Note that the transfer function of the hydraulic actuator is similar to that of the electric motor. Also, for an actuator operating at high pressure levels and requiring a rapid response of the load, the effect of the compressibility of the fluid must be accounted for [4, 5].

The M.K.S. units of the variables are given in Table B-1 in the Appendix. Also the conversion factors for the English system of units are given in Table B-2.

The transfer function concept and approach is very important since it provides the analyst and designer with a useful mathematical model of the system elements. We shall find the transfer function to be a continually valuable aid in the attempt to model dynamic systems. The approach is particularly useful since the s-plane poles and zeros of the transfer function represent the transient response of the system. The transfer functions of several dynamic elements are given in Table 2-4.

2-6 Block diagram models

The dynamic systems that comprise automatic control systems are represented mathematically by a set of simultaneous differential equations. As we have noted in the previous sections, the introduction of the Laplace transformation reduces the problem to the solution of a set of linear algebraic equations. Since control systems are concerned with the control of specific variables, the interrelationship of the controlled variables to the controlling variables is required. This relationship is typically represented by the transfer function of the subsystem relating the input and output variables. Therefore, one can assume, correctly, that the transfer function is an important relation for control engineering.

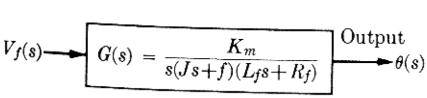


FIG. 2-17. Block diagram of dc-motor.

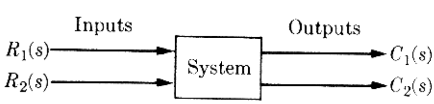


FIG. 2-18. General block representation of 2-input, 2-output system.

The importance of the cause and effect relationship of the transfer function is evidenced by the interest in representing the relationships of system variables by diagrammatic means. The *block diagram* representation of the systems relationships is prevalent in control system engineering. Block diagrams consist of *unidirectional*, operational blocks that represent the transfer function of the variables of interest. A block diagram of a field controlled dc-motor and load is shown in Fig. 2-17. The relationship between the displacement $\theta(s)$ and the input voltage $V_f(s)$ is clearly portrayed by the block diagram.

In order to represent a system with several variables under control, an interconnection of blocks is utilized. For example, the system shown in Fig. 2-18

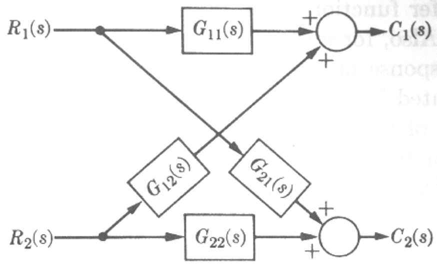


FIG. 2-19. Block diagram of interconnected system.

has two input variables and two output variables. Using transfer function relations, we can write the simultaneous equations for the output variables as

$$C_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s), \tag{2-77}$$

$$C_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s), \tag{2-78}$$

where $G_{ij}(s)$ is the transfer function relating the ith output variable to the jth input variable. The block diagram representing this set of equations is shown in Fig. 2-19. In general, for J inputs and I outputs, we write the simultaneous equation in matrix form as

$$\begin{bmatrix} C_1(s) \\ C_2(s) \\ \vdots \\ C_I(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & \cdots & G_{1J}(s) \\ G_{21}(s) & \cdots & G_{2J}(s) \\ \vdots & & \vdots \\ G_{I1}(s) & \cdots & G_{IJ}(s) \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_J(s) \end{bmatrix}, \tag{2-79}$$

