ON THE REPRESENTATION AND COMPLEXITY OF COOPERATIVE GAMES

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Overview

Aims of this presentation:

- review *coalitional games*: representation & complexity;
- introduce *qualitative coalitional games* (QCGs);
- extend QCGs with preferences (QCGPs);
- consider *coalitional resource games (CRGs*);
- conclusions & future work.

Coalitional Games

- A *coalitional game* is game where agents can *benefit from cooperating*.
- Issues in coalitional games (Sandholm et al, 1999):
  - *Coalition structure generation*:
    Which coalition should I join?
    The partitioning of a group of agents into coalitions, where the overall partition is a *coalition structure*.
  - *Solving the optimization problem of each coalition*:
    Solving the "joint problem" of a coalition, i.e., finding the best way to maximise the utility of the coalition itself.
  - *Dividing the value of the solution for each coalition*:
    Deciding "who gets what" in the payoff (e.g., Shapley value).

Formalising Coalitional Games

Many models of coalitional games, but simplest is a structure:

$$\langle Ag, \nu \rangle$$

where:

- $Ag = \{1, \ldots, n\}$ is a set of *agents*;
- $\nu : 2^{Ag} \to \mathbb{R}$ is the *characteristic function* of the game.

Usual interpretation: if $\nu(C) = k$, then coalition C can cooperate in such a way they will obtain utility k, which may then be distributed amongst team members.

What Coalition Should I Join?

- Most important question in coalitional games:

  is a coalition stable?

  that is,

  is it rational for all members of coalition to stay with the coalition, or could they benefit by defecting from it?

- Possible solutions:
  - nonemptiness of the core of a coalition,
  - the kernel,
  - the nucleolus.

How To Share Benefits of Cooperation?

- The *Shapley value* is best known attempt to define how to divide benefits of cooperation.

  The Shapley value of agent i is the average amount that i contributes by joining a coalition, assuming all coalitions are equally likely.

- Axiomatically: a value which satisfies axioms:

  symmetry, *dummy player*, and *additivity of games*.

Shapley Value Defined

- Let $\Delta_i(S)$ be the amount that i adds by joining $S \subseteq Ag$:

$$\Delta_i(S) = \nu(S \cup \{i\}) - \nu(S)$$

  ... the *marginal contribution of i to S*.

- Then the Shapley value for i, denoted φ_i is:

$$\varphi_i = \frac{1}{|Ag|!} \sum_{r \in R} \Delta_i(S_i(r))$$

  where R is the set of all permutations of Ag and $S_i(r)$ is the set of agents preceding i in ordering r.

Representing Coalitional Games

- It is important for an agent to know (eg) whether the core of a coalition is non-empty ...

  so, how hard is it to decide this?
- Problem: naive, obvious representation of coalitional game is *exponential* in the size of Ag!
- Now such a representation is:
  - *utterly* infeasible in practice; and
  - so large that it renders comparisons to this input size meaningless: stating that we have an algorithm that runs in (say) time *linear* in the size of such a representation means it runs in time *exponential* in the size of Ag!

How to Represent Characteristic Functions?

Two approaches to this problem:

- try to find a *complete* representation that is succinct in “most” cases
- try to find a representation that is not complete but is always succinct
- A common approach:

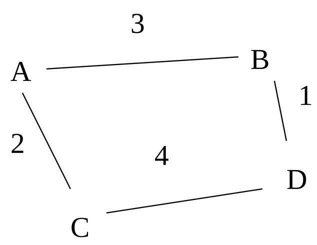
  interpret characteristic function over combinatorial structure.

Representation 1: Induced Subgraph

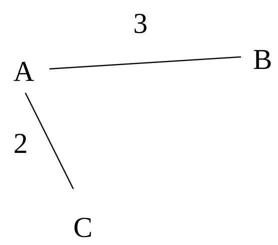
- Represent ν as an undirected graph on Ag, with integer weights $w_{i,j}$ between nodes $i,j \in Ag$.
- Value of coalition C then:

$$\nu(C) = \sum_{\{i,j\} \subseteq Ag} w_{i,j}$$

i.e., the value of a coalition $C \subseteq Ag$ is the weight of the subgraph induced by C.



the original graph defining v



subgraph induced by {A,B,C}
giving v({A,B,C}) = 3 + 2 = 5

Representation 1: Induced Subgraph

(Deng & Papadimitriou, 94)

- Computing Shapley:
  in polynomial time.
- Determining emptiness of the core:
  NP-complete
- Checking whether a specific distribution is in the core
  co-NP-complete

But this representation is not *complete*.

Representation 2: Weighted Voting Games

- For each agent $i \in Ag$, assign a weight w_i, and define an overall *quota*, q.

$$\nu(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

- Shapley value:

  #P-complete, and "hard to approximate" (Deng & Papadimitriou, 94).

- Core non-emptiness:

  in polynomial time.

Not a complete representation.

Representation 3: Marginal Contribution Nets

(Ieong & Shoham, 2005)

- Characteristic function represented as rules:

$$\text{pattern} \longrightarrow \text{value}.$$

- Pattern is conjunction of agents, a rule *applies* to a group of agents C if C is a superset of the agents in the pattern.

