 Marginal Cost Pricing Mechanism

We now discuss the class of strategyproof and efficient mechanisms under quasilinear utilities, namely the Clarke-Groves (CG) mechanisms. By the standard characterization result of CG mechanisms (*e.g.*, Green and Laffont [1979]) the property of strategyproofness combined with the efficiency of $S(u)$ implies the following form for $x(u)$:

$$x_i(u) = u_i q_i(u) - (w(N,u) - h_i(u_{-i})) \tag{6}$$

[17] Monderer and Shapley [1993] establish a seemingly unrelated property of the demand game associated with an arbitrary cost function C (not necessarily submodular, not even nondecreasing) and an arbitrary cost sharing method. They show that the demand game is a "potential game" if and only if the method is the Shapley value.

where h_i is a function that depends on all the willingness to pay values u_j excepting u_i.

In order to define the Marginal Cost Pricing mechanism (MC), we state without proof two easy and important consequences of the submodularity of C for the surplus function $w(S, u)$ and the maximizing coalitions.

Fact 1: For any u, $w(\cdot, u)$ is supermodular: $w(S, u) + w(T, u) \leq w(S \cup T, u) + w(S \cap T, u)$ for any $S, T \subseteq N$.

Fact 2: If any two coalitions S and T are efficient (*i.e.*, $u_S - C(S) = u_T - C(T) = w(N, u)$) then so is $S \cup T$; we denote by $S^*(u)$ the largest efficient coalition at u, and denote by $q^*(u)$ the corresponding allocation vector.

The MC mechanism picks the coalition $S(u) = S^*(u)$ and charges the cost shares $x^*(u)$ given by the formula:

$$x_i^* = u_i q_i^*(u) - (w(N, u) - w(N - i, u)) \tag{7}$$

We call it the marginal cost pricing mechanism because, in the case of a symmetric cost function, $C(S) = c(|S|)$, this mechanism charges approximately the marginal cost $c(|S^*|) - c(|S^*| - 1)$ to all agents and this approximation becomes an equality when each agent becomes a vanishingly small part of N (we omit the proof for brevity). [18]

Our choice of MC mechanism out of the whole CG class is by no means arbitrary. In fact, once we require NPT and VP[19] it is essentially the only available CG mechanism.

Proposition 3 *Consider a strategyproof revelation mechanism M selecting an efficient allocation (not necessarily the largest) at all profiles, meeting NPT and VP. Then M is welfare equivalent to MC: for all u, all i*

$$u_i q_i(u) - x_i(u) = u_i q_i^*(u) - x_i^*(u)$$

Conversely, the MC mechanism meets NPT and VP (as well as CS) and is strategyproof.

[18] The MC mechanism is quite similar in form to the traditional *pivotal* mechanism, which assigns the net utility $u_i - x_i = w(N) - w^*(N - i)$, where $w^*(N - i) = \max_{T \subseteq N - i}[u_T - \frac{|T|}{|T|+1} C(T + i)]$ (see, for example, Green and Laffont [1979]). In substance, however, they are quite different; the pivotal mechanism never runs a budget deficit (i.e, it either is budget balanced or runs a surplus) and fails VP, whereas the MC mechanism never runs a budget surplus (i.e, it either is budget balanced or runs a budget deficit) and satisfies VP.

[19] We need not require CS since all CG mechanisms satisfy CS; the set of efficient coalitions is independent of the mechanism, and if $u_i > C(i)$ all efficient coalitions will include agent i when the cost is submodular.

The proof of this Proposition is presented in Appendix C.

Remark 4: Both assumptions NPT and VP are needed in this characterization result. If we drop VP, the pivotal mechanism (see Footnote 18 meets all other assumptions. If we drop NPT then we can modify the MC mechanism by adding an arbitrary cash payment (from the banker to each agent) independent of the reported profile. The proof of Proposition 3 makes clear that if a strategyproof mechanism selects an efficient allocation and satisfies VP, then it must be "Pareto superior" to MC in the sense that at all profiles every agent incurs a smaller charge x_i. Such a mechanism obviously runs an unambiguously higher budget deficit at all profiles.

