Solving autonomous LTI Systems in Matrix Form
(& all their possible behaviors)

- Laplace transform of *n* 1st order linear ODEs in matrix form
- The inverse transform and *matrix exponential*
- Possible system behaviors
- Discrete-time LTI system (sampling at constant intervals)

Solving n 1st order linear ODEs in matrix form

- An autonomous (undriven, no u) LTI system with an n dimensional state:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- Its Laplace transform is:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

- Rearranging:

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}(0) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \end{aligned}$$

- $(s\mathbf{I} - \mathbf{A})^{-1}$ is known as the *resolvent*.
- To solve: simply find the inverse transform of the resolvent:

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]\mathbf{x}(0)$$
$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

- The inverse transform of the resolvent, $\mathbf{\Phi}(t)$is known as the *state transition matrix* because the system is linearly transformed by it from its state at time 0 (either forwards or backwards in time)

- Note the dependence on **A**'s *characteristic polynomial*: |s**I**-**A**|
  j,*i* entry of resolvent (j_{th} row, i_{th} column) can be expressed via Crammer's rule as:

$$(-1)^{i+j}\frac{\det \Delta_{ij}}{\det(sI-A)}$$

where Δ_{ij} is $sI - A$ with jth row and ith column deleted

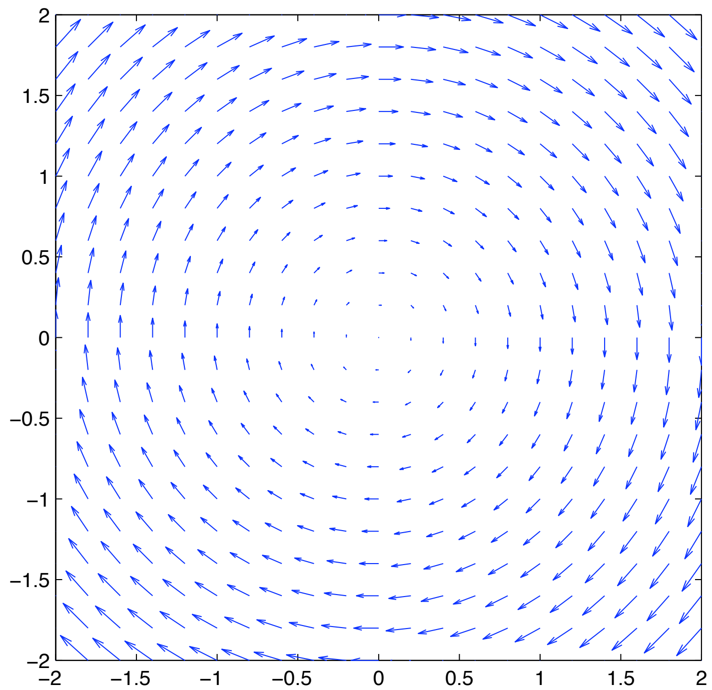
- So each of the entries in the resolvent matrix is a *polynomial fraction* in *s* whose denominator is **A**'s characteristic polynomial

- Since |s**I**-**A**| has real coefficients, its roots (the eigenvalues of **A**) are either real or are complex conjugate pairs...

Example: Harmonic Oscillator

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x} \qquad , \mathbf{x} \in R^2$$

The *phase plane* looks like:



The resolvent:

$$s\mathbf{I} - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

$$(s\mathbf{I} - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

$|s\mathbf{I}\text{-}\mathbf{A}| = s^2+1$ so the eigenvalues of **A** are *i*,-*i*

We could now use the Laplace transform tables to show that:

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

A rotation matrix

(but we show a general solution method instead).

Matrix Exponential

- For $c < 1$

$$(1-c)^{-1} = 1 + c + c^2 + c^3 + \dots$$

- Similarly, for $\mathbf{C} \in R^{n \times n}$ (with small enough real parts of its eigenvalues)

$$(\mathbf{I}-\mathbf{C})^{-1} = \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots$$

- Plug in $\mathbf{C} = \frac{\mathbf{A}}{s}$ and (for large enough s)

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s}\left(\mathbf{I} - \frac{\mathbf{A}}{s}\right)^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \cdots$$

- We've seen that

$$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

- So (due to the linearity of Laplace tr.) we can apply the inverse transform and get..

$$\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \mathbf{I} + t\mathbf{A} + \frac{(t\mathbf{A})^2}{2!} + \cdots$$

- This (infinite series), multiplied by $\mathbf{x}(0)$, is a general solution to LTI systems !

