Classical Dynamics: Lagrangian Formulation

Deriving a System's Equations Using the Principle of Least Action

Newton's 2nd Law

- Given a non-dissipative system of N particles with coordinates $\mathbf{r}_i \in R^3$, $i=\{1...N\}$, and masses m_i, we denote by $\mathbf{x}=[\mathbf{r}_1^T ... \mathbf{r}_N^T] \in R^{3N}$ the vector of all the position coordinates. In this notation, Newton's 2nd law can be stated as:

$$f_k = \dot{p}_k = -\frac{\partial V}{\partial x_k} \underbrace{=}_{constant\ mass} m_k \ddot{x}_k$$

f=force, V=potential ,p = momentum

$$p_k = m_k \dot{x}_k$$

- we have one equation per coordinate dimension of a particle

Lagrangian

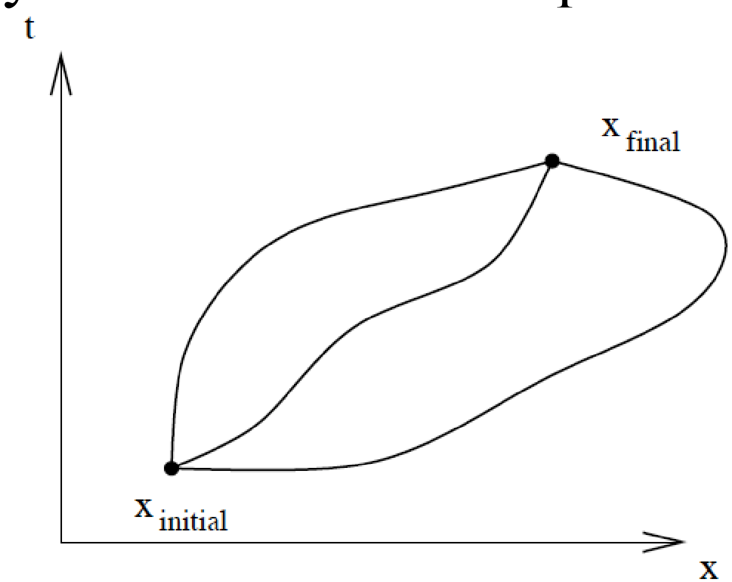
- Define the Lagrangian to be a function of the total kinetic and potential energies:

$$L(\mathbf{x}, \dot{\mathbf{x}}) = T(\dot{\mathbf{x}}) - V(\mathbf{x})$$

  Where T is the sum of kinetic energies and V is the sum of potential energies.

- Consider all the smooth paths, $\mathbf{x}(t)$ of the system with fixed end points $\mathbf{x}(t_0)$ and $\mathbf{x}(t_f)$

- We wish to find some function of the path (a functional) that has an extremum on the path that was taken by the system.



Theorem
(Principle of Least Action)

theorem: Define the *action S* as as functional of the Lagrangian:

$$S[\mathbf{x}(t)] = \int_{t_0}^{t_f} L(\mathbf{x}, \dot{\mathbf{x}})dt$$

Then the actual path taken by the system is an extremum of *S*.

proof: consider varying a given path slightly, so

$$\mathbf{x}(t) \longrightarrow \mathbf{x}(t) + \delta\mathbf{x}(t)$$

where the end points stay fixed:

$$\delta\mathbf{x}(t_0) = \delta\mathbf{x}(t_f) = 0$$

The change in the action is

$$\delta S = \delta\left[\int_{t_0}^{t_f} L dt\right] = \int_{t_0}^{t_f} \delta L dt$$

$$= \int_{t_0}^{t_f} \left(\sum_i \frac{\partial L}{\partial \mathbf{x}_i}\delta\mathbf{x}_i + \sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\delta\dot{\mathbf{x}}_i\right) dt$$

We now integrate the second term by parts to get

$$= \int_{t_0}^{t_f} \sum_i \left(\frac{\partial L}{\partial \mathbf{x}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}\right)\right)\delta\mathbf{x}_i dt + \sum_i \underbrace{\left[\frac{\partial L}{\partial \dot{\mathbf{x}}}\delta\mathbf{x}\right]_{t_0}^{t_f}}_{=0}$$

If S has an extremum on the path then $\delta S=0$. Since $\delta\mathbf{x}$ is arbitrary, this must mean:

$$\frac{\partial L}{\partial \mathbf{x}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}\right) = 0$$

Known as the *Lagrange Equations*. To finish: show that they are equivalent to Newton's!

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$$\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) = \mathbf{0}$$

We start by deriving the 1st term:

$$\frac{\partial L(\mathbf{x},\dot{\mathbf{x}})}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left(T\left(\dot{\mathbf{x}}\right) + V\left(\mathbf{x}\right)\right) = -\frac{\partial V}{\partial \mathbf{x}} = \dot{\mathbf{p}}$$

For the 2nd term:

$$\begin{aligned}\frac{\partial L(\mathbf{x},\dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}} &= \frac{\partial}{\partial \dot{\mathbf{x}}}\left(T(\dot{\mathbf{x}}) - V\left(\mathbf{x}\right)\right) \\ &= \frac{\partial}{\partial \dot{\mathbf{x}}}\frac{1}{2}m\dot{\mathbf{x}}^2 = m\dot{\mathbf{x}} = \mathbf{p}\end{aligned}$$

So $$\frac{d}{dt}\frac{\partial L(\mathbf{x},\dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{p}}$$ and the two terms cancel out

Interm' Summary

- We saw that an extremum of the action yields Newton's equations. Since they hold it follows that the action functional must be at an extremum when it is applied to the system's path.
- We will now see that Lagrange's equations hold in any coordinate system while Newton's equations are true only in an inertial frame of reference.
- When the system dynamics are constrained, it is usually easier to find the dynamics equations by using Lagrange's equations rather than with Newton's equations by deriving them in a non-constrasined coordinate system whose dimension is the number of degrees of freedom that the system has.