Note that the MC mechanism, like any mechanism in the CG class, is *not* group strategyproof. Moreover, there is a special kind of "altruistic" manipulation (a violation of group strategyproofness) that does not require any coordination between agents and to which the MC mechanism is always vulnerable. Fix a profile u (the true profile) and a willingness to pay u'_i for agent i such that $u_i < u'_i$. Denoting $u' = (u_{-i}, u'_i)$ we observe the following facts:[20]

Fact 1: $S^*(u) \subseteq S^*(u')$

Fact 2: no agent j, $j \neq i$, is hurt: $w(N,u) - w(N-j,u) \leq w(N,u') - w(N-j,u')$

Fact 3: if $i \in S^*(u)$ then $S^*(u') = S^*(u)$ and $x_i^*(u) = x_i^*(u')$

Facts 2 and 3 tell us that if an agent i is served at profile u, reporting a higher willingness to pay is a matter of indifference to her, and it can never hurt any other agent. To see that in some cases it can strictly help other agents, consider the following example with two agents:

$$C(1) = C(2) = 6\ ;\quad C(12) = 8\ ;\quad u_1 = u_2 = 5$$

Truthful reports yield $S^*(u) = \{1,2\}$ and cost shares $x_1^*(u) = x_2^*(u) = 3$. By raising her willingness

[20] To check Fact 1, denote $S = S^*(u)$, $T = S^*(u')$ and compute

$$u_{S\cap T} - C(S\cap T) \leq u_S - C(S) \Rightarrow C(S\cup T) - C(T) \leq C(S) - C(S\cap T) \leq u_{S-T} \leq u'_{S-T} \Rightarrow u'_T - C(T) \leq u'_{S\cup T} - C(S\cup T)$$

implying, by definition of T, $S \cup T = T$. Similar reasoning establishes $S^*(u, N-j) \subseteq S^*(u,N)$ for all j. To prove Fact 2, namely $w(N-j,u') - w(N-j,u) \leq w(N,u') - w(N,u)$, we note that both sides of the inequality are nonnegative and we distinguish three cases. If $S^*(N-j,u)$ contains i, so do $S^*(N,u)$, $S^*(N,u')$, and $S^*(N-j,u')$ and both terms of the inequality equal $u'_i - u_i$. If $S^*(N-j,u')$ does not contain i, neither does $S^*(N-j,u)$ and the left hand term is zero. If $i \notin S^*(N-j,u)$ but $i \in S^*(N-j,u')$, call u_i^* the smallest value such that $S^*(N-j,(u_{-i},u_i^*))$ contains i. Because $S^*(N,(u_{-i},u_i^*))$ contains i as well, the righthand term in the inequality is at least $u'_i - u_i^*$ whereas the lefthand term equals $u'_i - u_i^*$. The easy proof of Fact 3 is omitted.

to pay to $u_1' = 7$, agent 1 does not affect her allocation and reduces agent 2's cost share to $x_2^*(u_1', u_2) = 2$. The banker is the sucker!

In addition to being vulnerable to coalitional manipulations, the MC mechanism does not balance the budget. The maximal budget deficit (call it β) of MC can be computed with the help of (7) which defines the cost allocations of MC:

$$\beta = \sup_{u \in \Re_+^n} \left[(n-1)w(N,u) - \sum_{i \in N} w(N-i,u) \right] \tag{8}$$

Proposition 4 *For any nondecreasing and submodular cost function C such that $C(\emptyset) = 0$, we have:*

$$\beta = \left(\sum_{i \in N} C(N-i) \right) - (n-1)C(N) \tag{9}$$

Proof: Check first a consequence of submodularity: the quantity $\sum_{i \in N} C(S-i) - (n-1)C(S)$ is nondecreasing in S (using the convention that $S - i = S$ if $i \notin S$). We omit the straightforward proof. Then fix a profile u and set $S = S^*(u)$. Compute

$$(n-1)w(N,u) - \sum_{i \in N} w(N-i,u) \le (n-1)(u_S - C(S)) - \sum_{i \in N} (u_{S-i} - C(S-i))$$

$$= \sum_{i \in N} C(S-i) - (n-1)C(S) \le \sum_{i \in N} C(N-i) - (n-1)C(N)$$

In view of (8) this shows $\beta \le \sum_{i \in N} C(N-i) - (n-1)C(N)$. The converse inequality follows by choosing a profile with each u_i large enough so that $S^*(u) = N$ and $S^*(u_{-i}) = N - i$ for all i. This completes the proof. □