- The above series looks a lot like the Tailor expansion of an exponent:

$$e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots$$

- So we will borrow the exponent symbol and define a *matrix exponent* as

$$e^{\mathbf{M}} \equiv \mathbf{I} + \mathbf{M} + \frac{\mathbf{M}^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{\mathbf{M}^i}{i!}$$

- Using this new symbol we can write the solution

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]\mathbf{x}(0) = e^{t\mathbf{A}}\mathbf{x}(0)$$

- For a scalar system, $\dot{x} = ax$ the above solution is what we expect:

$$x(t) = e^{ta}x(0)$$

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

example: you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

however, we do have $e^{A+B} = e^A e^B$ if $AB = BA$, *i.e.*, A and B commute

- So for scalars t,s $e^{(t+s)\mathbf{A}} = e^{(t\mathbf{A}+s\mathbf{A})} = e^{t\mathbf{A}}e^{s\mathbf{A}}$

  since $(t\mathbf{A})(s\mathbf{A}) = (s\mathbf{A})(t\mathbf{A})$

  This is useful in understanding the LTI system solution...

Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

interpretation: the matrix e^{tA} propagates initial condition into state at time t

more generally we have, for *any* t and τ,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

interpretation: the matrix e^{tA} propagates state t seconds forward in time (backward if $t < 0$)

Autonomous LTI System Behaviors

or why are **A**'s eigenvalues interesting

- Given an autonomous LTI system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, if **A** is diagonalizable then there exists a matrix $\mathbf{P} \in C^{n \times n}$ whose columns are **A**'s eigenvectors and a diagonal matrix **D** of eigenvalues

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix}$$

  such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$

- Note the following useful property:

$$\mathbf{A}^n = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdots \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$$

  We will use it to rewrite the matrix exponential..

$$
\begin{aligned}
\mathbf{\Phi}(t) &= e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{(t\mathbf{A})^2}{2!} + \frac{(t\mathbf{A})^3}{3!} \cdots \\
&= \mathbf{P}\mathbf{I}\mathbf{P}^{-1} + \mathbf{P}t\mathbf{D}\mathbf{P}^{-1} + \frac{\mathbf{P}(t\mathbf{D})^2\mathbf{P}^{-1}}{2!} + \frac{\mathbf{P}(t\mathbf{D})^3\mathbf{P}^{-1}}{3!} \cdots \\
&= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\begin{bmatrix} \lambda_1 t & & \\ & \ddots & \\ & & \lambda_n t \end{bmatrix}}\mathbf{P}^{-1}
\end{aligned}
$$

- Using the definition of the matrix exponential again:

$$
\begin{aligned}
e^{\mathbf{D}} &= \mathbf{I} + \mathbf{D} + \frac{\mathbf{D}^2}{2!} + \cdots \\
&= \sum_{i=0}^{2} \frac{1}{i!} \begin{bmatrix} (\lambda_1 t)^i & & \\ & \ddots & \\ & & (\lambda_n t)^i \end{bmatrix} \\
&= \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}
\end{aligned}
$$

- So we see that (using matrix eigenvalue decomposition) we can write the solution of the system as simply

$$\mathbf{x}(t) = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}\mathbf{x}(0)$$

- Or, if we look at the coordinate system defined by **P**:

$$\mathbf{P}^{-1}\mathbf{x}(t) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}\mathbf{x}(0)$$

or, defining $\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}}$ such that $\mathbf{x} = \mathbf{P}\tilde{\mathbf{x}}$ we see that

$$\dot{\tilde{\mathbf{x}}} = \mathbf{D}\tilde{\mathbf{x}}$$

- So in these coordinates the system decomposes into separate scalar subsystems:

$$\tilde{x}_i = e^{\lambda_i t}\tilde{x}_i(0)$$

- If a system's eigenvalue is complex, $\lambda_i = \sigma + i\omega$ this corresponds to an exponentially decaying/expanding sinosoid:

$$e^{\lambda_i t} = e^{\sigma t} \left(\cos(\omega t) + i \sin(\omega t) \right)$$

- In the original (real) coordinate system the solution is simply a linear combination of expanding/decaying, possibly oscillating exponents:

