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Permanent Uncertainty: On the quantum evaluation of the determinant and the permanent of a matrix

Lidror Troyansky*and Naftali Tishby
Institute of Computer Science and Center for Neural Computation
The Hebrew University
Jerusalem 91904, Israel

We investigate the possibility of evaluating permanents and determinants of matrices by quantum computation. All current algorithms for the evaluation of the permanent of a real matrix have exponential time complexity and are known to be in the class $P^{\#P}$. Any method to evaluate or approximate the permanent is thus of fundamental interest to complexity theory. Permanents and determinants of matrices play a basic role in quantum statistical mechanics of identical bosons and fermions, as the only possible many particle states made of single particle wave functions.

We study a novel many-particle quantum-computational model in which an observable operator can be constructed, in polynomial-time complexity, to yield the determinant or the permanent of an arbitrary matrix as its expectation value. Both quantities are estimated, in this model, by quantum mechanics systems in a polynomial time, using fermions and bosons respectively. It turns out that the *variance* of the measurements in a "noise-free case" is zero for the determinant and exponential (in the rank of the matrix) for the permanent. The intrinsic measurement variance is therefore the quantum mechanical correspondence to the computational complexity gap.

1 Introduction

Quantum computational models have recently become important for the theory of computational complexity when it was shown that factorization can be performed in a polynomial time by a Quantum Turing machine [7, 6].

An intriguing computational question is naturally raised when considering quantum statistical mechanics of identical particles. By Pauli's "spin-statistics" theorem[5], the quantum state of identical particles is in general either symmetric with respect to exchange of particles for integer spin particles (bosons), or anti-symmetric for half-integer spin particles (fermions). The only possible completely symmetric or anti-symmetric combinations of general single particle functions are the permanent or determinant of those functions, respectively.

Whereas in QM the difference between permanents and determinants simply corresponds to the choice between the quantum statistics of bosons and fermions, determinants and permanents have very different computational complexity by all known algorithm. While the determinant can be easily evaluated in polynomial time in the size of the matrix (e.g. by diagonalization), no such polynomial time algorithm is known for the permanent. Moreover, there is no known algorithm for the evaluation of the permanent that can avoid the summation over exponential many terms, i.e. the permanent is known to be in the complexity class $\#P$-complete. In this paper we would like to understand the quantum mechanical correspondence to this dramatic computational complexity gap. We approach this problem by considering many particle quantum observable that correspond to the determinant of a given matrix in the fermion case, and to the permanent of the matrix in the boson case.

The permanent of a matrix $\mathbf{A} = [a_{ij}]$ is defined as:

$$\mathrm{per}[\mathbf{A}] \equiv \sum_{\{i_1..i_n\}} a_{1i_1} \cdot a_{2i_2} \ldots a_{ni_n} \ , \qquad (1)$$

where the set of indices $\{i_1 \ldots i_n\}$ denotes the set of all the permutations of $1 \ldots n$ [2]. The determinant is defined by the same sum of the permutations, where in addition each odd permutation is taken with a negative sign.

Permanents occur naturally in various counting problems in combinatorics[4], graph theory[1], and logic. If there was a way to evaluate, or even to approximate permanents in a polynomial time, then *every* $P^{\#P}$ (and thus every NP-complete) problem could have been evaluated in polynomial time [8] [3]. It can be shown that even the approximation of the log of the permanent is a hard problem[9]. The key technical reason for that computational difference between permanents and determinants is that determinants are invariants under similarity transformations of the matrix, while no such simple invariants are known for the permanent. So diagonalization of the matrix or similar transformations can not help for permanents.

Determinants and permanents are fundamental func-

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tions in quantum statistics of identical particles, as described below.

- The wave-function of indistinguishable fermions can be written in the Slater-determinant form. Namely, if there are n fermions in n single-particles states, $|1\rangle \ldots |n\rangle$, than the n-Particles wave-function is given by,

$$\begin{aligned}|\Psi_A(1\ldots n)\rangle &= \frac{1}{\sqrt{n!}}[|1,1\rangle|2,2\rangle\ldots|n,n\rangle \\ &- |1,2\rangle|2,1\rangle\ldots|n,n\rangle+\ldots] \\ &= \frac{1}{\sqrt{n!}}\sum_{\{i_1,\ldots i_n\}}(-1)^{\pi}|1,i_1\rangle \\ & \quad |2,i_2\rangle\ldots|n,i_n\rangle = \frac{1}{\sqrt{n!}}\det[|i,j\rangle]. \end{aligned} \tag{2}$$

  The sum is taken over all the permutations of $1\ldots n$, with parity π, where the first index denote the state and the second denote particles (i_j is the permutation index).