We now compute the respective losses, budget imbalance and inefficiency, of the SH and MC mechanisms with symmetric cost functions where $C(S) = c(|S|)$, c is concave on $\{0, 1, 2, \ldots, n\}$ and $c(0) = 0$. Let $\gamma(u)$ and $\beta(u)$ denote the loss at a particular utility profile u for SH and MC respectively, and let γ and β denote the respective worst case losses. These worst case losses take the form:

$$\frac{\beta}{n} = \left(\frac{c(n)}{n} - (c(n) - c(n-1)) \right) \tag{10}$$

$$\frac{\gamma}{n} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{c(i)}{i}\right) - \frac{c(n)}{n} \tag{11}$$

A consequence of equations (10) and (11) is that if $(ac - mc)(s)$ increases (respectively, does not decrease) in s then $\gamma < \beta$ (respectively, $\gamma \leq \beta$).[21] Conversely, if $(ac - mc)(s)$ decreases (respectively, does not increase) in s then $\gamma > \beta$ (respectively, $\gamma \geq \beta$).

Figures 1 and 2 illustrate these two cases in the case of a large n. For instance, if $c(n) = n^p$, $0 < p < 1$, then $ac - mc$ decreases and we have $\beta = p\gamma$ for large n. On the other hand, if mc is linearly decreasing up to n, then $ac - mc$ is increasing and we have $\gamma = \frac{\beta}{2}$.

The above comparison bears on the worst case losses (deficit and inefficiencies). On the other hand, the comparison of $\gamma(u)$ and $\beta(u)$ (welfare loss and deficit loss, respectively, at a given profile u) is unambiguously more favorable to the SH (or, equivalently, average cost pricing in this context) mechanism if we assume that the demand function is concave, as the reader will easily check by drawing pictures like Figures 1 and 2 with a concave demand function D. For instance, in the case where both D and mc are linear, straightforward computations (omitted for the sake of brevity) show that $\gamma(u) \leq \frac{\beta(u)}{8}$, a far cry from the worst case comparison $\gamma < \frac{\beta}{2}$.

5 Concluding Comments

Comment 1: We mentioned in Section 2 that the assumption of quasi-linear utilities involves no loss of generality. Indeed in our model, the properties NPT and VP imply that $x_i = 0$ if $q_i = 0$ hence agent i's consumption set is the union of $(0,0)$ ($q_i = 0$, $x_i = 0$) and of the half line $(1, x_i)$, $x_i \geq 0$. On this set, a classical preference is entirely described by a simple number u_i (the willingness to pay) such that $(0,0)$ is indifferent to $(1, u_i)$. Hence the classical domain (monotonic preferences – strictly in money – continuous and convex) and that of preferences represented by quasi-linear utilities coincide. In particular we can apply Theorem 2 in Moulin [1999] developed for the classical domain of preferences. Moreover, our domain is monotonically closed (Dasgupta,

[21] Indeed, assume $ac - mc$ increases and so, for all $i = 1, \ldots, n-1$, $ac(i) < mc(i) + ac(n) - mc(n)$ which then implies:

$$\frac{\gamma}{n} < \frac{1}{n}(\sum_{i=1}^{n} mc(i)) + (ac(n) - mc(n)) - ac(n) = ac(n) - mc(n) = \frac{\beta}{n}$$

Other cases are similar.

Hammond and Maskin [1979]), a property that fails for general quasilinear utilities.

On the other hand, if preferences are not quasi-linear, efficiency is no longer equivalent to maximization of joint surplus, therefore our Propositions 2 and 4 are no longer valid.