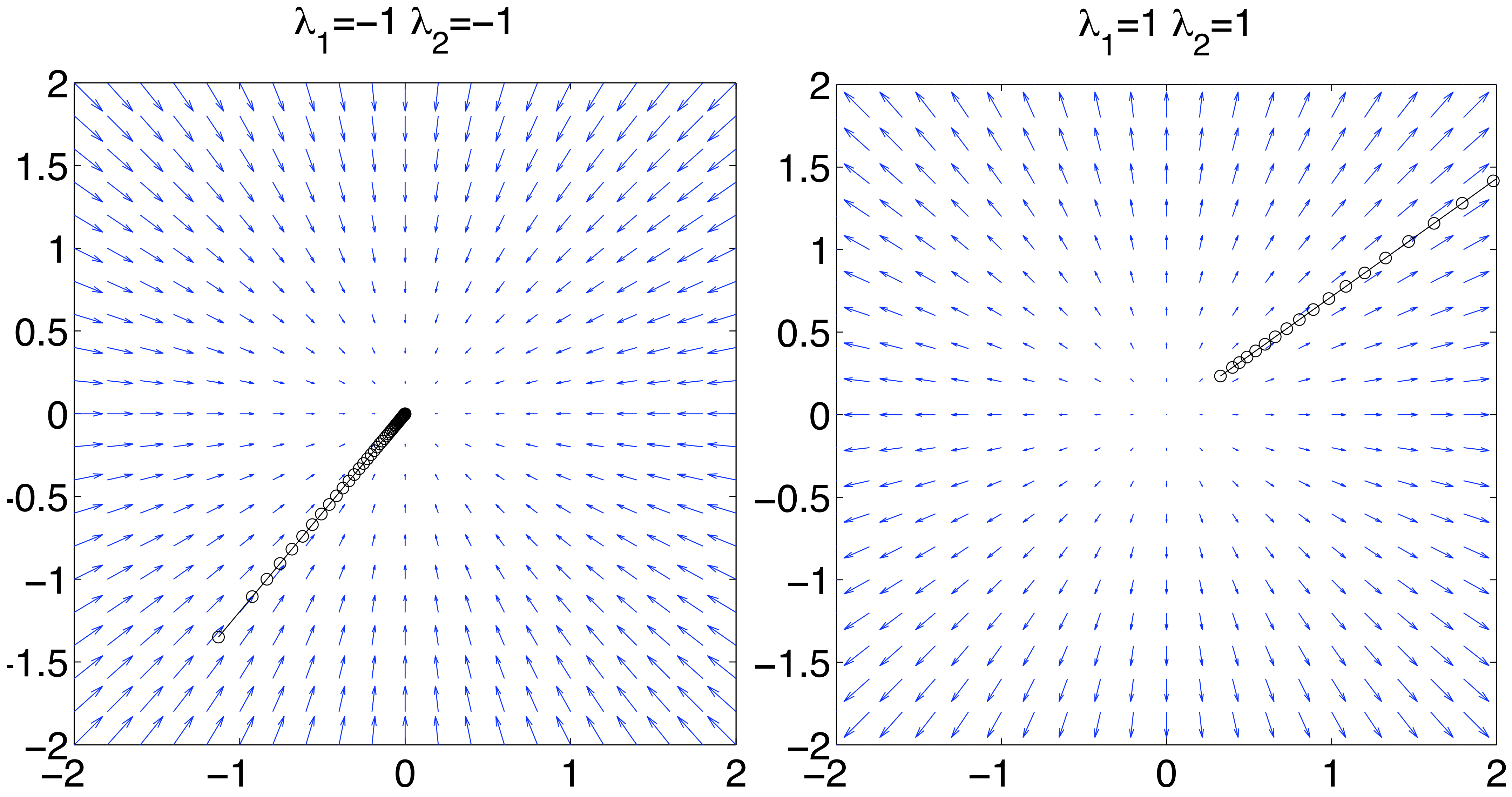
$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

- Define a *stable system* to be one in which **x**(t) goes to **0** as time passes. Then the system is stable only if all the eigenvalues of **A** have real parts that are negative, i.e for each i