- Particles with an integer spin ("bosons") obey the Bose-Einstein statistics: the many-particles wave-function is completely symmetric under exchange of any two particles,

$$\begin{aligned}|\Psi_S(1\ldots n)\rangle &= \frac{1}{\sqrt{n!}}[|1,1\rangle|2,2\rangle\ldots|n,n\rangle \\ &+ |1,2\rangle|2,1\rangle\ldots|n,n\rangle+\ldots] \\ &= \frac{1}{\sqrt{n!}}\cdot\sum_{\{i_1,\ldots i_n\}}|1,i_1\rangle|2,i_2\rangle\ldots|n,i_n\rangle \\ &= \frac{1}{\sqrt{n!}}\cdot \mathrm{per}\,[|\mathrm{i},\mathrm{j}\rangle]. \end{aligned} \tag{3}$$

In what follows we address two natural questions: (a) is there a quantum mechanical observable (or measurement) that can be constructed in order to evaluate the permanent or determinant of an arbitrary matrix; (b) what is the difference between the evaluation of determinants and permanents from the quantum computation perspective.

2 Measuring the permanent of an arbitrary Hermitian matrix

Let $\mathbf{A} = [a_{ij}]$ denote the Hermitian matrix whose permanent is required. If $\mathbf{A}$ is not Hermitian it can always be installed into an Hermitian matrix $\tilde{\mathbf{A}}$ of the form,

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix} \tag{4}$$

and for this matrix, $\mathrm{per}[\tilde{\mathbf{A}}] = \mathrm{per}[\mathbf{A}])^2$. So every permanent of a real matrix can be evaluated, up to sign, by the permanent of an Hermitian matrix.

Let ω be any single particle QM observable, with continuous unbounded spectrum, such as the linear momentum operator $\mathbf{p}$.

We denote by

$$\Omega = \prod_{i=1}^{n} \omega_i \ , \tag{5}$$

the product of these single particle observables, which is now a many particle observable on its own. Since single-particle operators that operate on different particles are always commutative the order of the operators in the product is not important.

The Hermitian matrix $\mathbf{A}$ can be diaganolized by a unitary transformation,

$$\mathbf{U}^{\dagger}\mathbf{A}\mathbf{U} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & \ldots \end{pmatrix} \ , \tag{6}$$

where $\mathbf{U} = [u_{ij}]$ is a unitary matrix, $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$.

We denote by $|\lambda_j\rangle$ the eigenstate of the single particle observable ω that corresponds to the eigenvalue λ_j, $\langle\lambda_i\lambda_j\rangle + \delta_{ij}$.

The idea is to prepare the many particle system in a special superposition, that depend on the given matrix $\mathbf{A}$, and then perform a measurement of the product operator Ω. We prepare the many particle system such that there is a single boson/fermion in each of the following single-particle superposition of the eigenstates of $\omega_{\mathbf{i}}$,

$$|i\rangle = \sum_{l} u_{li}|\lambda_l\rangle \ . \tag{7}$$

In that state the matrix elements of ω, in this single particle basis, are given by,

$$\langle i|\omega|j\rangle = \left(\sum_{l}\langle\lambda_l|u^*_{il}\right)\omega\left(\sum_{k}u_{kj}|\lambda_k\rangle\right) = \left(\sum_{l}\langle\lambda_l|u^*_{il}\right)\left(\sum_{k}\lambda_k u_{kl}|\lambda_k\rangle\right) = \sum_{k}u^*_{ik}u_{kj}\lambda_k = a_{ij} \ . \tag{8}$$

The new basis is orthonormal due to the unitarity of $\mathbf{U}$. We denote by $|\Psi\rangle$ the n-particle state of the above single particle superpositions. Due to the indistinguishability of the particles, $|\Psi\rangle$ must be symmetric/antisymmetric with respect to particle permutations.