Comment 2: We have focused on the maximal welfare loss as the metric to be minimized. Of course, one might be more interested in other metrics like the average welfare loss. Consider, for example, the case where the u_i are all independent and uniformly distributed in $[0, 1]$, and the cost function is $C(S) = 1$ for all nonempty S (this is the case of an excludable public good, as in Deb and Razzolini [1996, 1999]). For $n = 2$ the average welfare loss resulting from SH is easily calculated to be $\frac{1}{4}$. With somewhat more effort we can calculate the average behavior in the limit of large n. The SH average welfare loss can be computed in two parts. The welfare loss due to members being excluded when $S(u) \neq \emptyset$ (*i.e.*, at least some agents are given service) is given by the sum

$$\sum_{k=0}^{n-1} \frac{k}{2(n-k+1)} \frac{n!}{k!(n-k)!} (n-k+1)^{-k} (1 - \frac{1}{n-k})^{n-k} \approx \frac{1}{2n}$$

which vanishes for large n. The other component of welfare loss is when $S(u) = \emptyset$, but $S^*(u) \neq \emptyset$. We can bound this portion of the welfare loss by $f(n)$ (the maximal loss when no one gets service) times the probability $P(n)$ that $S(u) = \emptyset$. We can bound $P(n)$ by the probability that there is no more than one agent with $u_i \geq 0.5$, so $P \leq 2^{-n}(1 + n)$. Thus, $f(n)P(n)$ is asymptotically bounded by $2^{-n} n \log n$ and so vanishes rapidly as n diverges. Therefore, the asymptotic average welfare loss vanishes in large systems for the SH method.[22]

On the other hand, it is easy to show that the expected revenue from the MC mechanism falls at least as quickly as $n^{0.5}$, and so asymptotically the budget deficit is the full cost of the efficient set. Thus, at least in this case, for large populations the SH mechanism approaches perfect efficiency whereas the MC mechanism approaches maximal budget imbalance (no revenue collected).

Note that the above comparison is completely turned around if we go back to the worst case metric. Indeed, the worst case deficit of the MC mechanism is $\beta = 1$ (when the mechanism collects no money at all) whereas the worst case welfare loss of the average cost pricing mechanism, computed from equation (11), is $\gamma = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n$ for large n. In fact, this choice of cost function ($C(S) = 1$ for all nonempty S) yields the largest ratio for $\frac{\gamma}{\beta}$.

Comment 3: Our results extend directly to the case where there are positive externalities of

[22] Deb *et al.* [1997] present a similar calculation of average losses.

consumption; that is, agent i's utility for the service is a function $u_i(S)$ where S is the set of agents receiving service (and $u_i(S) = 0$ if $i \notin S$). If $u_i(\cdot)$ is supermodular then all mechanisms described in Proposition 1 keep the same strategic features; the characterization still holds as long as the case of no consumption externalities is a subdomain of the full utility domain. In models of networks (of many varieties) the assumption of positive externalities is commonplace.

Comment 4: One can also consider situations where the cost function is supermodular rather than submodular. Supermodularity occurs, for example, in tree networks with concave congestion (*i.e.*, the cost of the link is a concave function of the number of agents "downstream"). For supermodular costs, the MC mechanism remains the canonical CG mechanism, as Proposition 3 and its proof remain valid without change. However, the computation of its maximal budget imbalance becomes hard (and the imbalance can be a surplus or a deficit). On the other hand, the demand game associated with the Shapley value cost sharing method does not, in general, have a unique strong equilibrium (the best reply functions are decreasing, so multiple noncomparable equilibria are possible), hence it does not result in a strategyproof revelation mechanism. The (small) class of cost sharing methods that do result in a strategyproof mechanisms (because their demand games always have a unique strong equilibrium) has been characterized in Moulin [1999]. It reduces essentially to the family of hierarchical stand alone methods discussed in Section 2, where agents pay their incremental costs according to a fixed ordering of N: $\xi_{\sigma(i)}(N) = C(E(\sigma, i)) - C(E(\sigma, i-1))$ for some ordering σ (where $E(\sigma, i)$ is the set of the first i agents in ordering σ). The criterion of minimizing the maximal welfare loss does not seem to select a clearly identifiable method in this family. Thus, in the supermodular case, the combination of group strategyproofness and budget balance proscribes equal treatment of equals, whereas the combination of strategyproofness and efficiency (as in the submodular case) prescribes equal treatment of equals. While the former (group strategyproofness and budget balance) still has superior incentive properties in the supermodular case, it no longer has claim to greater flexibility.