or, simply,

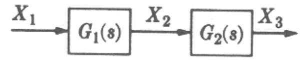
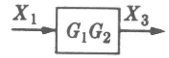
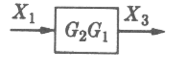
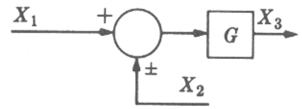
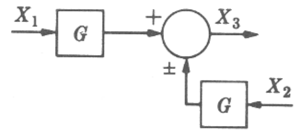
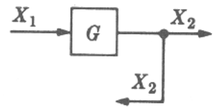
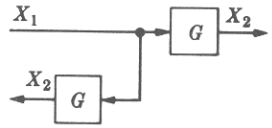
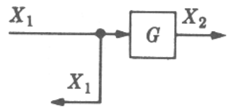
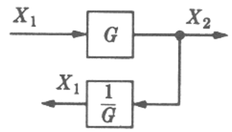
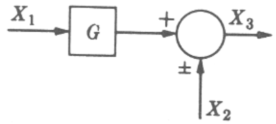
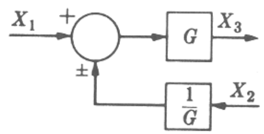
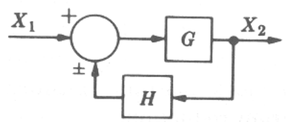
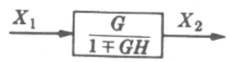
$$\mathbf{C} = \mathbf{G}\mathbf{R}. \tag{2-80}$$

Here the **C** and **R** matrices are column matrices containing the I output and the J input variables, respectively, and **G** is an I by J transfer function matrix. The matrix representation of the interrelationship of many variables is particularly valuable for complex multivariable control systems. An introduction to matrix algebra is provided in Appendix C for those unfamiliar with matrix algebra or who would find a review helpful [31].

The block diagram representation of a given system may often be reduced by block diagram reduction techniques to a simplified block diagram with fewer blocks than the original diagram. Since the transfer functions represent linear systems, the multiplication is commutative. Therefore, as in Table 2-5, Item 1,

Table 2-5

BLOCK DIAGRAM TRANSFORMATIONS

Transformation	Original diagram	Equivalent diagram
1. Combining blocks in cascade		 or 
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

Block diagram transformations and reduction techniques are derived by considering the algebra of the diagram variables. For example, consider the block diagram shown in Fig. 2–20. This negative feedback control system is described by the equation for the actuating signal

$$\begin{aligned} E_a(s) &= R(s) - B(s) \\ &= R(s) - H(s)C(s). \end{aligned} \tag{2–81}$$

Since the output is related to the actuating signal by $G(s)$, we have

$$C(s) = G(s)E_a(s), \tag{2–82}$$

and therefore

$$C(s) = G(s)\big(R(s) - H(s)C(s)\big). \tag{2–83}$$

Solving for $C(s)$, we obtain

$$C(s)\big(1 + G(s)H(s)\big) = G(s)R(s). \tag{2–84}$$

Therefore the transfer function relating the output $C(s)$ to the input $R(s)$ is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}. \tag{2–85}$$

This *closed-loop transfer function* is particularly important since it represents many of the existing practical control systems.

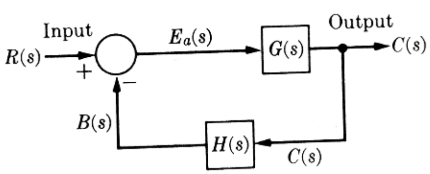


FIG. 2–20. Negative feedback control system.

The reduction of the block diagram shown in Fig. 2–20 to a single block representation is one example of several useful block diagram reductions. These diagram transformations are given in Table 2–5. All the transformations in Table 2–5 can be derived by simple algebraic manipulation of the equations representing the blocks. System analysis by the method of block diagram reduction has the advantage of affording a better understanding of the contribution of each component element than is possible to obtain by the manipulation of equations. The utility of the block diagram transformations will be illustrated by an example of a block diagram reduction.

EXAMPLE 2–4. The block diagram of a multiple-loop feedback control system is shown in Fig. 2–21 [8]. It is interesting to note that the feedback signal

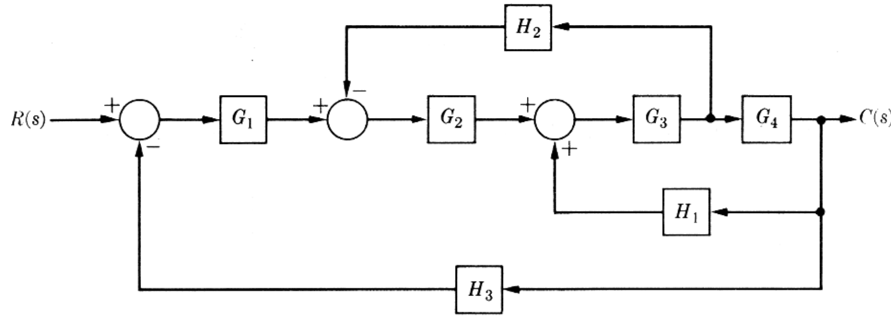


FIG. 2–21. Multiple loop feedback control system.