The expectation value of the operator Ω in the n-particle state $|\Psi\rangle$ is given, for the boson case, by:

$$\begin{aligned}
&\langle\Psi|\Omega|\Psi\rangle = \\
&= \frac{1}{n!}\cdot\left(\sum_{\{i_1..i_n\}}\langle 1,i_1|\ldots\langle n,i_n|\right) \\
&\cdot \left(\prod_{i=1}^{n}\omega_{\mathbf{i}}\right)\left(\sum_{\{i_1...i_n\}}|1,i_1\rangle\ldots|n,i_n\rangle\right) \\
&= \frac{1}{n!}\cdot n!\cdot\sum_{\{i_1...i_n\}}\langle 1|\omega|i_1\rangle\ldots\langle n|\omega|i_n\rangle \\
&= \mathrm{per}[\mathrm{A}] \ . \qquad (9)
\end{aligned}$$

Similarly, this expectation is det[A] for the fermion case.

For illustration, consider a 2×2 matrix. The symmetric two-particles wave-function is,

$$|\Psi(1,2)\rangle = \frac{1}{\sqrt{2!}}(|1,1\rangle|2,2\rangle + |1,2\rangle|2,1\rangle)\ , \qquad (10)$$

and the expectation value of Ω is:

$$\begin{aligned}
\langle\Psi|\Omega|\Psi\rangle &= \frac{1}{2!}[(\langle 1,1|\langle 2,2| + \langle 1,2|\langle 2,1|) \\
&\quad \omega_{\mathbf{1}}\omega_{\mathbf{2}}(|1,1\rangle|2,2\rangle + |1,2\rangle|2,1\rangle)] \\
&= \frac{1}{2!}[\langle 1|\omega|1\rangle\langle 2|\omega|2\rangle + \langle 2|\omega|1\rangle\langle 1|\omega|2\rangle \\
&\quad +\langle 2|\omega|2\rangle\langle 1|\omega|1\rangle + \langle 1|\omega|2\rangle\langle 2|\omega|1\rangle] \\
&= [\langle 1|\omega|1\rangle\langle 2|\omega|2\rangle + \langle 2|\omega|1\rangle\langle 1|\omega|2\rangle \\
&\quad +\langle 2|\omega|2\rangle\langle 1|\omega|1\rangle + \langle 1|\omega|2\rangle\langle 2|\omega|1\rangle] \\
&= a_{11}a_{22} + a_{12}a_{21} = \mathrm{per}[\mathbf{A}]\ . \qquad (11)
\end{aligned}$$

Similar expression holds for the determinant, where all the antisymmetric (odd) permutations switch sign for fermions.

Thus, both the *permanent* and the determinant of an *arbitrary hermitian matrix* can be expressed as the *expectation value* of an n-particle quantum mechanical observable in a state of n identical bosons or fermions.

The computation required for the preparation of the system are only the computations required for the diagonalization of the matrix $\mathbf{A}$ and the preparation of the single-particle superpositions, and therefore can be done in a polynomial number of steps. The evaluation of the determinant/permanent of the states is "done" by the spin-statistics by itself.

Quantum mechanical measurement can yield only one of the eigenvalues of the measured operator. Thus, measuring ω in a superposition of its eigenstates $|\lambda_1\rangle\ldots|\lambda_n\rangle$ can therefore yield one of the n possible eigenvalues, $\lambda_1..\lambda_n$. Similarly, measuring the n-particle observable $\Omega = \prod_{i=1}^{n}\omega_{\mathbf{i}}$ in the state $|\Psi\rangle$ can therefore yield only one of the possible outcomes,

$$O_J \doteq \lambda_1^{n_1^J}\lambda_2^{n_2^J}\ldots\lambda_n^{n_n^J}\ , \qquad (12)$$

where $n_i^J = 0,1,\ldots n$ and for all J and $\sum_{i=1}^{n} n_i^J = n$. The total number of possible outcomes of this measurement is

$$\binom{2n-1}{n} \approx 2^{2n}\ ,$$

i.e., the number of possible ways to distribute n identical balls into n distinct cells. The expectation value of Ω is obtained, asymptotically in the number of measurements,

$$\langle\Omega\rangle = \mathrm{per}[\mathbf{A}] = \sum_{J=1}^{\binom{2n-1}{n}}\lambda_1^{n_1^J}\lambda_2^{n_2^J}\ldots\lambda_n^{n_n^J}P(n_1^J\ldots n_n^J). \quad (13)$$

Whereas the permanent include $n! \approx \left(\frac{n}{e}\right)^n$ terms with equal weights, the decomposition in eq. (13) contains only $\approx 2^{2n}$ different terms, each with a different weight. This decomposition can be considered as the probability-amplitude space induced by the permanent.