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A Proof of direct portion of Proposition 1

We fix a cross monotonic method ξ and check that $M(\xi)$ is group stategyproof (all other statements are trivial).

Let u be the (true) profile of willingness to pay and $S = \tilde{S}(\xi, u)$. Assume a certain coalition T manipulates at u by u' (where $u_j = u'_j$ for all $j \notin T$) and set $S' = \tilde{S}(\xi, u')$. Finally, set

$S'' = S \cup S'$ and denote by q (respectively, q' and q'') the binary inclusion vector corresponding to S (respectively, S' and S''). Check first

$$for\ all\ i \in T:\ u_i q_i'' - \xi_i(S'') \geq u_i q_i' - \xi_i(S') \tag{12}$$

This is clear if $i \notin S''$ (implying $i \notin S'$). If $i \in S'$ this follows from $S' \subseteq S''$ and cross monotonicity. Lastly if $i \in S'' - S'$ (implying $i \in S$) then $u_i - \xi_i(S) = 0$ because the manipulation property requires $u_i q_i' - \xi_i(S') \geq u_i q_i - \xi_i(S)$ and so, by cross monotonicity, $u_i - \xi_i(S'') \geq 0$ as desired.

Combined with the manipulation assumption, formula (12) implies

$$for\ all\ i \in T:\ u_i q_i'' - \xi_i(S'') \geq u_i q_i - \xi_i(S) \tag{13}$$

with at least one strict inequality. We claim that (13) holds true as well for $i \notin T$. This is clear if $i \in S$ (implying $i \in S''$) by cross monotonicity, or if $i \notin S''$ (in which case both sides of the inequality are zero); if $i \in S'' - S$ (implying $i \in S'$) observe that $u_i - \xi_i(S') \geq 0$ (because q' is Nash equilibrium at u' and $u_i' = u_i$) so by cross monotonicity $u_i - \xi_i(S'') \geq 0$ as desired.

We have proven that (13) holds for all i with at least one strict inequality. Therefore $S'' - S$ is nonempty (or (13) would be an equality for all i) and there is a largest integer t such that $S'' \subseteq S^t$ (where the sequence S^t is derived by (3) at u). Pick an agent i in $S'' \cap (S^t - S^{t+1})$; we have

$$u_i < \xi_i(S^t) \leq \xi_i(S'') \Rightarrow u_i q_i'' - \xi_i(S'') < 0$$

But this contradicts (13) as q is a Nash equilibrium at u. $\square$

B Proof of Proposition 2

The proof is divided into 5 steps. In the first three steps we assume a given cross monotonic cost sharing method ξ and a utility profile u. We write $S = \tilde{S}(\xi, u)$ and $T = S^*(u)$.

Step 1: $S \subseteq T$

Assume, to the contrary, that $S - T$ is nonempty. Observe that for all $i \in S$, we have $u_i \geq \xi_i(S)$ (by construction of $\tilde{S}$ as in (3)). Hence, we compute:

$$(u_{S\cup T} - C(S \cup T)) - (u_T - C(T)) = u_{S-T} + C(T) - C(S \cup T) \geq \xi_{S-T}(S) + C(T) - \xi_{S\cup T}(S \cup T)$$

$$= (\xi_{S-T}(S) - \xi_{S-T}(S \cup T)) + (C(T) - \xi_T(S \cup T)) \geq 0$$

But this is a contradiction because T is the largest efficient coalition.

The inclusion just established implies in particular the following expression for the efficiency loss:

$$\gamma(\xi, u) = w(N, u) - w(\tilde{S}(\xi, u), u)$$

To see this, apply Step 1 to the restriction of the game to $\tilde{S}(\xi, u)$.

Step 2: *For all $R \subseteq N$, $R \cap S = \emptyset$, there exists $i \in R$ such that $u_i < \xi_i(S \cup R)$*

This follows from the definition of S; if we have for all $i \in R$, $\xi_i(S \cup R) \leq u_i$ then every coalition S^t in the sequence defined by (3) must contain $S \cup R$ (by cross monotonicity and an obvious induction argument), hence a contradiction.