  Value of a coalition is then sum over the values of all the rules that apply to the coalition.

  Example:

$$\begin{aligned} a \wedge b &\longrightarrow 5 \\ b &\longrightarrow 2 \end{aligned}$$

  We have: $\nu(\{a\}) = 0$, $\nu(\{b\}) = 2$, and $\nu(\{a, b\}) = 7$.

- We can also allow negations in rules (agent not present).

Representation 3: Marginal Contribution Nets

- Shapley value:
  in polynomial time
- Checking whether distribution is in the core:
  co-NP-complete
- Checking whether the core is non-empty:
  co-NP-hard.

A complete representation, but not necessarily succinct.

Qualitative Coalitional Games

- Often not interested in utilities, but in *goals* – either the goal is satisfied or not
  - TL specifications in CAV
  - state oriented domains (Rosenschein, 1994)
- QCGs are a type of coalitional game in which each agent has a set of goals, and wants one of them to be achieved (doesn't care which)

  Agents cooperate in QCGs to achieve mutually satisfying sets of goals.

  Coalitions have *sets of choices* representing the different ways they could cooperate

  Each choice is a set of goals.

QCGs

A *Qualitative Coalitional Game* (QCG) is a structure:

$$\Gamma = \langle G, Ag, G_1, \ldots, G_n, V \rangle$$

where

- $G = \{g_1, \ldots, g_m\}$ is a set of *possible goals*;
- $Ag = \{1, \ldots, n\}$ is a set of *agents*;
- $G_i \subseteq G$ is a set of goals for each agent $i \in Ag$, the intended interpretation being that any of G_i would satisfy i;
- $V : 2^{Ag} \to 2^{2^G}$ is a *characteristic function*, which for every coalition $C \subseteq Ag$ determines a set $V(C)$ of *choices*, the intended interpretation being that if $G' \in V(C)$, then one of the choices available to coalition C is to bring about *all* the goals in G' simultaneously.

Feasible/Satisfying Goal Sets

- Goal set $G' \subseteq G$ *satisfies* an agent i if $G_i \cap G' \neq \emptyset$.
  Goal set $G' \subseteq G$ satisfies a coalition $C \subseteq Ag$ if
  $$\forall i \in C, G_i \cap G' \neq \emptyset$$
- A goal set G' is *feasible* for C if $G' \in V(C)$.

Representing QCGs

- So, how do we represent the function $V : 2^{Ag} \to 2^{2^G}$?
- We use a formula Ψ_V of propositional logic over propositional variables Ag, G, such that:

$$\Psi[C, G'] = \top \text{ if and only if } G' \in V(C)$$

- "Often" permits *succinct* representations of V.
- Note that given Ψ_V, C, G', determining whether $G' \in V(C)$ can be done in time polynomial in size of C, G', Ψ_V.

Fourteen QCG Decision Problems (AIJ, Sep 2004)

Problem	Description	Complexity	q^{mono}
SC	SUCCESSFUL COALITION	NP-complete	NP-complete
SSC	SELFISH SUCCESSFUL COALITION	NP-complete	NP-complete
UGS	UNATTAINABLE GOAL SET	NP-complete	NP-complete
MC	MINIMAL COALITION	co-NP-complete	co-NP-complete
CM	CORE MEMBERSHIP	co-NP-complete	co-NP-complete
CNE	CORE NON-EMPTINESS	D^p-complete	D^p-complete
VP	VETO PLAYER	co-NP-complete	-
MD	MUTUAL DEPENDENCE	co-NP-complete	-
GR	GOAL REALISABILITY	NP-complete	P
NG	NECESSARY GOAL	co-NP-complete	-
EG	EMPTY GAME	co-NP-complete	co-NP-complete
TG	TRIVIAL GAME	Π_2^p-complete	Π_2^p-complete
GU	GLOBAL UNATTAINABILITY	Σ_2^p-complete	NP
IG	INCOMPLETE GAME	D_2^p-complete	-

Introducing Preferences

A *Qualitative Coalitional Game with Preferences* (QCGP) is a $2n + 3$-tuple:

$$\Gamma = \langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle,$$

where:

$$\langle G, Ag, G_1, \ldots, G_n, \Psi \rangle$$

is a QCG, and

$$\rhd_i \subseteq G_i \times G_i$$

is a partial order over G_i representing i's *preference relation*, so that $g_1 \rhd_i g_2$ indicates that i would rather have goal g_1 satisfied than goal g_2.

Problems with QCGPs

- Although they seem a simple extension of QCGs, it turns out that QCGPs are much harder to deal with, technically:
  - issue of lifting preference relations to coalitions;
  - outcomes in QCGs/QCGPs *have structure*: they are sets of goals.
- So, *how does the complexity compare*?

Core Membership

<u>CORE MEMBERSHIP</u>: (CM)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, coalition $C \subseteq Ag$, goal set $G' \subseteq G$.
Question: Is G' in the core of C?