$H_1(s)C(s)$ is a positive feedback signal and the loop $G_3(s)G_4(s)H_1(s)$ is called a *positive feedback loop*. The block diagram reduction procedure is based on the utilization of rule 6 which eliminates feedback loops. Therefore, the other transformations are used in order to transform the diagram to a form ready for eliminating feedback loops. First, in order to eliminate the loop $G_3G_4H_1$, we move H_2 behind block G_4 by using rule 4, and therefore obtain Fig. 2–22(a). Eliminating the loop $G_3G_4H_1$ by using rule 6, we obtain Fig. 2–22(b). Then, eliminating the inner loop containing H_2/G_4, we obtain Fig. 2–22(c). Finally, by reducing the loop containing H_3 we obtain the closed-loop system transfer function as shown in Fig. 2–22(d). It is worthwhile to examine the form of the numerator and denominator of this closed-loop transfer function. We note that the numerator is composed of the cascade transfer function of the feedforward elements connecting the input $R(s)$ and the output $C(s)$. The denominator is comprised of 1 minus the sum of each loop transfer function. The sign of the loop $G_3G_4H_1$ is plus, since it is a positive feedback loop, while the loops $G_1G_2G_3G_4H_3$ and $G_2G_3H_2$ are negative feedback loops. The denominator can be rewritten as

$$q(s) = 1 - (+G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4H_3) \tag{2–86}$$

in order to illustrate this point. This form of the numerator and denominator is quite close to the general form for multiple-loop feedback systems as we shall find in the following section.

The block diagram representation of feedback control systems is a very valuable and widely used approach. The block diagram provides the analyst with a graphical representation of the interrelationships of controlled and input variables. Furthermore, the designer can readily visualize the possibilities for adding blocks to the existing system block diagram in order to alter and improve the system performance. The transition from the block diagram method to a method utilizing a line path representation instead of a block representation is readily accomplished and is presented in the following section.

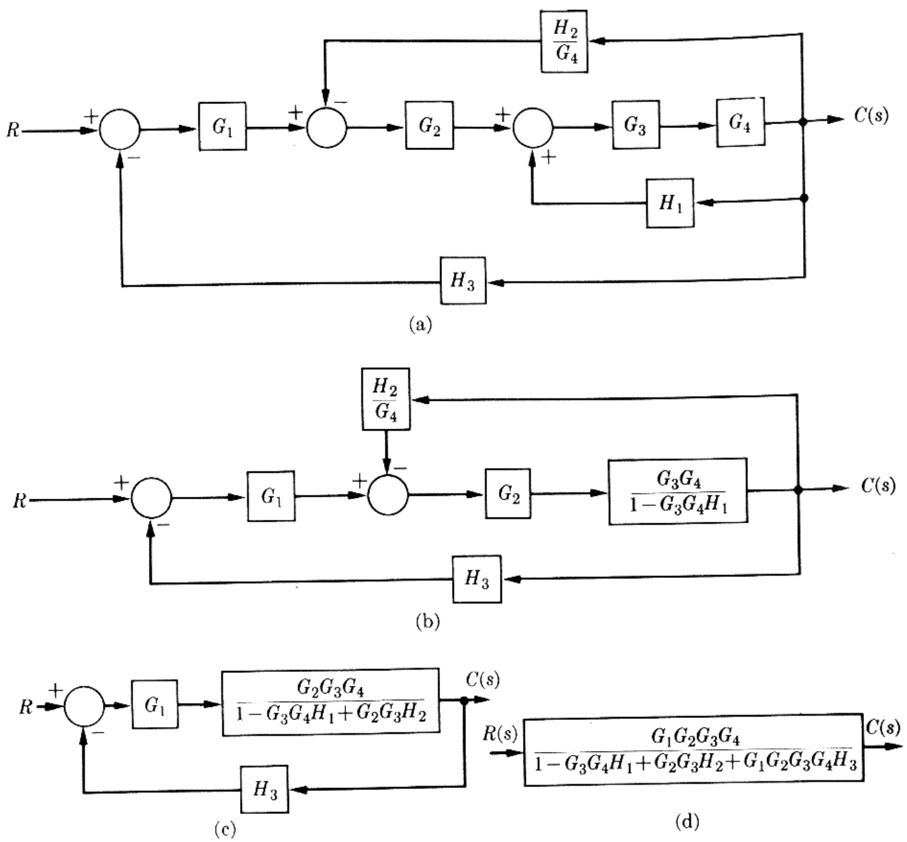


FIG. 2-22. Block diagram reduction of the system of Fig. 2-21.