Step 3: *Bounding $\gamma(\xi, u)$*

We represent an ordering of $N = \{1, 2, 3, \ldots, n\}$ by a permutation σ of N, with the interpretation that $\sigma(1)$ is the first in the ordering, $\sigma(2)$ is second, and so forth. We write $E(k, \sigma) = \{i | \sigma^{-1}(i) \geq k\}$ the set of agents ranked k or higher by the permutation σ. We set:

$$\delta(\xi) = \max_{\sigma} \left(\sum_{i=1}^{n} \xi_{\sigma(i)}(E(i, \sigma)) \right) - C(N) \tag{14}$$

where the maximum is taken over all permutations of N.

We now show that $\gamma(\xi, u) \leq \delta(\xi)$ for the arbitrary profile u fixed at the beginning of the proof. If $S = T$, we have $\gamma(\xi, u) = 0$ and the inequality follows from the nonnegativity of δ (a consequence of cross monotonicity). From now on, assume $S \neq T$ and so, by Step 1, we must have $S \subset T$. We label (arbitrarily) the agents in $N - T$ as $\{1, 2, \ldots, k\}$ and use cross monotonicity to see that $C(N) - C(T) \leq \xi_{N-T}(N)$ and:

$$C(N) - C(T) \leq \sum_{i=1}^{k} \xi_i(N - \{1, \ldots, i-1\})$$

Next we label the elements in $T - S$ such that $T - S = \{k+1, k+2, \ldots, k+k'\}$. We apply Step 2 repeatedly, first to $R_1 = T - S$ to pick an agent labeled $k+1$ in R_1, then to $R_2 = T - (S \cup \{k+1\})$ to pick an agent $k+2$ in R_2, and so on, with agent $k+i$ selected in $R_i = T - (S \cup \{k+1, \ldots, k+i-1\})$ such that

$$u_{k+1} < \xi_{k+1}(T), \ldots, u_{k+i} < \xi_{k+i}(T - \{k+1, \ldots, k+i-1\}), \ldots, u_{k+k'} < \xi_{k+k'}(S \cup \{k+k'\})$$

Finally, we label (arbitrarily) the agents in S as $\{k + k' + 1, \ldots, n\}$ and repeatedly use cross monotonicity to show:

$$C(S) \leq \sum_{i=k+k'+1}^{n} \xi_i(S - \{k + k' + 1, \ldots, i - 1\})$$

Summing the above inequalities yields

$$C(N) - C(T) + u_{T-S} + C(S) \leq \sum_{i=1}^{n} \xi_{\sigma(i)}(E(i, \sigma))$$

where σ is the ordering $\sigma(i) = i$, $i = 1, \ldots, n$. The lefthand term in the inequality above equals

$$(u_T - C(T)) - (u_S - C(S)) + C(N) = \gamma(\xi, u) + C(N)$$

hence the desired conclusion $\gamma(\xi, u) \leq \delta(\xi)$ follows.

Step 4: $\sup_{u \in \Re^n_+} \gamma(\xi, u) = \delta(\xi)$

In view of Step 3, it is enough to show that for any ordering σ of N, we have

$$\sup_{u \in \Re^n_+} \gamma(\xi, u) \geq \delta(\xi, \sigma) = \sum_{i=1}^{n} \xi_i(E(i, \sigma)) - C(N)$$

We pick an arbitrary ordering σ, denoted for simplicity $\sigma(i) = i$, and a number ϵ, $0 < \epsilon < 1$. Consider the utility profile

$$u_i = (1 - \epsilon)\xi_i(E(i, \sigma)) \quad for\ all\ i \tag{15}$$

Denote by S^t the inclusion monotonic sequence (3). Note that $\tilde{S}(\xi, u)$ is typically empty; it will be empty if $\xi_i(S)$ is positive for all S. However, if some of the cost shares $\xi_i(S)$ are zero, we may have $\tilde{S}(\xi, u) \neq \emptyset$.