Theorem 1 CM *is co-*NP*-complete.*

Thus no worse than QCGs.

Maximal Goal Sets

The next problem is whether or not a goal set is maximally preferred by a coalition, i.e., whether this goal set both satisfies every member of the coalition, and there is no other goal set that satisfies the coalition that is strictly preferred by it.

> MAXIMAL GOAL SET: (MGS)
> *Instance*: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \triangleright_1, \ldots, \triangleright_n \rangle$, coalition $C \subseteq Ag$, goal set $G' \subseteq G$.
> *Question*: Is $G' \in \mu^{\triangleright}(C)$?

Theorem 2 MGS *is co-*NP*-complete.*

Pareto Optimality

<u>PARETO OPTIMAL GOAL SET</u>: (PO)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, coalition $C \subseteq Ag$, goal set $G' \subseteq G$.
Question: Is G' P.O. for C?

Theorem 3 PO *is co-*NP*-complete.*

Core Completeness

CC is concerned with the question of whether C is successful *and* every feasible goal set that satisfies each member of C is in the core.

> CORE COMPLETENESS: (CC)
> *Instance*: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, $C \subseteq Ag$.
> *Question*: Is $\kappa^{\rhd}(C) = \mathcal{X}(C)$ and $\mathcal{X}(C) \neq \emptyset$?

Theorem 4 *CC is D^p-complete.*

Coalitional Resource Games (CRGs)

- Problem:

  where does characteristic function come from?

- One answer provided by *Coalitional Resource Games (CRGs)*.
- Key ideas:
  - achieving a goal requires *expenditure of resources*;
  - each agent *endowed* with a profile of resources;
  - coalitions form to pool resource so as to achieve mutually satisfactory set of goals.

CRGs

A *coalitional resource game* Γ is an $(n+5)$-tuple:

$$\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req} \rangle$$

where:

- $Ag = \{a_1, \ldots, a_n\}$ is a set of *agents*;
- $G = \{g_1, \ldots, g_m\}$ is a set of *possible goals*;
- $R = \{r_1, \ldots, r_t\}$ is a set of *resources*;
- for each $i \in Ag$, $G_i \subseteq G$ is a set of goals, as in QCGs;
- $\mathbf{en} : Ag \times R \to \mathbb{N}$ is an *endowment function*,
- $\mathbf{req} : G \times R \to \mathbb{N}$ is a *requirement function*.

Nine Decision Problems for CRGs

Problem	Complexity
SUCCESSFUL COALITION	NP-complete
MAXIMAL COALITION	co-NP-complete
NECESSARY RESOURCE	co-NP-complete
STRICTLY NECESSARY RESOURCE	D^p-complete
(C, G', r)-OPTIMAL	NP-complete
R-PARETO OPTIMALITY	co-NP-complete
SUCCESSFUL COALITION WITH RESOURCE BOUNDS	NP-complete
CONFLICTING COALITIONS	co-NP-complete
ACHIEVABLE GOAL SET	in P

QCG and CRG Equivalence

- We can define a notion of "equivalence" ($\equiv$) between QCGs and CRGs:

  $\Gamma_1 \equiv \Gamma_2$ means that QCG Γ_1 and CRG Γ_2 agree on the goal sets that are feasible for coalitions

- Given a QCG Γ_1 and CRG Γ_2, the problem of determining whether $\Gamma_1 \equiv \Gamma_2$ is co-NP-complete.

Can we translate between QCGs and CRGs?

Four questions suggest themselves:

1. Given a *crg*, Γ, is there always a QCG, Q_Γ such that $Q_\Gamma \equiv \Gamma$?
2. Given a *qcg*, Q, is there always a CRG, Γ_Q such that $\Gamma_Q \equiv Q$?
3. How "efficiently" can a given CRG be expressed as an equivalent QCG in those cases where such an equivalent structure exists?
4. How "efficiently" can a given QCG be expressed as an equivalent CRG in those cases where such an equivalent structure exists?

$\boxed{\text{Translating CRGs} \rightarrow \text{QCGs}}$

- We can always translate a CRG into an equivalent QCG.
- More interestingly, we can do this *efficiently*:

  for every CRG Γ_1 there exists an equivalent QCG Γ_2 such that $|\Gamma_2| \leq |\Gamma_1|^2$.

Translating QCGs to CRGs

- We *cannot* always translate QCGs to equivalent CRGs.
- Moreover, even when we *can* translate, we can't always do it efficiently:

  there exist QCGs Γ for which equivalent CRGs exist but for which the size of the *smallest* equivalent CRG is at least $2^{|\Gamma|}$

- We can identify some necessary and some sufficient conditions on QCGs for the existence of equivalent CRGs.

Conclusions & Future Work

- QCGs, QCGPs, + CRGs allow us to formally frame meaningful questions about cooperation & cooperative systems, and allow us to investigate their complexity.
- Future work:
  - temporal QCGs (AAMAS06) – use ATL variants to classify solution concepts
    "resource automata" to model temporal resource requirements of goals
  - new (qualitative) solution concepts
  - social choice mechanisms

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Find Out More

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