2-7 Signal flow graph models

Block diagrams are adequate for the representation of the interrelationships of controlled and input variables. However, for a system with reasonably complex interrelationships, the block diagram reduction procedure is cumbersome and often quite difficult to complete. An alternative method for determining the relationship between system variables has been developed by Mason and is based on a representation of the system by line segments [9]. The advantage of the line path method, called the signal flow graph method, is the availability of a flow graph gain formula which provides the relation between system variables without requiring any reduction procedure or manipulation of the flow graph.

The transition from a block diagram representation to a directed line segment representation is easy to accomplish by reconsidering the systems of the previous section. A *signal-flow graph* is a diagram consisting of nodes which are connected

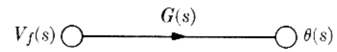


FIG. 2-23. Signal flow graph of the dc-motor.

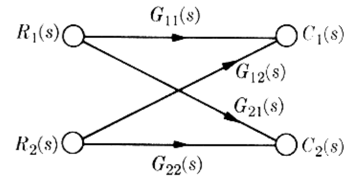


FIG. 2-24. Signal flow graph of interconnected system.

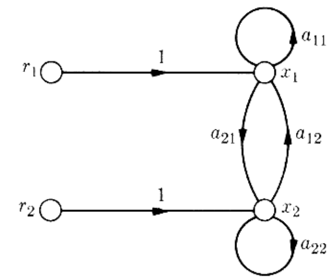


FIG. 2-25. Signal flow graph of two algebraic equations.

by several directed branches and is a graphical representation of a set of linear relations. Signal-flow graphs are particularly useful for feedback control systems because feedback theory is primarily concerned with the flow and processing of signals in systems. The basic element of a signal flow graph is a unidirectional path segment called a *branch* which relates the dependency of an input and an output variable in a manner equivalent to a block of a block diagram. Therefore, the branch relating the output of a dc-motor, $\theta(s)$, to the field voltage, $V_f(s)$, is similar to the block diagram of Fig. 2-17 and is shown in Fig. 2-23. The input and output points or junctions are called *nodes*. Similarly, the signal-flow graph representing Eqs. (2-77) and (2-78) and Fig. 2-19 is shown in Fig. 2-24. The relation between each variable is written next to the directional arrow. All branches leaving a node pass the nodal signal to the output node of each branch (unidirectionally). All branches entering a node summate as a total signal at the node. A *path* is a branch or a continuous sequence of branches which can be traversed from one signal (node) to another signal (node). A *loop* is a closed path which originates and terminates on the same node, and along the path no node is met twice. Therefore, reconsidering Fig. 2-24, we obtain

$$C_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s), \tag{2-87}$$

$$C_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s). \tag{2-88}$$

The flow graph is simply a pictorial method of writing a system of algebraic equations so as to indicate the interdependencies of the variables. As another example, consider the following set of simultaneous algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1, \tag{2-89}$$

$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2. \tag{2-90}$$

The two input variables are r_1 and r_2, and the output variables are x_1 and x_2. A signal flow graph representing Eqs. (2-89) and (2-90) is shown in Fig. 2-25.

Loops L_1 and L_2 do not touch L_3 and L_4. Therefore, the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4). \tag{2-99}$$

The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from Δ. Therefore, we have

$$L_1 = L_2 = 0 \quad \text{and} \quad \Delta_1 = 1 - (L_3 + L_4).$$

Similarly, the cofactor for path 2 is

$$\Delta_2 = 1 - (L_1 + L_2).$$

Therefore, the transfer function of the system is

$$\frac{C(s)}{R(s)} = T(s) = \frac{P_1\,\Delta_1 + P_2\,\Delta_2}{\Delta} \tag{2-100}$$

$$= \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}. \tag{2-101}$$

The signal-flow graph gain formula provides a reasonably straightforward approach for the evaluation of complicated systems. In order to compare the method with block diagram reduction, which is really not much more difficult, let us reconsider the complex system of Example 2-4.