We now prove $\gamma(\xi, u) \geq \delta(\xi, \sigma)(1 - \epsilon) - \epsilon C(N)$. As the choice of σ and ϵ were arbitrary this will conclude the proof of Step 4. We shall use the following fact: for all S, all $i \in S$

$$\xi_i(S) = 0 \Rightarrow [C(S) = C(S - i)\ and\ \xi_j(S) = \xi_j(S - i)\ for\ all\ j \in S - i] \tag{16}$$

To see this, sum up the inequalities $\xi_j(S) \leq \xi_j(S - i)$ and invoke the monotonicity of the cost function.

If $\tilde{S} = \emptyset$ we have

$$\gamma(\xi, u) = w(N, u) \geq u_N - C(N) = \delta(\xi, \sigma)(1 - \epsilon) - \epsilon C(N)$$

and so we are done. However, as we noted above, $\tilde{S}$ need not be empty. We now show that this does not alter our conclusion. Define $A = \{i \in N | i \in S^i\}$, where the S^i are defined in (3). If $A = \emptyset$, then $\tilde{S} = \emptyset$ and we are done. Suppose $A \neq \emptyset$ and let i be its smallest element. For all j smaller than i we have $j \notin S^j$ hence $j \notin S^{i-1}$; thus, $S^{i-1} \subseteq E(i, \sigma)$ and the assumption $i \in S^i$ implies

$$(1 - \epsilon)\xi_i(E(i, \sigma)) = u_i \geq \xi_i(S^{i-1}) \geq \xi_i(E(i, \sigma))$$

Hence $u_i = 0$ and $\xi_i(E(i, \sigma)) = 0$. Now we can "eliminate" agent i and consider the reduced problem over $N - i$ with the ordering induced by σ; let σ_{-i} denote this induced ordering. In view of (16), we have $\delta(N, \xi) = \delta(N - i, \xi)$ (noting $\xi_i(N)$ must be zero, hence $C(N) = C(N - i)$). Similarly, denoting by u_{-i} the $N - i$ profile obtained by removing u_i, we have, by (16):

$$for\ all\ j < i, \quad \xi_i(E(j, \sigma)) = 0 \Rightarrow \xi_j(E(j, \sigma)) = \xi_j(E(j, \sigma_{-i}))$$

where $E(j, \sigma_{-i})$ are the successors of j in $N - i$. Therefore, the profile u_{-i} is defined in exactly the same way (namely by property (15)) in the reduced problem as u was in the original problem.

Consider again the sequence $S^t(\xi, u)$ defined in (3). Dropping the dependence on ξ and u temporarily, we denote by $S^t(T)$ the t'th term in the sequence defined by the program (3) with the starting point $S^0(T) = T$. Note that in general $T \subseteq T' \Rightarrow S^t(T) \subseteq S^t(T')$; thus, denoting the limit of these sequences by $\tilde{S}(T')$ and $\tilde{S}(T)$, we have $\tilde{S}(T') \neq \tilde{S}(T) \Leftrightarrow T' - T \cap \tilde{S}(T') \neq \emptyset$.

Applying this, and (16), to the two sequences $S^t(N)$ and $S^t(N - i)$, we see that either $\tilde{S}(N) = \tilde{S}(N - i)$ or $\tilde{S}(N) = \tilde{S}(N - i) \cup \{i\}$. Therefore

$$w(N, u) = w(N - i, u_{-i}); w(\tilde{S}(\xi, u), u) = w(\tilde{S}(\xi, u_{-i}), u_{-i})$$

implying $\gamma(\xi, u) = \gamma(\xi, u_{-i})$. We can now repeat the above argument for the reduced problem, and use an obvious induction argument to arrive at a reduced problem where indeed $A = \emptyset$. This completes the proof of Step 4.

Step 5:

We have proven $\gamma(\xi) = \delta(\xi)$. It remains to show the inequality $\delta(\xi^*) < \delta(\xi)$ when ξ^* is the Shapley value and ξ is another cost sharing method. First, one checks easily:

$$\frac{1}{n!}\sum_{\sigma}\delta(\xi,\sigma)=\left(\sum_{S\subseteq N}\frac{(|S|-1)!(n-|S|)!}{n!}C(S)\right)-C(N) \tag{17}$$

where the lefthand sum is taken over all permutations of N. Recall that the righthand sum is the potential function $P(N)$ of Hart and Mas-Colell [1989] who proved that the Shapley value method ξ^* can be written as:

$$\xi_i^*(S)=P(S)-P(S-i)\ for\ all\ i,\ all\ S,\ with\ P(S)=\sum_{T\subseteq S}\frac{(|T|-1)!(|S|-|T|)!}{|S|!}C(T) \tag{18}$$