The probability to get the outcome

$$O_J \doteq n_1^J n_2^J\ldots n_n^J$$

in a measurement of Ω in the boson case, is given by,

$$\left|\frac{1}{\sqrt{n_1!..n_n!}}\sum_{\{i_1..i_n\}}(\langle\lambda_{1,,i_1}|)^{n_1^J}..(\langle\lambda_{n,i_n}|)^{n_n^J}|\Psi\rangle\right|^2\ , \qquad (14)$$

where $\langle\lambda_{1,i_1}|$ denotes that the i_1 particle is in the eigenstate $\langle\lambda_1|$, etc.

The composite n-particles wave-function Ψ can be written as

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{n!}}\sum_{\{i_1...i_n\}}|1,i_1\rangle|2,i_2\rangle\ldots|n,i_n\rangle \qquad (15) \\
&= \frac{1}{\sqrt{n!}}\sum_{\{i_1..i_n\}}\prod_{k=1}^{n}(\sum_{j=1}^{n}u_{kj}|\lambda_{j,i_k}\rangle)\ .
\end{aligned}$$

Since:

$$\langle\lambda_{j,i_n}||\lambda_{l,i_m}\rangle = \delta_{jl}\delta_{i_n i_m}\ ,$$

we obtain sum of the permutations as,

$$\begin{aligned}
&\sum_{\{i_1...i_n\}}\frac{1}{\sqrt{n_1^J!\ldots n_n^J!}}((\langle\lambda_{1,,i_1}|)^{n_1^J}\ldots((\langle\lambda_{n,i_n}|)^{n_n^J} \\
&(\sum_{j=1}^{n}u_{1j}|\lambda_{j,i_1}\rangle)\ldots(\sum_{j=1}^{n}u_{nj}|\lambda_{j,i_n}\rangle) \\
&= \frac{1}{\sqrt{n_1^J!\ldots n_n^J!}}\mathrm{per}[\mathbf{U}^J]\ . \qquad (16)
\end{aligned}$$

The matrix $\mathbf{U^J}$ is derived from the unitary matrix, $\mathbf{U}$, with the first row taken n_1^J times, the second row n_2^J times, etc. [4].

The probability amplitude for any of the outcomes of the measurement of the product is by itself proportional to a permanent of a matrix, and therefore a direct evaluation of the joint probabilities $P(n_1^J, n_2^J\ldots n_n^J)$ is a again in $P^{\#P}$, i.e., exponential in the rank of the matrix.

3 Physical implementation

While in general, the evaluation of an expectation value of an operator which is constructed from a product of many single-particle observables may be difficult, an actual, rather simple physical device which allows the evaluation of permanents of $n \times n$ matrices can be constructed, using a scattering experiment with n input channels and n output channels, as schematically shown in figure 1. In this device, amplitude of the scattering from the i-th input channel to the j output channel is just u_{ij}.

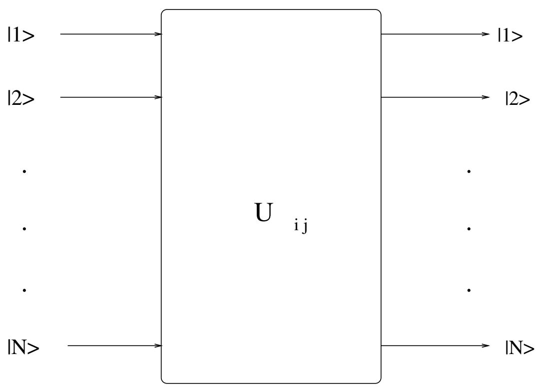

Figure 1: A sketch of the physical implementation.

If a single boson is enters each of the input channels, than each of the output channels can contain from 0 to n bosons. The probability for each of these $J = 1 \ldots \binom{2n-1}{n}$ possible scattering outcomes is given by,

$$P(n_1^J, n_2^J, \ldots n_n^J) = \frac{1}{\sqrt{n_1! n_2! \ldots n_n!}} \mathrm{per}[\mathbf{U}^\mathbf{J}] \ . \tag{17}$$

If M such scattering experiments are performed and for each outcome, O_J, the term $\frac{1}{M}\lambda_1^{n_1^J} \cdot \lambda_2^{n_2^J} \ldots \lambda_n^{n_n^J}$ is added, then in the limit $M \to \infty$ this sum converges, by the law of large numbers, to the expectation value, i.e., the real value of the permanent.