Example 2-6. A multiple-loop feedback system is shown in Fig. 2-22(a) in block diagram form. There is no reason to redraw the diagram in signal flow graph form so we shall proceed as usual by using the signal flow gain formula, Eq. (2-98). There is one forward path $P_1 = G_1G_2G_3G_4$. The feedback loops are

$$L_1 = -G_2G_3H_2, \qquad L_2 = G_3G_4H_1, \qquad L_3 = -G_1G_2G_3G_4H_3.$$

All the loops have common nodes and therefore are all touching. Furthermore, the path P_1 touches all the loops and hence $\Delta_1 = 1$. Thus, the closed-loop transfer function is

$$T(s) = \frac{C(s)}{R(s)} = \frac{P_1\,\Delta_1}{1 - L_1 - L_2 - L_3}$$

$$= \frac{G_1G_2G_3G_4}{1 + G_2G_3H_2 - G_3G_4H_1 + G_1G_2G_3G_4H_3}. \tag{2-102}$$

Example 2-7. Finally, we shall consider a reasonably complex system which would be difficult to reduce by block diagram techniques. A system with several feedback loops and feedforward paths is shown in Fig. 2-27. The

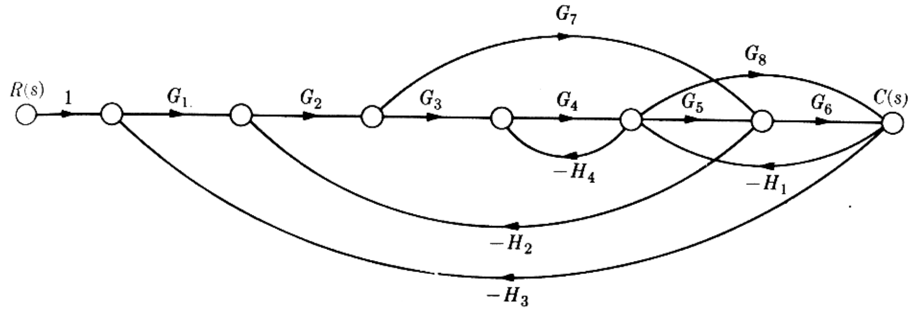


Fig. 2-27. Multiple-loop system.

forward paths are

$$P_1 = G_1G_2G_3G_4G_5G_6, \qquad P_2 = G_1G_2G_7G_6, \qquad P_3 = G_1G_2G_3G_4G_8.$$

The feedback loops are

$$L_1 = -G_2G_3G_4G_5H_2, \qquad L_2 = -G_5G_6H_1, \qquad L_3 = -G_8H_1,$$
$$L_4 = -G_7H_2G_2, \qquad L_5 = -G_4H_4, \qquad L_6 = -G_1G_2G_3G_4G_5G_6H_3,$$
$$L_7 = -G_1G_2G_7G_6H_3, \qquad L_8 = -G_1G_2G_3G_4G_8H_3.$$

Loop L_5 does not touch loop L_4 and loop L_7; loop L_3 does not touch loop L_4; and all other loops touch. Therefore the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5L_7 + L_5L_4 + L_3L_4). \tag{2-103}$$

The cofactors are

$$\Delta_1 = \Delta_3 = 1 \quad \text{and} \quad \Delta_2 = 1 - L_5 = 1 + G_4H_4.$$

Finally, the transfer function is then

$$T(s) = \frac{C(s)}{R(s)} = \frac{P_1 + P_2\,\Delta_2 + P_3}{\Delta}. \tag{2-104}$$

Signal-flow graphs and the signal-flow gain formula may be used profitably for the analysis of feedback control systems, analog computer diagrams, electronic amplifier circuits, statistical systems, mechanical systems, among many other examples [10].

2-8 Simulation of control systems

When a model is available for a component or system, a computer can be utilized to investigate the behavior of the system. A computer model of a system in a mathematical form suitable for demonstrating the system's behavior may be