We show below, by induction on $|N|$, that the Shapley value is characterized by the property that $\delta(\xi^*,\sigma)$ is independent of σ. In view of (17) this will imply the desired conclusion, namely

$$\delta(\xi^*)=P(N)-C(N)\ and\ \delta(\xi)>P(N)-C(N)\ for\ all\ \xi,\ \xi\neq\xi^*$$

This claim is obvious for $n=1$ (there is only one cost sharing method) and very easy to check for $n=2$. Assume it holds for all N' such that $|N'|\leq n-1$ and consider N such that $|N|=n$. Fix an ordering σ such that $\sigma(1)=i$, and denote by σ_{-i} the induced ordering on $N-i$.

Check first that $\delta(\xi^*,\sigma)$ is independent of σ by computing:

$$\delta(\xi^*,\sigma)=\xi_i^*(N)+\delta(\xi^*,\sigma_{-i})+(C(N-i)-C(N))$$

$$=(P(N)-P(N-i))+(P(N-i)-C(N-i))+(C(N-i)-C(N))=P(N)-C(N)$$

Conversely, assume that ξ is a method such that $\delta(\xi,\sigma)$ is independent of σ. Check that $\delta(\xi,\sigma_{-i})$ is independent of σ_{-i} (this is obvious from the definition of $\delta(\xi,\sigma)$), therefore, for any given σ such that $\sigma(1)=i$, we compute:

$$P(N)-C(N)=\delta(\xi,\sigma)=\xi_i(N)+\delta(\xi,\sigma_{-i})+(C(N-i)-C(N))=\xi_i(N)+P(N-i)-C(N)$$

so $\xi_i(N)=P(N)-P(N-i)$. The conclusion $\xi=\xi^*$ then follows from the potential formula for the Shapley value (18).□

C Proof of Proposition 3

To check that MC satisfies NPT, first pick a profile u and an agent i in an efficient coalition, $i \in S^*(u)$; nonnegativity of $x_i^*(u)$ amounts to $w(N,u) \leq u_i + w(N-i,u)$, which follows from

$$w(N,u) = u_{S^*(u)} - C(S^*(u)) \leq u_i + u_{S^*(u)-i} - C(S^*(u)-i) \leq u_i + w(N-i,u)$$

VP is a straightforward consequence of $w(N-i,u) \leq w(N,u)$. CS follows (as it does for all CG mechanisms) from the fact (due to submodularity) that, for a given i, if $u_i \geq C(\{i\})$ then $u_{S+i} - C(S+i) \geq u_S - C(S)$ for all $S \subseteq N$ and all $u_{-i} \in \Re_+^{n-1}$.

Conversely, choose a mechanism M as in the premises of the first statement. As in equation (6) we know that $x(u)$ must have the form:

$$x_i(u) = u_i q_i(u) - (w(N,u) - h_i(u_{-i}))$$

Consider an arbitrary $(N-i)$ profile u_{-i} and denote by u^0 its completion to an N-profile such that $u_i^0 = 0$. We must have $x_i(u^0) = 0$ (by NPT and VP, as noted at the beginning of Section 2) and the above equation yields

$$h_i(u_{-i}) = w(N,u^0) = w(N-i,u^0) = w(N-i,u)$$

for any profile u with $(N-i)$ projection u_{-i}. Thus, the mechanism M takes the same form as MC, except for the fact that $S(u)$ can be any efficient coalition (not necessarily the largest one as in MC). In order to check that the two mechanisms are welfare equivalent, we need only to consider an agent i in $S^*(u) - S(u)$ because both mechanisms give the precisely same allocation to every agent in $S(u)$ or outside of $S^*(u)$. For any agent i in $S^*(u) - S(u)$, we have $w(N,u) = w(N-i,u)$ hence she receives from the two mechanisms the (q_i, x_i) allocations of $(0,0)$ and $(1,u_i)$ respectively, which are welfare equivalent.$\square$