$$.\Re e(\lambda_i) < 0$$

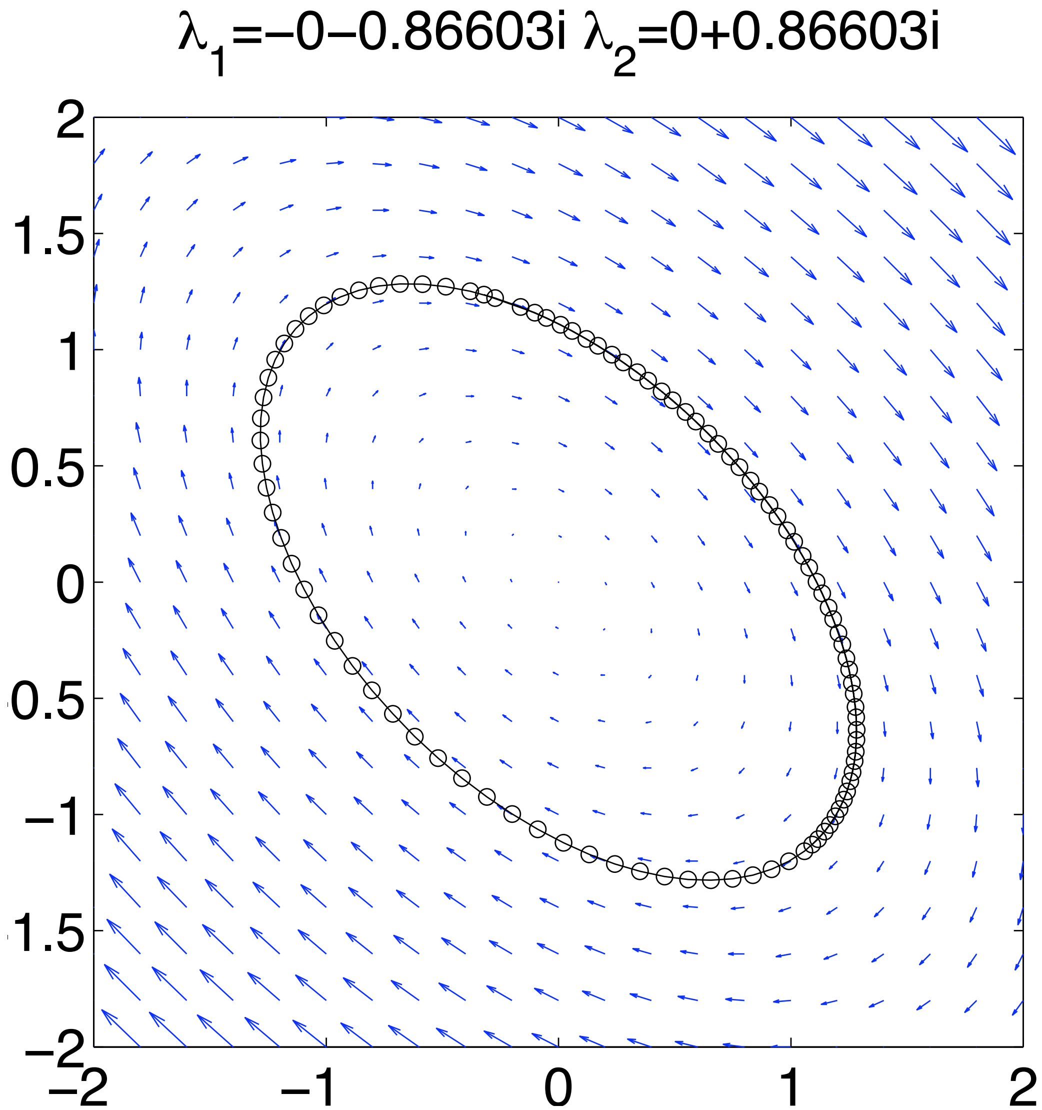
2D examples of system behaviors

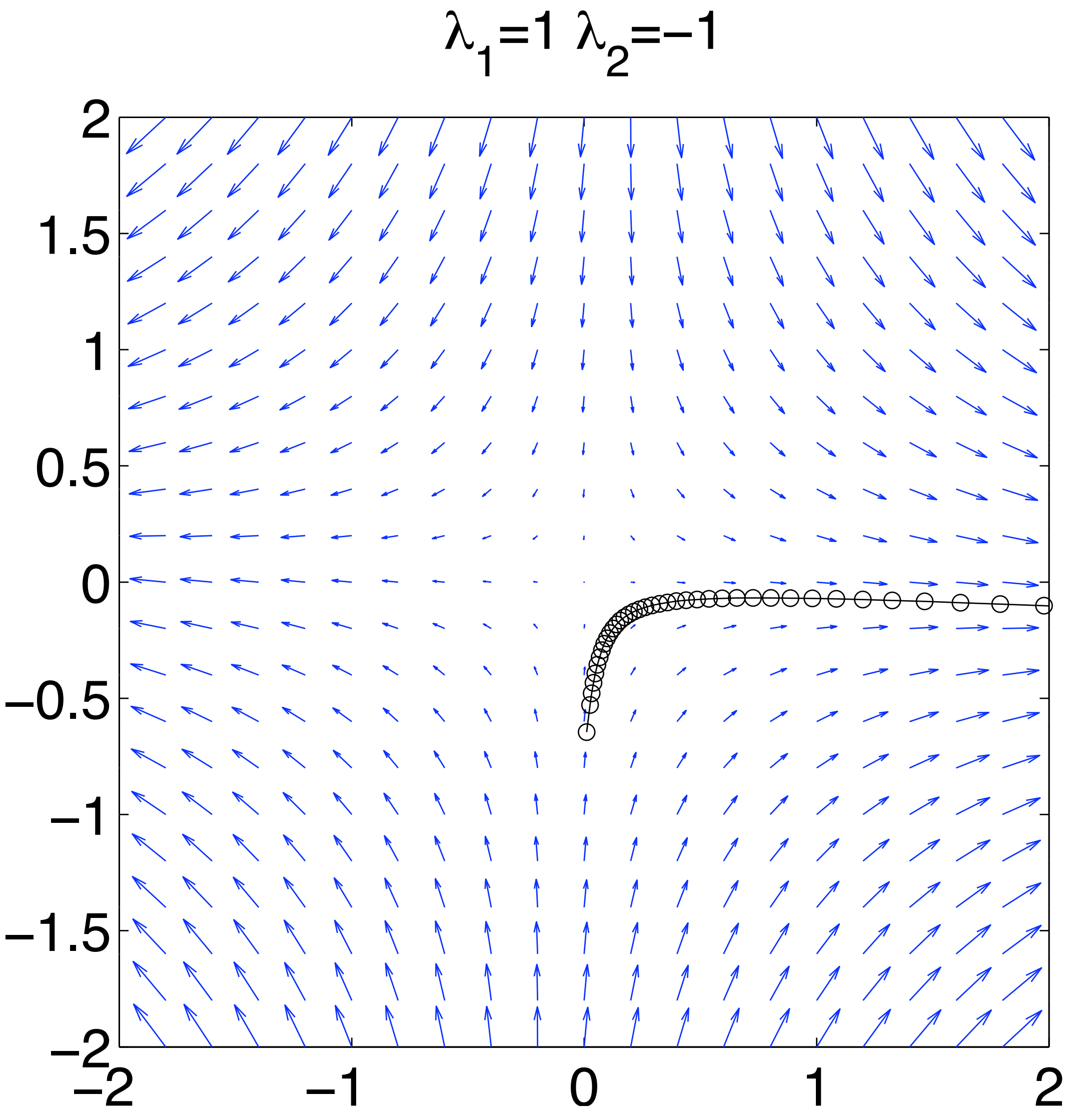
- Real eigenvectors: no oscillation



2D examples of system behaviors

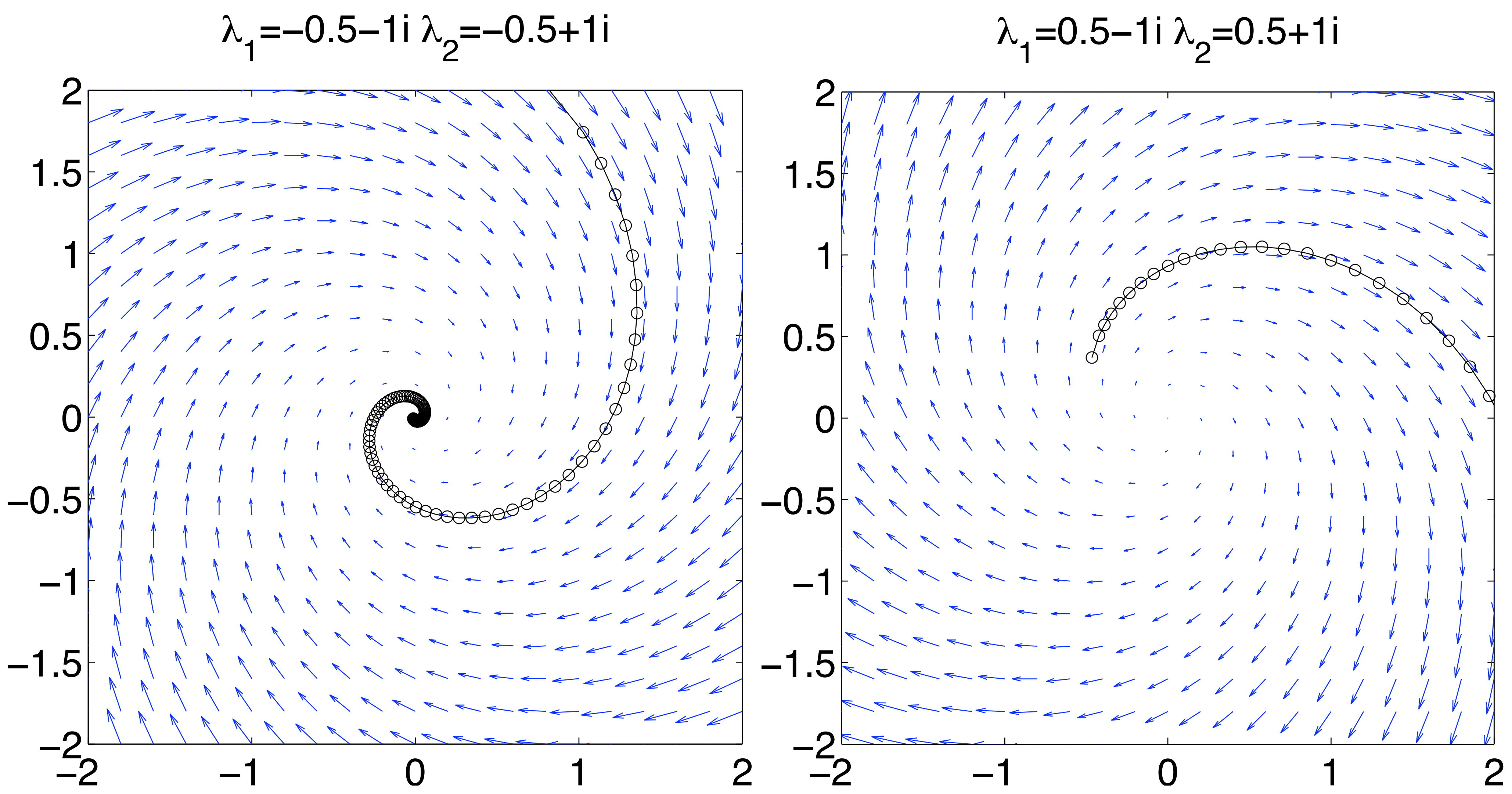
- Purely imaginary complex conjugate pair: permanent oscillation
- one positive real part and one negative (must be purely real): saddle point





2D examples of system behaviors

- Complex conjugate pair: either exponentially decaying (negative real part) or expanding (negative real part) oscillations.



Sampling a Continuous LTI System at Constant Intervals