4 The accuracy of the measurement

As can be expected, while there is nothing so far that discriminates the permanent from the determinant in the many-particle quantum measurement, the caveat must be in the variance, or precision of our estimate. The accuracy in which the above procedure yields an estimate to the value of the permanent, in a finite number of measurements, depends on the variance in the measurement, or the intrinsic quantum mechanical uncertainty of our product observable Ω.

The second moment of Ω, in the many-particle state $|\Psi\rangle$ is given by,

$$\langle \Omega^2 \rangle = \langle \prod_{i=1}^{n} \omega_i^2 \rangle = \mathrm{per}[\mathbf{B}]. \tag{18}$$

where $\mathbf{B}$ is the matrix whose elements are:

$$b_{ij} = \langle i | \omega^2 | j \rangle.$$

However, this expectation value is simply related to the original matrix, $\mathbf{A}$, through,

$$\begin{aligned}[\mathbf{A}^2]_{ij} &= \sum_l a_{il} a_{lj} = \sum_l \langle i|\omega|l\rangle\langle l|\omega|j\rangle = \\ &= \langle i|\omega^2|j\rangle = b_{ij}\end{aligned} \tag{19}$$

and therefore,

$$\langle \Omega^2 \rangle_\Psi = \mathrm{per}[\mathbf{A}^2] \ . \tag{20}$$

Since each of the elements of $\mathbf{A}^2$ is a sum of n products of pairs of elements of the Hermitian matrix $\mathbf{A}$ – the permanent of $\mathbf{A}^2$ might be exponentially larger than $(\mathrm{per}[\mathbf{A}])^2$. This means that the intrinsic variance can be exponential in the rank of the matrix, so an exponential number of experiments might be needed for a good approximation of the permanent.

If the same scheme is applied to fermions, the second moment in the measurements of Ω will be

$$\langle \Omega^2 \rangle_\Psi = \det[\mathbf{A}^2] \ . \tag{21}$$

However, since

$$\det[\mathbf{A}^2] = (\det[\mathbf{A}])^2$$

the variance in this measurements of Ω is identically *zero*. Thus for the fermionic case, the exact value of the determinant is obtained in a single(!) scattering experiment, if we ignore all other noise sources.

The inherent difference in the quantum computational complexity of permanents and determinants is therefore expressed, in our particular setup, in the *variance* of the possible outcomes of measurements. Although we have discussed a specific physical setup, we believe that this is a generic result, up to polynomial factors, in any QM many-particle experiment.

5 Permanent of a unitary Hermitian Matrix

The eigenvalues of a unitary Hermitian matrix can be either $+1$ or -1. In this case the decomposition in eq. (13) contained only two terms:

$$\mathrm{per}(\mathbf{U}) = P(+1) - P(-1). \tag{22}$$

In this case, in order to approximate the permanent we need to estimate just two probabilities, at least one of them is of $O(1)$. The accuracy of such an estimation will be of order $1/M$, were M is the number of experiments. Since the value of a permanent of a unitary, Hermitian matrix can be any real value between -1 and $+1$, this may give a good approximation for certain matrices. However, the value of the permanent any be exponentially small, and in this case the relative approximation may not be good enough. Still, there is no way that we know of to get a better approximation.

6 Conclusions

In this work we consider the possibility of using the symmetry properties of quantum mechanical systems of indistinguishable particles to evaluate the determinant or permanent of a given real valued Hermitian matrix. We show that it is rather easy to construct a quantum mechanical scattering experiment that yields the permanent and determinant of a matrix, as its expectation value, for bosons and fermions respectively.

While such an experiment can be prepared and performed in a polynomial time in the size of the matrix, we traced the manifestation of the notorious difference between the computational complexity of determinants and permanents to the intrinsic variance of the measurement. While there is usually an exponentially large variance in the measurements of the product observable $\Omega = \prod_{i=1}^{n} \omega_i$ in the setup we described for an n-boson system, there will be no variance at all for an n-fermion system, or the estimation of the determinant. Since this is an intrinsic QM uncertainty, we consider this result to be generic for any possible many-particle quantum mechanical computation of the permanent, but proving it remains an open problem.

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