Changing Coordinate Systems

- Claim: Lagrange's equations hold even when coordinate systems are changed.
- Let $\mathbf{q} = \mathbf{q}(\mathbf{x}, t) \in R^m$ be a new (possibly time varying) set of coordinates. The number of these coordinates may be different than the Cartesian set so as to reflect the true number of Degrees Of Freedom available (e.g, a pendulum is limited to two DOF. A group of points on a rigid body is limited to 6 DOF).
- Assume also that the inverse relation exists: $\mathbf{x} = \mathbf{x}(\mathbf{q}, t)$ i.e. $\left|\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right| \neq \mathbf{0}$

- The velocities(in terms of $\mathbf{q}$) are:

$$\dot{\mathbf{x}} = \sum_j \frac{\partial \mathbf{x}}{\partial \mathbf{q}_j}\dot{\mathbf{q}}_j + \frac{\partial \mathbf{x}}{\partial t}$$

- Note that we can "cancel the dots"

$$\frac{\partial \dot{\mathbf{x}}}{\partial \dot{\mathbf{q}}_i} = \frac{\partial}{\partial \dot{\mathbf{q}}_i}\left(\sum_j \frac{\partial \mathbf{x}}{\partial \mathbf{q}_j}\dot{\mathbf{q}}_j + \frac{\partial \mathbf{x}}{\partial t}\right) = \frac{\partial \mathbf{x}}{\partial \mathbf{q}_i}$$

$$\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) = \mathbf{0} \qquad \frac{\partial \dot{\mathbf{x}}}{\partial \dot{\mathbf{q}}_i} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}_i} \qquad \dot{\mathbf{x}} = \sum_j \frac{\partial \mathbf{x}}{\partial \mathbf{q}_j}\dot{\mathbf{q}}_j + \frac{\partial \mathbf{x}}{\partial t}$$

- To show that Lagrange's equations hold in the new generalized coordinate frame ($\mathbf{x}$) we express them using the new coordinates:
  - $$\frac{\partial L}{\partial \mathbf{q}_j} = \sum_i \frac{\partial L}{\partial \mathbf{x}_i}\frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j} + \cancel{\sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\frac{\partial \dot{\mathbf{x}}_i}{\partial \mathbf{q}_j}}$$
  - $$\frac{\partial L}{\partial \dot{\mathbf{q}}_j} = \sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\frac{\partial \dot{\mathbf{x}}_i}{\partial \dot{\mathbf{q}}_j} = \sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_j}\right) = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}\right)\frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j} + \sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\frac{d}{dt}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j}\right)$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}\right)\frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j} + \cancel{\sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\frac{\partial \dot{\mathbf{x}}_i}{\partial \mathbf{q}_j}}$$

- summing to get Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_j}\right) - \frac{\partial L}{\partial \mathbf{q}_j} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}\right) - \frac{\partial L}{\partial \mathbf{x}_i}\right]\frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_j}\right) - \frac{\partial L}{\partial \mathbf{q}_j} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}\right) - \frac{\partial L}{\partial \mathbf{x}_i}\right] \frac{\partial \mathbf{x}_i}{\partial \mathbf{q}_j}$$

i.e. if Lagrange's equation is solved in coordinate system **x** the RHS vanishes and therefore, it is also solved in **q** and vise-versa (given that the Jacobian has an inverse).

Example: Pendulum

- Unforced planar pendulum
- The cartesian coordinated
  can be stated in terms of θ:

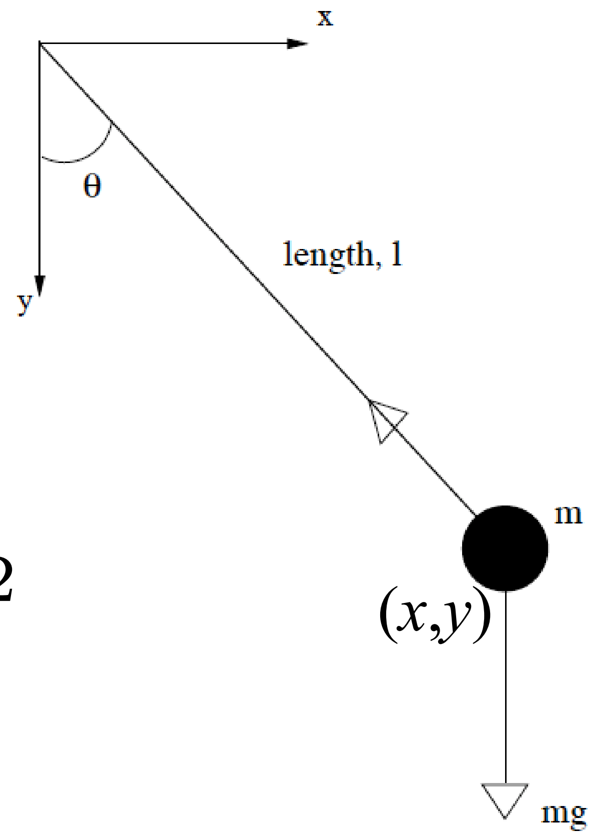
$$y = l\cos\theta \quad x = l\sin\theta$$

Kinetic energy:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

Potential energy

$$V = -mgl\cos\theta$$



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- We plug in to $$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$L = T(\dot{\theta}) - V(\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(ml^2\dot{\theta}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

Subtracting, we get:

$$\ddot{\theta} = -\frac{g}{l}sin\theta$$

- This was easier that deriving the equations using tension