- Given a continuous (autonomous) LTI system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
- We'd like to sample it at constant time intervals $n\Delta t$
- We've seen that

$$\mathbf{x}(n\Delta t) = e^{\mathbf{A}n\Delta t}\mathbf{x}(0)$$

- So, for *n*+1

$$\begin{aligned}\mathbf{x}((n+1)\Delta t) &= e^{\mathbf{A}(n+1)\Delta t}\mathbf{x}(0) \\ &= e^{\mathbf{A}n\Delta t}e^{\mathbf{A}\Delta t}\mathbf{x}(0) \\ &= e^{\mathbf{A}\Delta t}\mathbf{x}(n\Delta t)\end{aligned}$$

- We can therefore represent the system in discrete time as:

$$\begin{aligned}\mathbf{x}(n) &= \tilde{\mathbf{A}}\mathbf{x}(n-1) \\ \tilde{\mathbf{A}} &= e^{\mathbf{A}\Delta t}\end{aligned}$$

Stability in a Discrete system

- Let $\mathbf{x}(k) = \mathbf{A}\mathbf{x}(k\text{-}1) = \mathbf{A}^k\mathbf{x}(0)$
- Decompose $\mathbf{A}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, then

$$\underbrace{\mathbf{P}^{-1}\mathbf{x}(k)}_{\bar{\mathbf{x}}(k)} = \underbrace{\begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix}}_{\mathbf{D}^k} \underbrace{\mathbf{P}^{-1}\mathbf{x}(0)}_{\bar{\mathbf{x}}(0)}$$

- So in the new coordinate system it is clear that the state goes to zero only if for all eigenvalues

$$|\lambda_i| < 1$$

- This, of course, holds for the original system as well.